# The Duration of Liabilities with Interest Sensitive Cash Flows


## Abstract

In order to apply asset-liability management techniques to property-liability insurers, the sensitivity of liabilities to interest rate changes, or duration, must be calculated. The current approach is to use the Macaulay or modified duration calculations, both of which presume that the cash flows are invariant with respect to interest rate changes. Based on the structure of liabilities for property-liability insurers, changes in interest rates -- given that interest rates are correlated with inflation -- should affect future cash flows on existing liabilities. This paper analyzes the effect that interest rate changes can have on these cash flows, shows how to calculate the resulting effective duration of these liabilities, and demonstrates the impact of failing to use the correct duration measure on asset-liability management for property-liability insurers.

## 1. Introduction

Asset-liability management (ALM), as used in the insurance industry, is a process by which insurers attempt to evaluate and adjust the exposure of the net value of the company (assets minus liabilities) to interest rate changes. Although, in theory, the volatility of other factors -- e.g., catastrophes, changes in unemployment rates[1] -- can also affect both asset and liability values, the current focus of ALM for insurers, as for most other financial institutions, is on interest rate risk. Life insurers were the first in the industry to apply ALM techniques, since they have significant exposure to interest rate risk due to the long payout patterns of losses and their high leverage. However, this approach is now being applied to the property-liability insurance industry as well.

The general approaches used by life insurers to measure the sensitivity of assets to interest rate risk are applicable to property-liability insurers to the extent that they have similar asset portfolios. In general, property-liability companies invest more heavily in equities and less in mortgages, but the overall structure of the investment portfolio is roughly similar. However, the *liabilities* of property-liability insurers are different enough that the approaches used by life insurers are simply not applicable to them, and new techniques must be developed.

The basic approach of ALM involves measuring the durations of assets and liabilities, and then adjusting one or both until the insurer is not significantly affected by interest rate changes (essentially, this involves setting the duration of surplus, $D_S$, equal to zero). If the duration of liabilities is measured incorrectly, then an insurer trying to immunize itself from interest rate risk

[1]For example, an increase in the unemployment rate is likely to increase the severity of workers compensation losses and also alter the prepayment patterns on mortgage-backed securities.

based on the incorrect measure will actually still be exposed to interest rate risk. Much research has been done on determining the duration of complex financial instruments held by insurers, such as collateralized mortgage obligations (CMOs) (Fabozzi 1995, Chapter 25), and corporate bonds with callability provisions. Attention has also been given to determining the appropriate duration measure of life insurance liabilities (Babbel 1995). However, much less attention has been paid to the duration of liabilities of property-liability insurers. (The issue has been briefly discussed or alluded to in, for example, Butsic, 1981; D'Arcy, 1984; Ferguson, 1983; and Noris, 1985.) The general approach to measuring the duration of liabilities for property-liability insurers has been to calculate a weighted average of the time to payment for loss reserves (Campbell, 1995; Hodes and Feldblum, 1996; Staking and Babbel, 1995). This approach is patterned after the work by Frederick Macaulay (1938), which determined that the sensitivity of the price of non-callable fixed income securities to changes in interest rates was approximated by this duration measure:

$$Macaulay\ Duration = \sum_{i=1}^{n} \frac{t \times PVCF_t}{PVTCF}$$

where PVCF = the Present Value of the Cash Flow at time t,

PVTCF = the Present Value of the Total Cash Flow, and

t = time to payment of the cash flow.

Additional analysis (Panning 1995) has been based on the modified duration measure (Fabozzi 1995), which is the Macaulay duration value divided by $1+r$ (where $r$ is the current interest rate):

$$Modified\ Duration = \frac{Macaulay\ Duration}{1+r}$$

or alternatively a measure of the slope of the price vs yield curve (see Appendix).

Both the Macaulay and modified duration calculations depend on three basic assumptions:

1. The yield curve is flat
2. Any change in interest rates is a parallel yield curve shift
3. The cash flows do not change as interest rates change

In practice, none of these assumptions is correct. A number of researchers have examined the effect of the first two assumptions in general. (See Klaffky, Ma, and Nozari, 1992; Ho, 1992; and Babbel, Merrill, and Panning, 1997.) In addition, the issue of variable cash flows has been widely recognized for specific classes of assets. Bonds with embedded options (such as call provisions) and mortgage-backed securities (where prepayments depend on the interest rate level) are examples of assets on which the expected cash flows change as interest rates change. A measure termed effective duration has been developed to express the sensitivity of the present value of the expected cash flows with respect to interest rate changes; this measure specifically reflects the fact that the cash flows can change as interest rates change (Fabozzi 1995). For assets with variable cash flows, it is appropriate to calculate the effective duration rather than the modified duration.

The liabilities of property-liability insurers also vary with interest rates, due to the correlation of interest rates with inflation. As explained by Hodes and Feldblum, "Personal auto loss reserves are at least partially inflation sensitive. Medical payments in tort liability states, for instance, depend in part upon jury awards at the date of settlement. The jury awards, in turn, are influenced by the rate of inflation, which is correlated (at least in the long run) with interest rates." (Hodes and Feldblum, 1996, p. 558.) Thus, the appropriate measure of interest rate sensitivity of the liabilities of property-liability insurers is one that reflects this interest rate-inflation relationship, or effective duration. Hodes and Feldblum indicate that "A mathematical determination of the loss reserve (effective) duration is complex." (Hodes and Feldblum, 1996, p. 559) This is the task that is addressed in the remainder of this paper.

In order to accommodate non-parallel yield curve shifts, stochastic interest rate models must be used. This approach has been advocated for insurance applications by Tilley (1988), Reitano (1992), and Briys and de Varenne (1997 ). However, as pointed out by Litterman and Scheinkman (1991), parallel shifts explain over 80% of historical yield curve movements. Although hypothetical portfolios can be constructed that show significant differences in duration values under parallel versus non-parallel yield curve shifts, for the asset and liability portfolios of typical property-liability insurers these differences are likely to be far less important than the impact of variable cash flows. Thus, this paper focuses on analyzing liability cash flows that vary with interest rate changes. Further research will explore the impact of stochastic interest models for both assets and liabilities for representative property-liability insurers.

Section 2 of this paper discusses the nature and relative significance of property-liability insurance company liabilities. Section 3 examines the three major liability items, and discusses the timings of cash flows for each of these items. The natures of the cash flows have important implications for the type and level of impact on liability durations of changes in interest rates. Section 4 provides a mathematical derivation of a closed-form effective duration formula in a highly simplified framework. Section 5 describes a more detailed numerical model used to estimate effective durations. Section 6 summarizes the results of empirical estimates and sensitivity tests of effective duration measures. Section 7 demonstrates the impact on asset-liability management of using modified versus effective duration measures of liabilities. Section 8 concludes. In addition, an Appendix describes the mathematical underpinnings of duration.

## 2. The Liabilities of Property-Liability Insurers

The three major balance sheet liability items of property-liability insurers are the loss reserve, the loss adjustment expense reserve, and the unearned premium reserve. As of 12/31/97, for the industry in aggregate, these components totaled 84.8% of liabilities (A.M. Best 1998). All of these three reserves are subject to change, via inflationary pressures, as interest rates change. The remaining liabilities of property-liability insurers consist primarily of expenses payable, including taxes, reinsurance, contingent commissions, and declared dividends. These cash flows are not likely to be affected by interest rate changes. Thus, the remainder of this section describes the three major liability components for which we are attempting to measure effective duration.

The *loss reserve* is the estimate of future payments that will be made on losses that have already occurred. Insurers use a variety of techniques to arrive at this value. Some loss reserve estimates are based on a review of the specific circumstances of individual claims. Claims department personnel collect information about the claim, including estimates of the value of property damaged and the extent of bodily injuries, as well as the likelihood that the insurer will be required to pay the claim. For other claims, average values are established based solely on the type of coverage involved. In addition, some estimates have to be made for claims that have already occurred but on which the insurer has no information. These claims, termed "Incurred

But Not Reported" (IBNR) losses, have generally occurred so recently that the insurer has not had time to receive a claim report or perform any evaluation.

The total loss reserve, including a provision for IBNR, is determined based on an actuarial analysis of historical development patterns. For example, if a review of past data revealed that the individual claim estimates made by claims department personnel tended to be 10% too high, the overall reserve would adjust for this redundancy. Other adjustments include those for losses that have not yet been reported, or for trends in reopening claims currently considered closed. Historical data are examined to determine loss development trends, and these trends are then applied to current loss data. In general, the trends are based on aggregate historical information -- thus the combined impact of late reporting, inflation, new legal basis for liability, improvements in damage assessment and repair, etc., would all be aggregated. (There is one reserving technique that attempts to isolate the inflationary component from the other effects (Taylor 1986), but this approach is not widely used.)

The *loss adjustment expense reserve* is the estimate of expenses associated with settling claims that have already occurred. Loss adjustment expenses are classified into two types: allocated and unallocated. Allocated loss adjustment expenses (ALAE) consist of items such as legal fees, independent adjuster costs, court costs, and investigation expenses, which can be assigned to specific claims. Unallocated loss adjustment expenses (ULAE) consist of the expenses of the company in operating a claims department that cannot be accurately allocated to an individual claim. ULAE are assigned to particular lines of business and accident years based on a statutory formula.[2] The reserves for both ALAE and ULAE are also set on the basis of historical patterns.

The *unearned premium reserve* is a statutory reserve requirement for insurers. For a given insurance policy, a company is generally required to reserve a pro-rated portion of the premium representing the part of the policy term that has not yet expired. This reserve is generally recognized as being excessive. Premiums are set to cover losses, loss adjustment expenses, and all other expenses. Whereas the losses and loss adjustment expenses occur throughout the term of the policy, other expenses (such as commissions, underwriting, and premium taxes) tend to occur at the inception of the policy. Thus, there is some "equity" in the unearned premium reserve. The cash flows that will emanate from the unearned premium reserve are essentially losses and loss adjustment expenses on claims that occur after the evaluation date but during the remaining policy term. Since these events have not yet occurred, they are completely sensitive to changes in inflation affecting the value of these future losses.

Since loss reserve estimates are based on historical development patterns, and the historical development patterns are affected by historical economic variables such as interest rates and inflation, the accuracy of the loss reserve is, in essence, path dependent with respect to those economic variables. In other words, the level of loss reserves calculated at any point in time will depend upon how economic variables have performed in prior years. However, it is not the accuracy of the current estimate that is of concern in measuring the effective duration, but how future cash flow patterns are influenced by future interest rate changes via inflation.

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[2] This formula assigns 45% of calendar year unallocated loss adjustment expense payments to the current accident year, 5% to the immediately prior accident year, and the balance in proportion to loss payments by accident year.

An added complication to the measurement of the sensitivity of insurer assets and liabilities to interest rate changes is the statutory accounting conventions of the insurance industry. Specifically, bonds are valued on a book, or amortized, basis. Also, loss liabilities are not discounted to reflect the time value of money until payment. Thus, *statutory* valuations are often not directly affected by changes in interest rates. However, the *economic* values of these assets and liabilities *are* affected by interest rate changes. It is the economic values that are considered here, since these reflect the true worth of the company to its owners.

Each of the three major liability items is discussed in greater detail below. More specifically, Section 3 sets the groundwork for evaluating the impact of future interest rate changes and inflation on the liabilities of property-liability insurers.

## 3. The Timing of Property-liability Insurer Liabilities

### *A. Loss Reserves*

A company's aggregate loss reserve represents the total amount to be paid in the future on all claims that have already occurred. However, a variety of different situations can exist with respect to these claims.

1) A loss reserve can reflect a claim on which the insurer is in the process of issuing a check -- the claim has already been fully investigated, and the insurer has agreed to a settlement amount with the claimant. The nominal value of the claim amount will not be affected by changes in interest rates, although the present value would change slightly.

2) Alternatively, a loss reserve can represent a claim that has caused a known amount of damage to property or to a person (the medical bills are complete). Thus, the amount of the loss to the claimant is determined and will not change. However, the insurer and the claimant are still in dispute over whether the incident is covered, or over the extent of the insurer's liability for payment. Again, the nominal amount of the payment should not change if interest rates change[3]. However, the economic value of the loss would change, since the future cash flow would be discounted by a different interest rate.

3) A third type of loss reserve is for damage that has yet to be incurred. The insurer will be liable for the loss when the claimant experiences it, but the value of the loss will only be known in the future. On an occurrence-based policy, this could apply for medical malpractice to a person who has not yet suffered the adverse consequences of an injury caused by a negligent physician (e.g., improper diagnosis, long term adverse consequences from prescribed medication, surgical errors that will lead to future complications). Or, in the case of workers compensation, if a former employee, exposed to a work related environmental hazard, first manifests the ailment at some future date, the claim will be assigned to policies in effect during the period of employment. For another example, a company may have sold a defective product, but the injuries have not yet occurred The insurer is required to establish loss reserves for these future losses because they will be paid based on prior policies. For these claims, the nominal value of the loss payment will be affected by interest rate changes to the extent that the interest rate change is correlated with inflation on the goods or services related

[3]One way this could happen is if the insurer's claim settlement philosophy were to change with interest rates -- for example, if the financial condition of the insurer were to become impaired in conjunction with an interest rate change and the company had to alter its claim settlement approach.

to the cost of the claim (property damage, medical expenses). The economic value of these losses will also change with interest rates.

4) The most common type of loss reserve is for losses on which some of the damages have already been fixed in value, but the remainder has yet to be determined. In addition, the question of the extent of the insurer's liability may not have been settled. This could apply to an automobile accident involving property damage and bodily injury in which the policyholder of the insurer may be liable. The damage to the claimant's vehicle is predetermined. The injured person has received some medical care, but that care will continue at least up until the settlement of the claim and perhaps beyond. The nominal value of a portion of these losses, termed "fixed," will not be affected by interest rate changes, but the remaining portion of the losses will be affected by future inflation.

Calculating the effect of inflation on *tangible* losses, such as medical expenses, wage losses, and property damage, although complicated, is relatively straightforward once the appropriate inflation indices are determined. However, quantifying the effect of inflation on the value of *intangibles* in a liability claim, termed "general damages" in a legal context, presents additional challenges. These components include items such as pain and suffering, loss of consortium, and hedonic losses. It is difficult to determine exactly how these values are established. Are they based on the value at the time of the loss or the time of the verdict in a jury trial? Is the pain and suffering of a broken arm that occurred in 1986 evaluated the same as, or less than, a similar broken arm that occurred in 1996, if both are being settled at the same time?

Due to the difficulty in putting a numerical value on an intangible such as pain and suffering, general rules of thumb arise that try to relate the pain and suffering award to the medical expenses incurred by the patient. Thus, a broken arm that generated $15,000 in medical bills is worth roughly three times as much as another broken arm that generated only $5,000 in medical bills. (This does not mean that the pain and suffering from a soft tissue injury, such as a sore neck, which generated $15,000 in medical expenses would be worth as much as a broken arm with the same amount of medical expenses.) On this basis, the general damages on liability claims will be impacted by interest rate changes to the same extent that medical expenses are affected. However, a typical question asked by a plaintiff's attorney in a bodily injury case is how much a member of the jury would require to be willing to undergo the same pain that the client has experienced. Since this is asked, rhetorically, near the end of the claim settlement process, conceivably the jury will implicitly adjust the value of the claim to the then-current cost of living. In this case, the entire loss reserve for general damages would be sensitive to future inflation changes.

Determining the effective duration of reserves will, therefore, depend on a model for dividing the future payments into a *fixed* component, which is not sensitive to future inflation, and an *inflation sensitive* component, which will vary with subsequent inflation. This model is developed and described below.

## *B. Loss Adjustment Expenses*

Loss adjustment expense reserves are established for future payments, in a manner similar to loss reserves. These expenses will be paid over the time during which the remaining losses are settled. Loss adjustment expenses are assigned to the accident year in which the loss that generated these expenses occurred; they are assigned either directly (for allocated loss adjustment expenses) or indirectly (for unallocated loss adjustment expenses). The same approach used for determining the proportion of loss reserves that are fixed in value can be used for loss adjustment expense reserves. However, since the rate with which these expenses become fixed in value can differ from the loss itself, they may be modeled separately using different parameter values.

Loss adjustment expenses are different from loss reserves in the following respect. As an insurer generates loss adjustment expenses, such as by hiring outside adjusters, it would generally pay these expenses shortly after the work is completed. The loss adjustment expense reserve, then, represents costs that are fixed in value to a much lower degree than loss reserves. Also, the legal costs associated with defending a claim that goes to court will not be established until the very end of the loss settlement process. In addition, the allocation process for unallocated loss adjustment expenses assigns a portion of the general claim department's expenses to the accident year of the claim when the loss is paid. Thus, for loss adjustment expense reserves, few of these costs will be fixed in value when the claim occurs and a relatively high portion of the total costs will be based on the cost of living when the claim is finally settled.

### C. *Unearned Premium Reserve*

Since the unearned premium reserve essentially represents exposure to losses that have not even occurred yet, this liability is fully sensitive to future inflation. The expected cash flow emanating from the unearned premium reserve will shift to the extent that any change in interest rates is correlated with inflation. If it is assumed that the insurer writes policies with terms not more than one year, then all of the claims emanating from the unearned premium reserve will occur in the next accident year. The payments on these losses will follow the claim payout pattern of the insurer, except that losses will occur approximately in the middle of the first half of the year (assuming annual policies written evenly throughout the year), as opposed to in the middle of the year as would be assumed for accident year data. Thus, the duration of the unearned premium reserve at the end of a year would be the weighted average of the time until payment of the most recent accident year, plus 3/4 of a year. For example, the unearned premium reserve as of 12/31/99 will cover losses that will occur, on average, on 4/1/00. For the loss reserve for accident year 1999, the average loss would have occurred at the middle of the year, or 7/1/99. Thus, the duration of the unearned premium reserve as of 12/31/99 is 3/4 of a year more than the duration of the accident year 1999 loss reserves.

## 4. Mathematical Model of the Effective Duration of Reserves

In Section 5, we will present a detailed numerical model for determining effective duration. In this section, we develop a simplified mathematical model of an effective duration formula. This formula will provide a method to determine the general value of the effective duration of insurance liabilities, as well as a point of reference for the more detailed calculations discussed later.

In this section, it is assumed that all payments are fully sensitive to inflation. In this case, the price level at which an insurer makes a claim payment depends only upon the date of that payment. Put in the context of "fixed" costs described in the last section, here it is assumed that there are no fixed costs. This provides a framework in which a closed-form solution can be easily derived, assuming an appropriate payment pattern. The measurement of duration assuming partial fixed costs will be derived in Section 5.

Define the following variables:

$R_t$ = the (correct) nominal reserve at time t,

$c$ = the (constant) annual payout ratio, and

$r$ = the relevant interest rate.

Assume that the payout, over time, of property-liability reserves is represented by a "proportional decay" model: each year, proportion $c$ of the beginning reserve is paid out.[4] Thus,

$$R_t = (1-c)R_{t-1}$$

Under this assumption, the present value of the initial reserve is expressed as

$$PV(R_0) = \sum_{t=1}^{\infty} \frac{(1-c)^{t-1}cR_0}{(1+r)^t} = \frac{cR_0}{1-c}\sum_{t=1}^{\infty}\left(\frac{1-c}{1+r}\right)^t = \frac{cR_0}{r+c}$$

where the final form of the equation is derived from the formula for an infinite geometric progression.[5] Now, we can derive an expression for the Macaulay duration as follows:

$$Macaulay\ duration = D_1 = \frac{\sum_{t=1}^{\infty} \frac{(1-c)^{t-1}cR_0 t}{(1+r)^t}}{PV(R_0)}$$

By again using the properties of infinite geometric progressions, the numerator of the Macaulay

[4]Theoretically, this assumes that payouts are made forever, although after some years they become negligible in size. Finite-length payout patterns are considered in Section 5.

[5]For $0<x<1$, the value of $x + x^2 + x^3 + \ldots = x/(1-x)$.

duration formula reduces to

$$\frac{cR_0(1-r)}{(r+c)^2}$$

Dividing by the previous expression for $PV(R_0)$, the Macaulay duration is

$$D_0 = \frac{1+r}{r-c}$$

Since the modified duration is the Macaulay duration divided by $(1+r)$, we have

$$\textit{Modified duration} = MD_0 = \frac{1}{r+c}$$

In order to determine the *effective* duration of property-liability insurer liabilities, we must calculate the present value of those liabilities in three different ways: with the original interest rate, with an increased interest rate, and with a decreased interest rate. Under this approach, after calculating the present value assuming the original interest rate, we assume that the interest rate increases (e.g., by 100 basis points), and then that the interest rate decreases (e.g., by 100 basis points). The effective duration is then calculated as

$$\textit{Effective duration} = ED_0 = \frac{PV_- - PV_+}{2PV_0(\Delta r)}$$

where $PV_-$ = the present value of the expected cash flows if interest rates decline by $\Delta r$,

$PV_+$ = the present value of the expected cash flows if interest rates increase by $\Delta r$,

and

$PV_0$ = the initial present value of the expected cash flows.

The key in calculating the effective duration is to account for the impact of hypothetical changes in the interest rate on the future cash flows emanating from the liability items. For property-liability reserves, the primary impact on cash flows of a change in interest rates is via inflation: since interest rates are correlated with inflation, and inflation increases future nominal claim payments, changes in interest rates will affect the level of future cash outflows, and thus the present value of those outflows. Therefore, in order to calculate the effective duration, we need to adjust the formulas above to reflect this inflationary impact.

Define the following additional variables:

$r_{+\text{ or }-} = r \pm \Delta r$ = the increased or decreased interest rate, and

$i_{+\text{ or }-}$ = the inflationary adjustment after the change in interest rate.

The inflationary adjustment contemplates the correlation between changes in interest rates and inflation (actually, not just overall inflation, but claim inflation for the specific type of insurance at issue).

We can now adjust the present value equation above in preparation for calculating the effective duration:

$$PV_{+}(R_0) = \sum_{t=1}^{\infty} \frac{(1-c)^{t-1} c R_0 (1+i_{+})^t}{(1+r_{+})^t} = \frac{cR_0}{1-c} \sum_{t=1}^{\infty} \left(\frac{(1-c)(1+i_{-})}{1+r_{+}}\right)^t = \frac{cR_0(1+i_{+})}{r_{+}+c+ci_{-}-i_{+}}$$

A similar equation applies for the present value of reserves under the assumption of an interest rate decrease. Thus, we derive the following formula for the effective duration:

$$ED_0 = \frac{r+c}{2\Delta r}\left[\frac{1+i_-}{r_-+c+ci_--i_-} - \frac{1+i_+}{r_++c+ci_+-i_+}\right]$$

These formulas can be used to indicate the relative magnitudes of the various duration measures. For example, assume the following illustrative parameter values: $r = 0.05$, $\Delta r = 0.01$, $c = 0.40$, and the correlation between interest rate and inflation changes is 0.50 (thus, $i_+ = 0.005$, and $i_- = -0.005$). Given these values, the formulas above provide the following duration measures: $D_0 = 2.333$, $MD_0 = 2.222$, and $ED_0 = 1.056$. This example illustrates the potentially significant differences between effective duration and the more common, traditional measures of duration.

## 5. Modeling the Interest Rate Sensitivity of Loss and LAE Reserves

One of the difficulties in measuring the interest rate sensitivity of liabilities is the need for extensive data. What information is publicly available to determine the impact of interest rate changes on the cash flows of losses? For the loss and loss adjustment expense reserve, the expected nominal cost of these amounts at the end of each year are reported in aggregate, by accident year, by line of business, in the Annual Statement. Although the expected payment dates for these values are not listed, the actual payments made in each historical year -- categorized by accident year and by line of business -- are included. This allows a comparison of the actual payments with the expected payments and permits the generation of a profile of when the aggregate loss reserves are likely to be paid in the future. However, there is no public information on when the value of an unpaid loss is set in value. Thus, this relationship needs to be modeled.

For this model, the following assumptions are made. At the time the loss occurs, proportion $k$ of the eventual cost of the claim is "determined" -- i.e., a proportion of the future cost is "fixed" and no longer open to change from interest rate and inflationary changes. In addition, proportion $m$ of the loss will not be determined until the time the claim is settled. Examples of loss costs that will go into $k$ are medical treatment sought immediately after the loss occurs, the wage loss component of an injury claim and property damage. Examples of loss costs that will go into $m$ are medical evaluations that are done immediately prior to determining the settlement offer, general damages to the extent they are based on the cost of living at the time of settlement, and loss adjustment expenses connected with settling the claim.

The remaining ($1$-$k$-$m$) portion of the expenses are modeled in three ways, to allow for differing rates at which the claim values could become fixed: these expenses could be fixed in value linearly over the time period from loss to settlement, or in a manner that would represent a concave function or a convex function. Figure 1 illustrates the three different functions proposed for the proportion of loss reserves that are fixed in value, and therefore not subject to inflation, over time.

A representative function that displays these attributes is:

$$f(t)=k+[(1-k-m)\times(t/T)^n]$$

where $f(t)$ represents the proportion of ultimate paid claims "fixed" at time $t$,

k = the proportion of the claim that is fixed in value immediately,

m = the proportion of the claim that is not fixed in value until the claim is settled,

n = 1 for the linear case.

$n < 1$ for the concave case,

$n > 1$ for the convex case, and

$T$ = the time at which the claim is fully and completely settled.

For example, assume an insured causes an automobile accident in the middle of 1997, and the victim requires immediate medical attention. This is the $k$ portion of the claim that is predetermined immediately; assume, for example, that it represents 15% of the total cost of the claim. Further, assume that $m$ is zero. After the accident, the victim receives medical care on an ongoing basis until the claim is eventually settled in the middle of the year 2000. These continuing care expenses will be influenced by inflation. At the end of 1997, half of a year of continuing expenses has been obtained. The total length of time before the claim will be settled is three years (2000-1997). Thus, for the linear case ($n$=1),

$$f(0.5) = .15+[(1-.15)\times(\frac{0.5}{3})^1]$$

In this case, $f(0.5) = 0.292$, meaning that at the end of 1997, 29.2% of the loss reserve for this 1997 accident year claim is fixed in value, with the remainder subject to future inflation.

## 6. Duration Measures for Insurer Liabilities

### *A. Empirical Estimates*

In order to implement our model of effective duration, values of several parameters must be determined:

- Loss payout pattern
- Economic parameters
    - Interest rate
    - Correlation between interest and inflation
    - Growth rate of insurance writings
- Cost determination parameters
    - k (the proportion of claim value that is fixed immediately)
    - m (the proportion of claim value that is not fixed until the claim is settled)
    - n (the shape parameter of the fixed-claim-proportion function)

Each of these parameter values is discussed in greater detail below.

A key component to determining effective duration is identifying the future cash flows. For property-liability insurance, this involves determining the timing of future loss payments as loss reserves run off. For a particular corporate application of this effective duration procedure, the company's historical loss payment information by line of business can be used as a basis for estimating future claim payouts. For purposes of this paper, we used aggregate industry information available from A.M. Best (1998). Due to their size and importance, two lines of business were used in our analysis: private passenger auto liability (PPAL) and workers compensation (WC). An additional advantage of using these two lines of business is that their cash flows have different timing characteristics: WC pays out more slowly, in general, than PPAL. This distinction allows us to test the potential impact of calculating effective duration under different payout environments.

Aggregate industry payout data for PPAL and WC were each used in two different ways. First, the raw empirical data were used. Empirical loss payment patterns were generated from an actuarial analysis of historical calendar and accident year payment data. The second approach was to fit statistical distributions to the raw empirical payment patterns. This was done using software called "BestFit" (a product of Palisade Corporation), which provides best-fit parameter values to sample data for a variety of theoretical distributions. For both PPAL and WC, a gamma distribution was used for illustrative purposes as the "smoothed alternative" to the raw empirical payment pattern.

The loss payment patterns used in our tests were as shown in Table 1.

[Insert Table 1 here]

This table reflects the payout patterns through ten years, which is the timeframe in which aggregate industry data is available in any particular edition of A.M. Best's *Aggregates and Averages*. For our purposes, the WC patterns are extrapolated out to 30 years, and the PPA patterns to 15 (empirical) and 19 (smoothed) years.

The selected economic parameters are based largely on current and historical economic relationships. A "base case" 5% interest rate was selected in accordance with the level of short-term government rates in effect during the late 1990s. A 40% relationship between interest rates and claim inflation was selected based on the historical relationship between these two economic variables.[6] Finally, a 10% growth rate is assumed, based on judgment. This growth rate parameter reflects the fact that a typical insurance company carries reserves for a number of different accident years. The distribution of reserves by accident year is a function of the growth rate in ultimate accident year incurred losses, and the runoff patterns. The 10% growth assumption assumes that ultimate accident year losses are growing at 10% per year, which reflects the growth in both the number of policies written and claim cost inflation.

The selection of cost determination parameters is very difficult. Publicly available loss development information (e.g., *Best's Aggregates and Averages* or the NAIC data tapes) includes loss payments made each year, by accident year, on a by-line basis. This is not sufficient to determine the fixed and variable portions of loss reserves. Even within a company, the data needed to determine these relationships is not generally maintained in an easily accessible format. To address this issue, several large insurers were approached and asked to participate in a study to help estimate the parameters used in this model. These companies were asked to report information on a small sample of claims that were settled several years after the date of loss. None of the companies could provide an answer to the question of when the general damages portion of a claim is fixed in value. It appears that there is simply too much uncertainty about the process used to establish this figure to know if it is based on costs at the time of the loss, the time of the settlement, or some interim time.

One company did provide especially detailed reports on a sample of auto liability insurance

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[6]The selected relationship is based upon regressions of historical annual changes in U.S. Treasury bill returns (independent variable) and historical annual changes in inflation rates (dependent variable) for CPI, PPA bodily injury liability, PPA property damage liability, and WC. The regression coefficients varied greatly -- by both magnitude and statistical significance -- according to the type of inflation and the period being tested. The test periods ranged from 20 to 60 years. The intercept term in the regressions were selected as zero. Insurance claim inflation data were taken from Masterson (1968 and subsequent); T-bill and CPI data were taken from Ibbotson (1996).

claims. These reports showed all the medical, wage loss, and property damage costs associated with the claims, the date any of these expenses were incurred by the claimant, and the total claim payment made by the company. For most of these cases, the final claim paid exceeded the total costs the claimant had incurred. This is expected, since the itemized expenses represented special damages, and the final payment would also include the intangible general damages. However, there was one case in which the policyholder was not fully liable for the claim and the total payment was less than the plaintiff's expenses.

The general pattern of the expenses was as follows. At the time of the loss, the plaintiff incurred significant medical expenses, property damage, and wage loss. After the initial medical treatment, the plaintiff incurred some continuing medical expenses, either for additional treatment or for rehabilitation. These expenses most frequently ended before the claim was finally settled. This would suggest that the function for the value of the fixed claim is concave ($n<1$), at least for the special damages portion of the claim.

The results of this sample indicate that a more extensive and detailed examination of this process would be very helpful in determining the appropriate parameters for measuring effective duration. For purposes of getting initial empirical estimates of effective duration, we have chosen to begin with $k = 0.15$, $m = 0.10$, and $n = 1.0$. These values will be varied in the next subsection, in order to determine the potential sensitivity of effective duration results to the magnitude of these parameters.

Based on these selected parameters, and a $\Delta r$ of 100 basis points, and using a spreadsheet model to implement the calculations, the effective duration indications in Table 2 were derived:

[Insert Table 2 here]

The essential finding is that effective duration measures -- which properly account for the inflationary impact of interest rate changes on future loss reserve payments -- are approximately 25% below their modified duration counterparts. This relationship appears to be consistent, based on the illustrative PPA and WC tests above, regardless of line of business, or whether empirical or smoothed payout patterns are utilized.

In addition to duration, another quantity that is important to asset-liability management -- convexity -- is also displayed in Table 2. Just as the impact of inflation on future cash flows must be measured via *effective* duration, the second derivative of the price/interest rate relationship (see appendix) is appropriately measured by *effective* convexity in an inflationary environment. The results in Table 2 show that there is a significant difference between the traditional and effective measures of convexity. The effective convexity formula used to derive the values in Table 2 was:

$$\textit{Effective convexity} = \frac{PV_{-} + PV_{+} - 2PV_0}{PV_0(\Delta r)^2}$$

*B. Sensitivity of Effective Duration to Parameter Values*

As indicated above, effective duration measures can provide significantly different evaluations of property-liability insurer interest rate sensitivity than the traditional modified duration measures. Use of the appropriate effective duration measure is therefore critical when utilizing asset-liability management techniques. Similarly, it is important to have an understanding of which parameter values have the greatest impact on the magnitude of the effective duration calculation. In Table 3, various parameters have been changed -- one at a time -- to demonstrate the level of sensitivity of effective duration values with respect to those parameters. (Since the empirical and smoothed pattern results were so similar above, to promote clarity only the empirical patterns were used for each line of business.)

[Insert Table 3 here]

The main result from Table 3 is the significant sensitivity of effective duration to the interest rate - inflation relationship. In particular, this parameter expresses how much inflationary pressure is associated with a 100 basis point change in interest rates. If there is no correlation between interest rates and inflation, the modified duration and effective duration are the same. If the correlation is as high as 80%, the effective duration is approximately one-half the modified duration. The relationship between changes in interest rates and changes in inflation -- both CPI and line of business claim inflation -- has historically been very volatile. Our results suggest that additional efforts to determine reasonable values for this relationship parameter would be worthwhile.

Another observation from the table is that the results are not overly sensitive to some of the cost determination parameters. Given the difficulties, mentioned above, of determining values for the parameters, this is a somewhat comforting finding. For companies undertaking asset-liability management, simply using effective duration measures of their liabilities is more important than having the exact parameter values. However, these companies should be encouraged collect data that will allow them to monitor the sensitivity of their results to different cost determination function specifications.

## 7. Use of Effective Duration in Asset-Liability Management

In previous sections, the deficiencies of traditional measures of duration in an inflationary world were identified, and an alternative measure -- effective duration -- was described. In this section, the impact of using effective, as opposed to modified, duration on a company's asset-liability management process is illustrated. The example used is a hypothetical workers compensation insurer; it is assumed that this company has asset and liability values which are related in a manner consistent with aggregate industry balance sheet figures.

The effective duration analysis in the prior section concentrates on loss and allocated loss adjustment expense reserves and runoffs. A complete asset-liability management analysis would also consider unallocated loss adjustment expenses and unearned premium reserves (the timings of which are described in Section 3 of this paper). For simplicity, and because they represent a relatively small part of an insurer's liabilities, unallocated loss adjustment expenses are considered together with losses and ALAE in the illustrative example in this section. However, the reasonableness of this assumption would need to be evaluated in any specific corporate application of asset-liability management.

The duration of the unearned premium reserve was described in Section 3. The one adjustment that must be made with respect to asset-liability management is to only consider the portion of the UPR which is associated with future losses and loss adjustment expenses -- it is only this portion which represents a liability for future cash flows which may be impacted by inflation. The duration for this portion of the UPR is calculated by determining the duration of the loss and LAE reserve for the most recent accident year, and adding 0.75. The other portion of the UPR -- the "equity" in the UPR -- represents prepaid expenses associated with prior writings of insurance policies, and is essentially an accounting construct which is unrelated to *future* cash flows. Thus, this portion of the UPR is not considered in the following illustration.

For illustrative purposes, all other liability items on the insurer's balance sheet are considered to have a Macaulay duration of 1.0 (and thus, at an interest rate of 5%, a modified duration of 0.952).

The duration of an insurer's surplus is as follows (Staking and Babbel, 1995):

$$D_S\, S = D_A\, A - D_L\, L$$

where $S$ = surplus,

$D$ = duration,

$A$ = assets, and

$L$ = liabilities.

In order to immunize its surplus (setting $D_s$=0) from interest rate risk, an insurer needs to set the duration of its assets as follows:

$$D_A = D_L \frac{L}{A}$$

Thus, the appropriate determination of the duration of liabilities is critical for asset-liability management.

Based on the aggregate industry balance sheet figures for WC insurers reported in A.M. Best (1998), Table 4 shows the liability distribution for an insurer with assets of $1 billion.[7]

[Insert Table 4 here]

The liability durations were calculated as described above and in Section 6 based on the empirical WC payout pattern. The resulting overall (value-weighted) liability modified duration is 3.823, while the effective duration of total liabilities is 2.801.

If the insurer wanted to immunize surplus from interest rate swings based on modified duration, the duration of assets would need to be 2.714. However, based on effective duration, the duration of assets should be 1.989. An insurer that attempted to immunize its exposure to interest rate risk by matching the duration of assets with the modified duration of liabilities, instead of effective duration, would find that it still would be exposed to interest rate risk. Based on these values, the insurer would have a duration of surplus of 2.501: each 1 percentage point increase in the interest rate would decrease surplus by 2.501 percent (where surplus here is defined as the economic value of statutory surplus plus the equity in the unearned premium reserve).

[7]Workers compensation insurers tend to have a slightly higher proportion of their liabilities in loss and loss adjustment expenses, and a much lower proportion in the unearned premium reserve, than other insurers. In applications of this technique, the actual values for these liabilities, and for the relationship between assets and liabilities, for the company should be used.

## 8. Conclusion and Future Research

This paper has demonstrated a method for determining the effective duration and convexity of property-liability insurer liabilities, and has provided some general estimates of these values. Based on the results derived, it appears that there can be significant differences between the traditional measures of duration -- i.e., Macaulay and modified duration -- and effective duration. Of these measures, only effective duration is capable of properly accounting for the impact of inflationary pressures on liability cash flows that are associated with potential changes in interest rates. This means that effective duration is the appropriate tool for measuring the sensitivity of the liabilities of property-liability insurers to interest rates when performing asset-liability management. Use of the wrong duration measure can lead to an unintended mismatch of assets and liabilities, and an unwanted exposure to interest rate risk.

In addition to inflation, interest rate changes may also be correlated with other financial and economic variables. For example, a decrease in interest rates is often -- on average -- associated with an increase in stock prices (since the discount rate on future dividends and capital gains is lower). Similarly, changes in interest rates in the U.S. may certainly impact international financial relationships. To the extent to which these other variables are factors in a jury's damage award considerations, they must also be contemplated in an effective duration framework. For example, if the stock market has increased in value significantly between the time of an accident and the final jury verdict, a well-structured comment from the plaintiff's attorney to the jury may lead to a higher award on the grounds that the plaintiff could have invested the monies lucratively if they had been available at the time of the accident.[8] These types of issues are beyond the analytical scope of this paper, and are left for future research.

In this paper, we have approached the measurement of effective duration from the standpoint of a shift in a constant interest rate. Future research should examine the impact of a stochastic interest rate model on effective duration and asset-liability management. Interesting and important work in the non-insurance literature on effective duration, yield curves, and stochastic interest rates (e.g., Babbel, Merrill, and Panning, 1997) has significant future applicability to the issues addressed in this paper. In addition, stochastic interest rate models are beginning to appear in the property-liability insurance industry, especially within the context of dynamic financial analysis (D'Arcy and Gorvett, et al, 1997 and 1998). In analyses in which assets are valued according to a stochastic rate assumption, it is appropriate to value liabilities on the same basis. This will be an important area for future research.

[8]The appropriate analytical framework in this case may involve option pricing theory -- it is possible that the jury award may depend on the maximization of alternatives involving such considerations as inflationary environment, stock market performance, etc.

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**Table 1**
**Cumulative Proportion of Ultimate Accident Year Losses Paid**
(Based on Age After Beginning of Accident Year)

| | PPA Liability | | Workers Compensation | |
|---|---|---|---|---|
| Age (Years) | Empirical | Smoothed | Empirical | Smoothed |
| 1 | .386 | .398 | .225 | .362 |
| 2 | .701 | .672 | .486 | .496 |
| 3 | .843 | .827 | .635 | .588 |
| 4 | .919 | .909 | .727 | .658 |
| 5 | .958 | .953 | .785 | .713 |
| 6 | .977 | .976 | .822 | .757 |
| 7 | .986 | .988 | .847 | .793 |
| 8 | .991 | .994 | .867 | .823 |
| 9 | .994 | .997 | .880 | .848 |
| 10 | .995 | .998 | .891 | .869 |

**Table 2**
**Summary of Duration Measures for Loss Reserves**
(Based on "Base Case" Parameter Assumptions)

| | PPA Liability | | Workers Compensation | |
|---|---|---|---|---|
| | Empirical | Smoothed | Empirical | Smoothed |
| Macaulay Duration | 1.516 | 1.511 | 4.485 | 4.660 |
| Modified Duration | 1.444 | 1.439 | 4.271 | 4.438 |
| Effective Duration | 1.089 | 1.085 | 3.158 | 3.285 |
| | | | | |
| Convexity | 5.753 | 5.214 | 50.771 | 45.060 |
| Effective Convexity | 1.978 | 1.807 | 16.038 | 14.383 |

**Table 3**
**Analysis of the Sensitivity of Effective Duration Measures of Loss Reserves**
(Based on Single-Parameter Changes From "Base Case" Values*)

| | | PPA Empirical | WC Empirical |
|---|---|---|---|
| Macaulay Duration** | | 1.516 | 4.485 |
| Modified Duration** | | 1.444 | 4.271 |
| *Effective Duration* | | | |
| Base Case | | 1.089 | 3.158 |
| Inflation-Interest Relationship: | | | |
| | 80% | 0.733 | 2.036 |
| | 60% | 0.911 | 2.596 |
| | 40% | 1.089 | 3.158 |
| | 20% | 1.267 | 3.721 |
| | 0% | 1.445 | 4.286 |
| k = | 0.25 | 1.128 | 3.284 |
| | 0.20 | 1.108 | 3.221 |
| | 0.15 | 1.089 | 3.158 |
| | 0.10 | 1.069 | 3.095 |
| | 0.05 | 1.049 | 3.032 |
| m = | 0.20 | 1.067 | 3.104 |
| | 0.15 | 1.078 | 3.131 |
| | 0.10 | 1.089 | 3.158 |
| | 0.05 | 1.099 | 3.185 |
| | 0.00 | 1.110 | 3.212 |
| n = | 1.40 | 1.045 | 3.040 |
| | 1.20 | 1.065 | 3.092 |
| | 1.00 | 1.089 | 3.158 |
| | 0.80 | 1.120 | 3.245 |
| | 0.60 | 1.160 | 3.362 |
| g = | 0.20 | 1.070 | 2.849 |
| | 0.15 | 1.079 | 2.985 |
| | 0.10 | 1.089 | 3.158 |
| | 0.05 | 1.101 | 3.367 |
| | 0.00 | 1.116 | 3.589 |

* Base case values are: k=0.15, m=0.10, n=1.00, g=0.10, a 5% interest rate, and a 40% relationship between interest rate and inflation movements.

** These duration figures reflect base case parameter values. When parameter g is changed according to the range above, Macaulay and modified durations also change slightly:
PPA: D = 1.501 to 1.540, and MD = 1.429 to 1.466
WC: D = 4.128 to 4.910, and MD = 3.932 to 4.676

**Table 4**
**Example of Asset-Liability Management**
**for a Hypothetical Workers Compensation Insurer**
($ figures are in millions)

| | Dollar Value | Modified Duration | Effective Duration |
|---|---|---|---|
| Loss and LAE Reserves | 590 | 4.271 | 3.158 |
| UPR (portion for losses and LAE only) | 30 | 3.621 | 1.325 |
| Other Liabilities | 90 | 0.952 | 0.952 |
| Total Liabilities | 710 | 3.823 | 2.801 |
| Total Assets | 1,000 | | |
| Indicated Asset Duration to Immunize Surplus: | | 2.714 | 1.989 |

## Appendix
## Derivation of the Formulae for the Sensitivity of a Cash Flow to Interest Rates Based on the Taylor Series

The basic objective of a duration measure is to determine the sensitivity of the present value of a cash flow to a change in interest rates. This is an important factor for financial institutions, since interest rate risk is such a significant source of uncertainty for these companies. Several measures of interest rate risk have been developed in finance. An understanding of these different measures can best be grasped by examining their basic mathematical foundations.

The Taylor series states that the value of a function at any point can be approximated by the value of a series of derivatives of the function valued at another point, with each derivative multiplied by the difference between the two points raised to the same power as the order of the derivative and divided by the factorial of that order. The accuracy of the approximation is determined by the number of derivatives taken. Mathematically, the Taylor series is represented as:

$$f(r_1) = f(r_0) + hf^{(1)}(r_0) + \frac{h^2}{2!}f^{(2)}(r_0) + \frac{h^3}{3!}f^{(3)}(r_0) + \ldots + \frac{h^m}{m!}f^{(m)}(r_0) + \ldots$$

where

$r_0$ = initial interest rate
$r_1$ = new interest rate

$h = r_1 - r_0$

The Taylor series will be used to evaluate a simple interest rate function, but one that is typical for fixed income assets. This function is the present value of a \$1 million ten year zero-coupon bond, which is simply the present value \$1,000,000, or:

$$f(r) = \frac{1{,}000{,}000}{(1+r)^{10}}$$

The first four derivatives of this function are:

$$f^{(1)} = \frac{(-10)(1{,}000{,}000)}{(1+r)^{11}} = \frac{-10{,}000{,}000}{(1+r)^{11}}$$

$$f^{(2)} = \frac{(-11)(-10)(1{,}000{,}000)}{(1+r)^{12}} = \frac{110{,}000{,}000}{(1+r)^{12}}$$

$$f^{(3)} = \frac{(-12)(-11)(-10)(1{,}000{,}000)}{(1+r)^{13}} = \frac{-132{,}000{,}000}{(1+r)^{13}}$$

$$f^{(4)} = \frac{(-13)(-12)(-11)(-10)(1{,}000{,}000)}{(1+r)^{14}} = \frac{17{,}160{,}000{,}000}{(1+r)^{14}}$$

Figure 1-A illustrates the present value function for this zero coupon bond. As can be seen from this figure, the present value of the bond is inversely related to interest rates and the function is convex. This represents the typical relationship for bonds with a fixed coupon rate and payment pattern.

In order to apply the Taylor series, we need an initial interest rate, which will be set at

10%, and a new interest rate to be used to value the function. The accuracy of each order of Taylor series approximation is illustrated graphically over the range of 0 to 30 percent.

Figure 2-A illustrates the bond value and the Taylor series approximations based on the first, second, third and fourth orders. For small changes, all of the approximations appear to provide a fairly reasonable fit. However, for larger changes -- for example, at an interest rate of 20 percent -- it is clear that the higher order approximations provide progressively better fits to the actual bond value.

Figure 3-A illustrates the present value function and the first four order approximations over the range of 8 to 12 percent. Within this range, the second, third and fourth order approximations are visually indistinguishable from each other and from the bond value itself. The first order approximation is clearly distinct, however. Based on this mathematical background, the attempts to quantify interest rate sensitivity can be examined.

The first attempt to develop a measure of the sensitivity of a cash flow to interest rates was performed by Frederick Macaulay in 1938. The measure he developed, termed the Macaulay duration, is the present-value-weighted average time to receipt of the cash flows, divided by the initial present value of the cash flows, and is expressed by the following formula:

$$D = \frac{\sum_{t=1}^{n} tCF_t / (1 + r_0)^t}{\sum_{t=1}^{n} CF_t / (1 + r_0)^t}$$

where $CF_t$ = the cash flow received at time t

$r_0$ = initial interest rate

The numerator of the Macaulay duration is close to the first derivative of the function for the present value of the cash flow. If the denominator of the expression in the numerator were $(1+r_0)^{t+1}$ instead of $(1+r_0)^t$, then the numerator would be equal to the negative of the first derivative of the function for the present value of the cash flow. When the Macaulay duration is used to estimate the effect of a change in interest rates, the Macaulay duration is multiplied by the change in the interest rate, or $r_1$-$r_0$, and, in recognition of the inverse relationship between the present value and interest rates, also by -1.

For example, the Macaulay duration of the zero-coupon bond is:

$$D = \frac{\sum 10CF_{10}/(1+r_0)^{10}}{\sum CF_{10}/(1+r_0)^{10}} = 10$$

The change in the present value of the cash flow is estimated to be $f(i_0)(-1)(10)(r_1-r_0)$. In this example, for every 1 percentage point increase in interest rates, the present value of the cash flow is estimated to decrease by 10 percent. The accuracy of this estimate, for an $r_0$ = 10 percent, is illustrated on Figure 4-A. Similar to the first order Taylor series approximation, this estimate is only accurate for very small changes in interest rates.

The next approach to measure the sensitivity of the present value of a cash flow to interest rates is termed modified duration. The formula for modified duration is:

$$MD = -\frac{f^{(1)}(r_0)}{f(r_0)}$$

This expression is mathematically equivalent to the Macaulay duration divided by $(1+r_0)$. Apparently, the negative of the first derivative is used simply to avoid having a negative number for the duration measure. Since the slope of the present value function for almost all financial assets is negative, then the first derivative would naturally be a negative. Taking the negative of this derivative transforms it to a positive value.

The modified duration of the ten year zero-coupon bond is, for an initial interest rate of 10%:

$$MD = (-1)\frac{(-10)(1{,}000{,}000)/(1.10)^{11}}{1{,}000{,}000/(1.10)^{10}} = (-1)\frac{-3{,}504{,}939}{385{,}543} = 9.0909$$

To use the modified duration estimate to determine the new present value of a cash flow, the change in the present value is:

$$\%\ \text{Price Change} = (-1)(MD)(r_1 - r_0)$$

Here, multiplying the modified duration by negative one cancels out the negative of the slope that was taken in determining modified duration. The accuracy of this estimate, for an $r_0$ = 10 percent, is also illustrated on Figure 4-A. Modified duration is exactly the same as a first order Taylor series approximation, and is also only accurate for small changes in interest rates.

In order to compare the Macaulay and modified duration measures to the Taylor series approximation requires the rearranging of terms. Since duration attempts to measure the sensitivity of the present value to interest rate changes based on a first order approximation, the correct duration measure would be as follows:

$$\frac{f(r_1) - f(r_0)}{f(r_0)} = \frac{(r_1 - r_0) f^{(1)}(r_0)}{f(r_0)}$$

Macaulay duration is $-f^{(1)}(r_0)(1+r_0)/f(r_0)$. Modified duration is $-f^{(1)}(r_0)/f(r_0)$. The determination of the present value of the cash flow based on a first order Taylor series approximation is shown below. Seen in this framework, it is apparent why the modified duration is multiplied by negative one, the initial present value and the difference in interest rates.

$$\begin{aligned} f(r_1) &= f(r_0) + (r_1 - r_0) f^{(1)}(r_0) \\ &= f(r_0) + (r_1 - r_0)(-1)(\text{Modified Duration}) f^{(1)}(r_0) \end{aligned}$$

In order to increase the accuracy of the interest rate sensitivity measure, an additional term, called convexity, has been introduced in finance. The measure for convexity is the second derivative of the price with respect to interest rates, divided by the price of the bond, or:

$$Convexity = \frac{f^{(2)}(r_0)}{f(r_0)}$$

Applying convexity to determine the price change:

$$\% \text{ Price Change} = (-1)(MD)(r_1 - r_0) + \frac{1}{2}\text{Convexity}(r_1 - r_0)^2$$

This expression is similar to the second order Taylor series approximation. Using the first and second derivatives only, and rearranging to determine the change in the value of the function, as a percentage of the original value, leads to:

$$\frac{f(r_1) - f(r_0)}{f(r_0)} = \frac{(r_1 - r_0)f^{(1)}(r_0)}{f(r_0)} + \frac{(r_1 - r_0)^2 f^{(2)}(r_0)}{2!f(r_0)}$$

Convexity is multiplied by ½ to correspond with the 2! in the Taylor series approximation and multiplied by the yield change squared also to correspond with the Taylor series determination.

The convexity of the ten year zero-coupon bond is, for an initial interest rate of 10%:

$$\text{Convexity} = \frac{(-11)(-10)(1{,}000{,}000)/(1.10)^{12}}{1{,}000{,}000/(1.10)^{10}} = \frac{35{,}049{,}390}{385{,}543} = 90.909$$

Figure 5-A illustrates the effect of introducing the convexity expression to determine the present value of the cash flow. This is exactly the same as the second order Taylor series illustrated in Figure 2-A.

Despite the importance of accurate measures for interest rate sensitivity, there are no specific financial expressions for third or higher order approximations of the Taylor series. These terms are used in some calculations of interest rate sensitivity, however. In practice, calculations of these more accurate expressions are no more difficult than the modified duration and convexity determinations.

Table 1-A shows the actual values for the present value of the $1 million ten year zero-coupon bond for interest rates ranging from 6 to 14 percent, and the estimated values based on modified duration, convexity, and the third and fourth order Taylor series expansions. For interest rates near the initial 10 percent interest rate, the duration and convexity estimates are very close to the actual values. For larger interest rate changes, the higher order terms provide a much better approximation.

Although typically the relationship between the present value of a cash flow and interest rates is convex, this is not always the case. In some instances the relationship is concave over part of the range. For example, if the maturity of the bond is a function of interest rates, a concave relationship can be obtained. In finance, this is termed ***negative convexity***, rather than the mathematical term ***concavity***. This type of function is illustrated in Figure 6-A, in which the maturity of a zero-coupon bond is 100 r. Thus, if interest rates are 10 percent, the bond matures in 10 years. If interest rates fall to 6 percent, the bond matures in six years; if interest rates rise to 12 percent, the bond matures in 12 years. This relationship is representative of actual payoff patterns for collateralized mortgage obligations and callable bonds.

As shown in this appendix, the financial measures used for interest rate sensitivity are all based on the Taylor series expansion. Although some of the terminology used in finance is new, the basic application is a standard mathematical technique.

**Figure 1**
**Formula for "Fixed" Costs**
$$f(t)=k+[(1-k-m)*(t/T)^n]$$

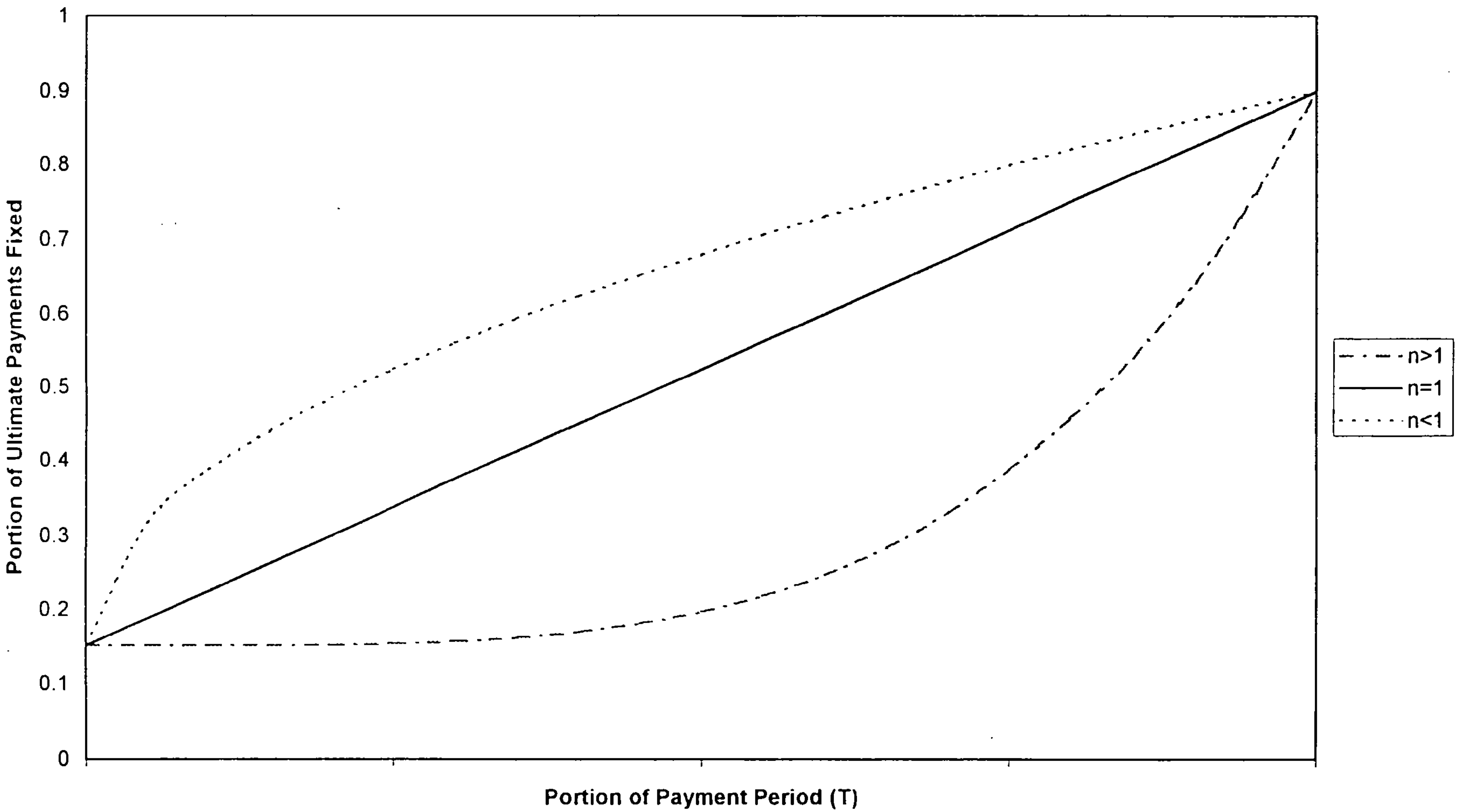

Figure 1-A
Present Value of a $1 Million Ten Year Zero Coupon Bond

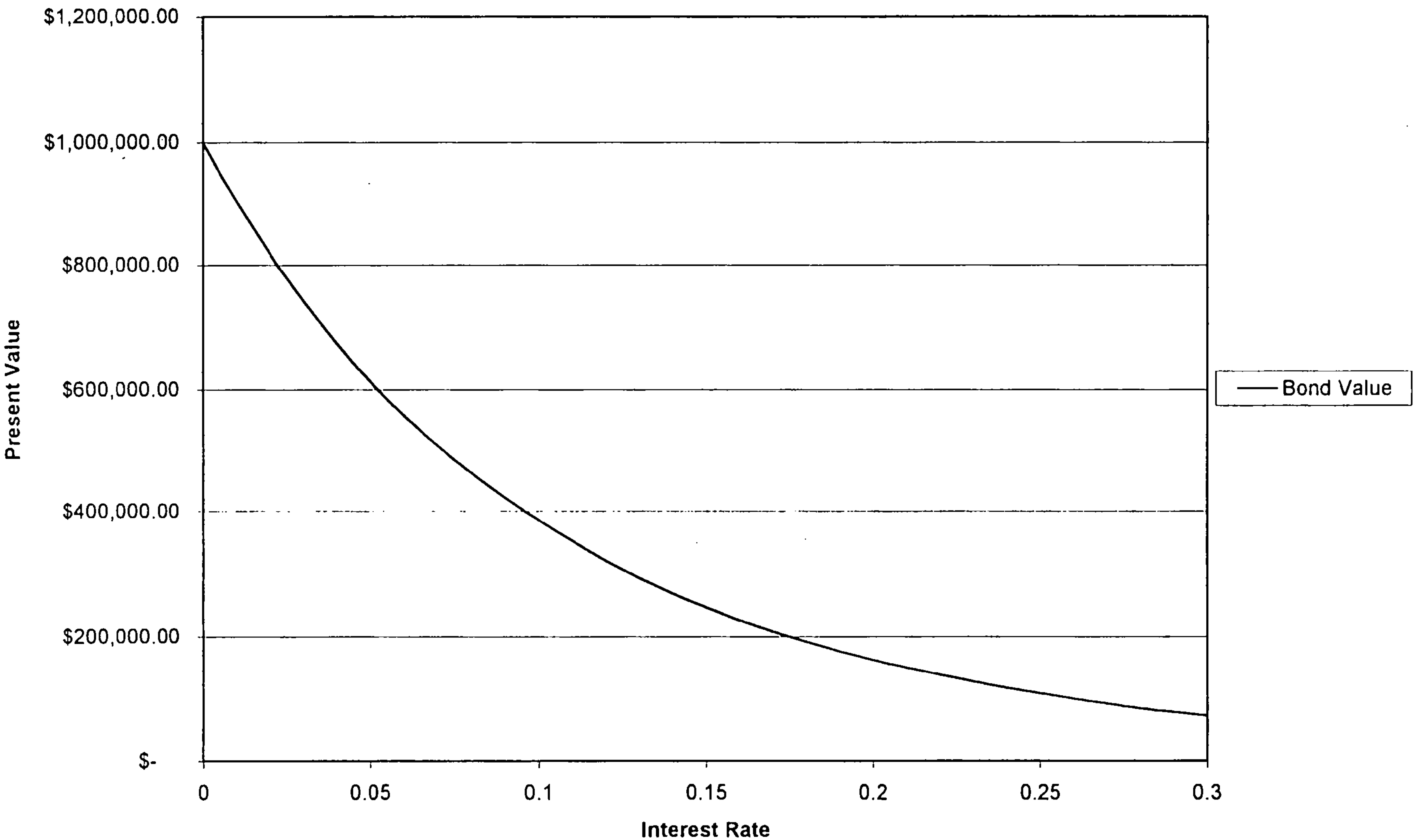

**Figure 2-A**
**Present Value of a $1 Million Ten Year Zero Coupon Bond with Taylor Series Approximations**

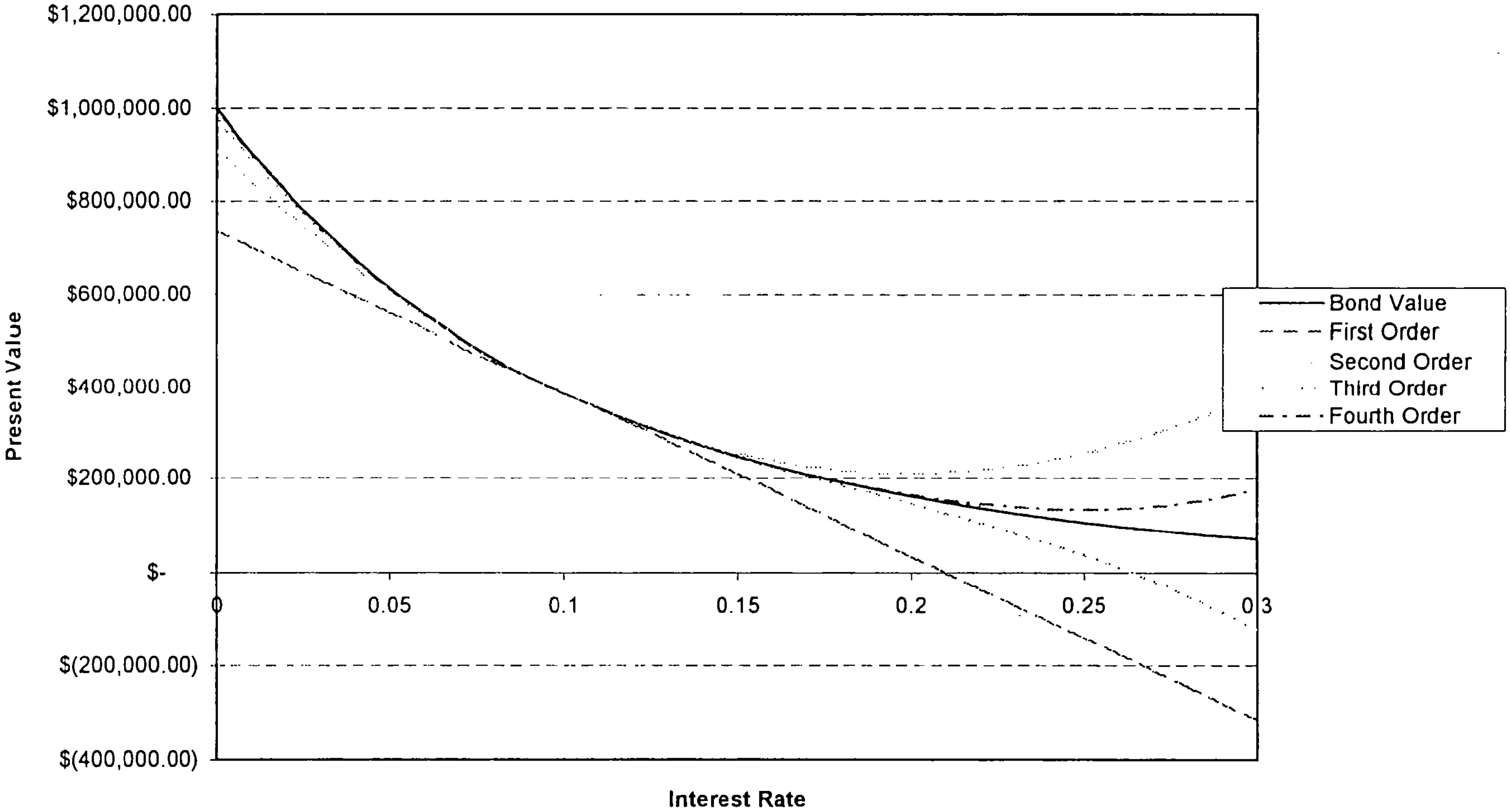

**Figure 3-A**
**Present Value of a $1 Million Ten Year Zero Coupon Bond with Taylor Series Approximations**

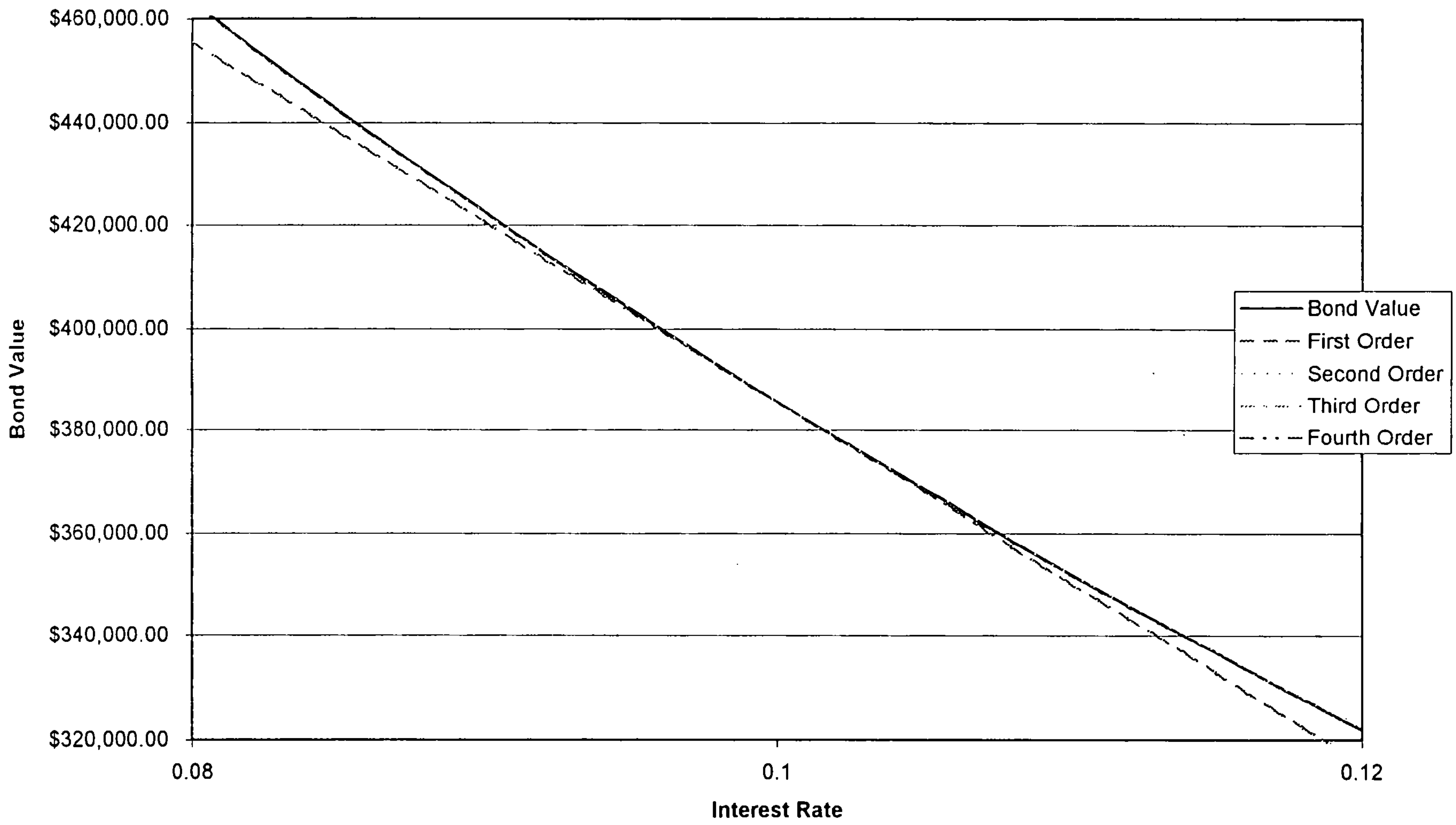

**Figure 4-A**
**Present Value of a $1 Million Ten Year Zero Coupon Bond with Macaulay and Modified Duration Estimates**

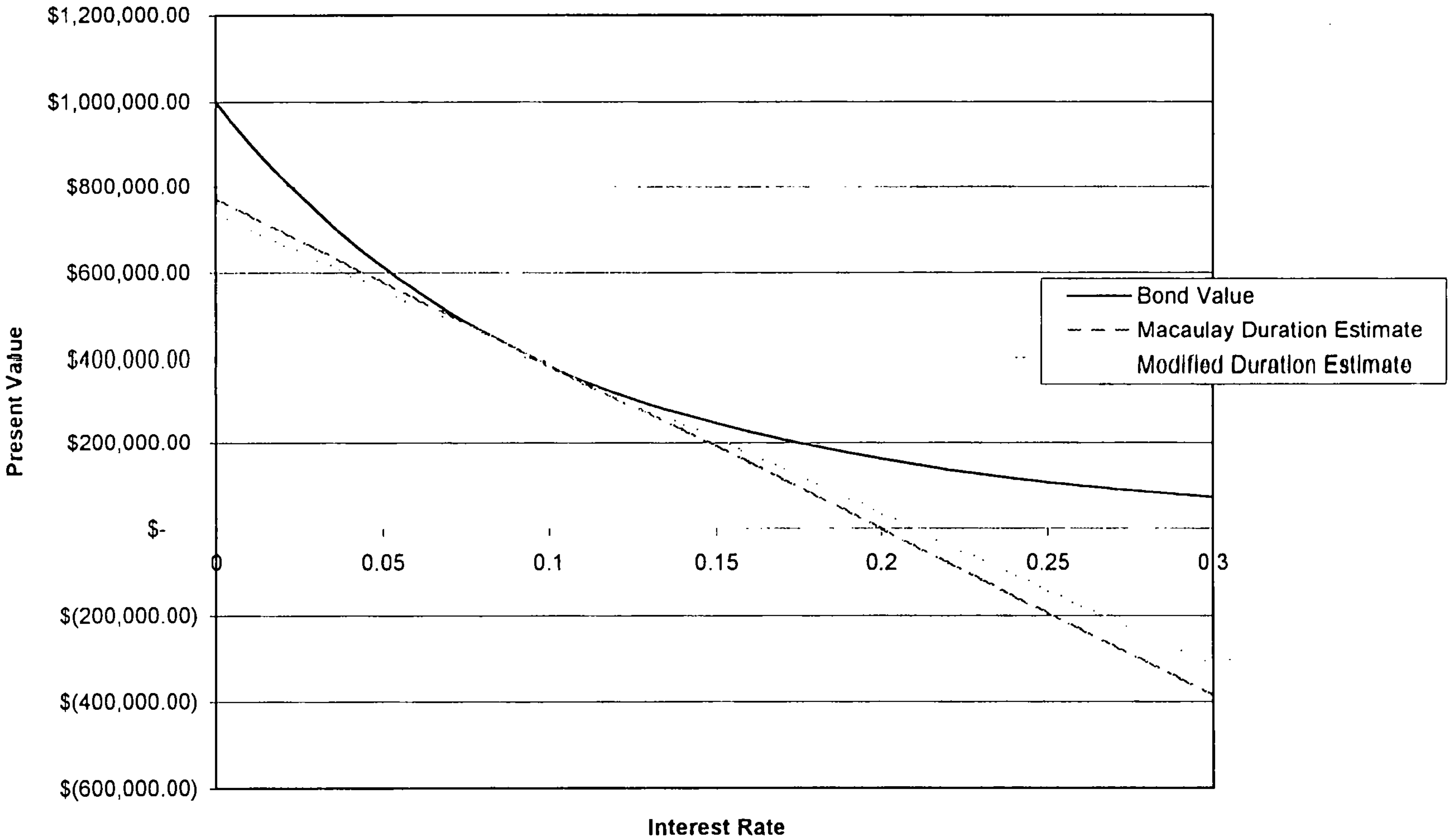

**Figure 5-A**
**Present Value of a $1 Million Ten Year Zero Coupon Bond with Modified Duration and Convexity Estimates**

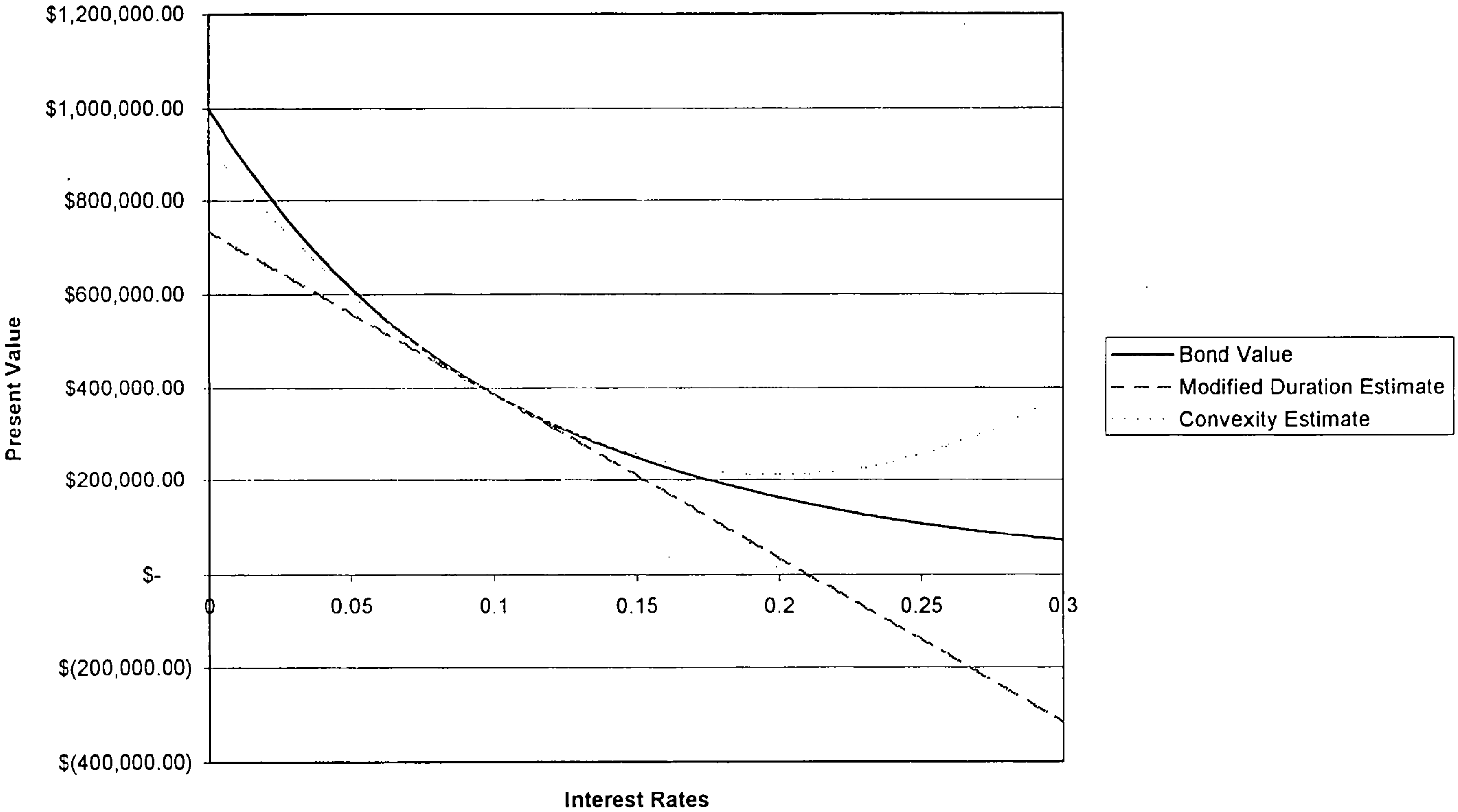

**Figure 6-A**
**Present Value of a $1 Million Zero Coupon Bond with Maturity a Function of Interest Rates**

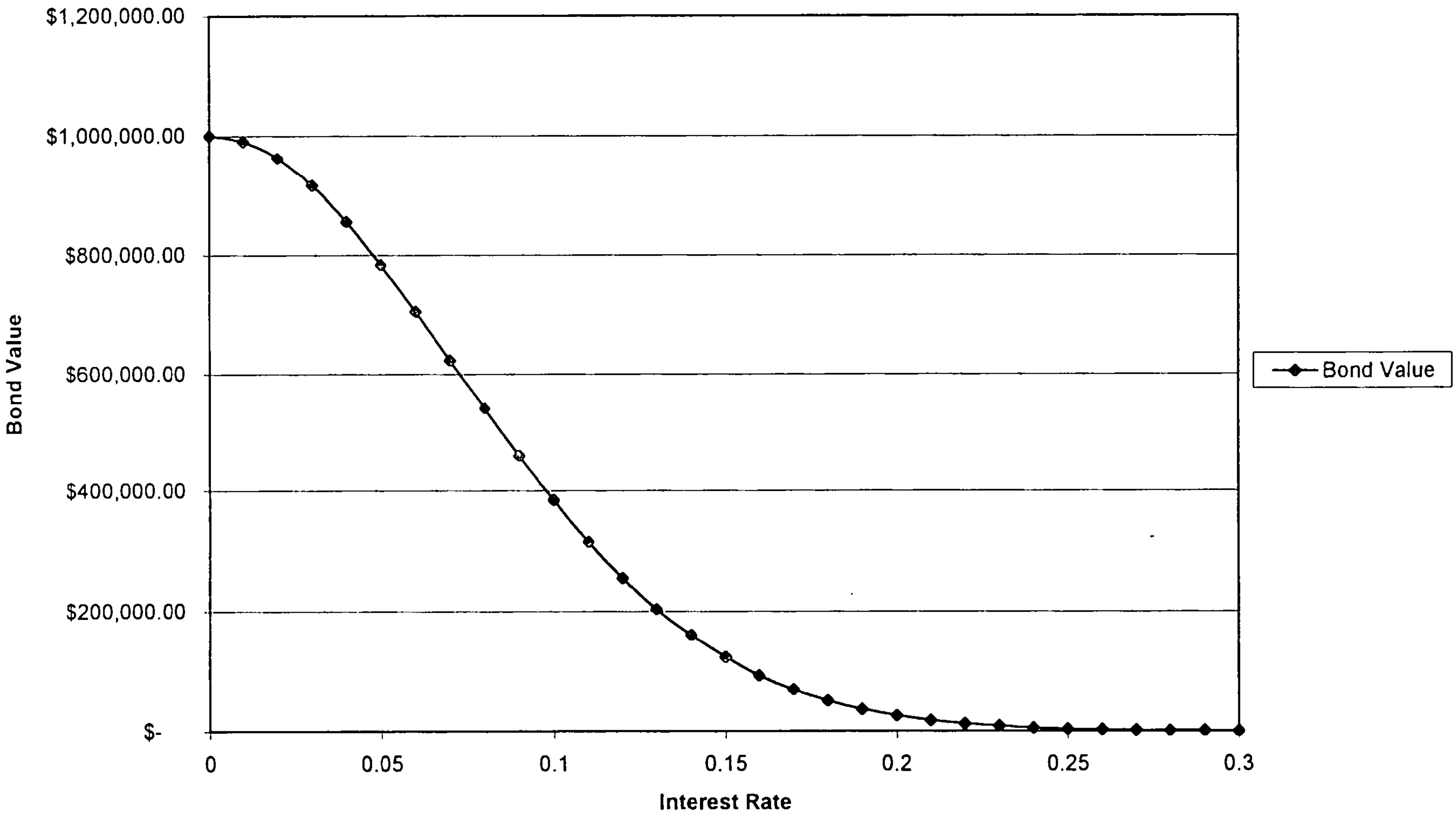

Table 1-A
Accuracy of Estimated Values of $1 Million Ten Year Zero-Coupon Bond
Initial Interest Rate = 10%

| | | | Estimates Based On: | | |
|---|---|---|---|---|---|
| Interest Rate | Actual Value | Modified Duration | Convexity | Third Order | Fourth Order |
| 6.00% | $ 558,394.78 | $ 525,740.85 | $ 553,780.36 | $ 557,858.84 | $ 558,340.84 |
| 6.25% | $ 545,394.32 | $ 516,978.50 | $ 541,622.60 | $ 544,983.16 | $ 545,355.50 |
| 6.50% | $ 532,726.04 | $ 508,216.15 | $ 529,683.91 | $ 532,416.16 | $ 532,698.71 |
| 6.75% | $ 520,380.68 | $ 499,453.81 | $ 517,964.27 | $ 520,151.87 | $ 520,361.92 |
| 7.00% | $ 508,349.29 | $ 490,691.46 | $ 506,463.68 | $ 508,184.29 | $ 508,336.80 |
| 7.25% | $ 496,623.19 | $ 481,929.11 | $ 495,182.16 | $ 496,507.47 | $ 496,615.15 |
| 7.50% | $ 485,193.93 | $ 473,166.76 | $ 484,119.70 | $ 485,115.42 | $ 485,188.97 |
| 7.75% | $ 474,053.34 | $ 464,404.42 | $ 473,276.29 | $ 474,002.17 | $ 474,050.43 |
| 8.00% | $ 463,193.49 | $ 455,642.07 | $ 462,651.95 | $ 463,161.76 | $ 463,191.88 |
| 8.25% | $ 452,606.67 | $ 446,879.72 | $ 452,246.66 | $ 452,588.19 | $ 452,605.85 |
| 8.50% | $ 442,285.42 | $ 438,117.37 | $ 442,060.43 | $ 442,275.51 | $ 442,285.04 |
| 8.75% | $ 432,222.47 | $ 429,355.03 | $ 432,093.26 | $ 432,217.73 | $ 432,222.32 |
| 9.00% | $ 422,410.81 | $ 420,592.68 | $ 422,345.15 | $ 422,408.88 | $ 422,410.76 |
| 9.25% | $ 412,843.59 | $ 411,830.33 | $ 412,816.10 | $ 412,842.98 | $ 412,843.58 |
| 9.50% | $ 403,514.19 | $ 403,067.98 | $ 403,506.10 | $ 403,514.07 | $ 403,514.19 |
| 9.75% | $ 394,416.17 | $ 394,305.64 | $ 394,415.17 | $ 394,416.16 | $ 394,416.17 |
| 10.00% | $ 385,543.29 | $ 385,543.29 | $ 385,543.29 | $ 385,543.29 | $ 385,543.29 |
| 10.25% | $ 376,889.48 | $ 376,780.94 | $ 376,890.47 | $ 376,889.48 | $ 376,889.48 |
| 10.50% | $ 368,448.86 | $ 368,018.59 | $ 368,456.71 | $ 368,448.75 | $ 368,448.86 |
| 10.75% | $ 360,215.71 | $ 359,256.25 | $ 360,242.01 | $ 360,215.13 | $ 360,215.72 |
| 11.00% | $ 352,184.48 | $ 350,493.90 | $ 352,246.37 | $ 352,182.64 | $ 352,184.53 |
| 11.25% | $ 344,349.77 | $ 341,731.55 | $ 344,469.79 | $ 344,345.32 | $ 344,349.92 |
| 11.50% | $ 336,706.36 | $ 332,969.20 | $ 336,912.26 | $ 336,697.19 | $ 336,706.72 |
| 11.75% | $ 329,249.16 | $ 324,206.86 | $ 329,573.79 | $ 329,232.26 | $ 329,249.92 |
| 12.00% | $ 321,973.24 | $ 315,444.51 | $ 322,454.39 | $ 321,944.58 | $ 321,974.70 |
| 12.25% | $ 314,873.78 | $ 306,682.16 | $ 315,554.04 | $ 314,828.16 | $ 314,876.41 |
| 12.50% | $ 307,946.15 | $ 297,919.81 | $ 308,872.75 | $ 307,877.03 | $ 307,950.58 |
| 12.75% | $ 301,185.80 | $ 289,157.47 | $ 302,410.52 | $ 301,085.21 | $ 301,192.89 |
| 13.00% | $ 294,588.35 | $ 280,395.12 | $ 296,167.35 | $ 294,446.74 | $ 294,599.25 |
| 13.25% | $ 288,149.52 | $ 271,632.77 | $ 290,143.23 | $ 287,955.63 | $ 288,165.69 |
| 13.50% | $ 281,865.15 | $ 262,870.42 | $ 284,338.15 | $ 281,605.92 | $ 281,888.46 |
| 13.75% | $ 275,731.22 | $ 254,108.08 | $ 278,752.15 | $ 275,391.62 | $ 275,763.95 |
| 14.00% | $ 269,743.81 | $ 245,345.73 | $ 273,385.24 | $ 269,306.77 | $ 269,788.77 |