Solvency Capital Estimation, Reserving Cycle and Ultimate Risk

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CAE Spring Meeting - June 5th, 2015

Agenda

Objective and Motivation

- Our objective is to estimate the Solvency Capital Requirement and the Risk Margin as prescribed in the Solvency II regulation for a non-life (re)insurance portfolio.

- The most common method used in practice for the SCR estimation is the Merz-Wüthrich formula.
  - The hypothesis behind the MW formula are often violated.
  - MW formula does not actually provide estimations for the SCR and the Risk Margin!
  - The MW formula is not robust if used outside its applicability perimeter
    (Dacorogna - Ferriero - Krief, “Taking the one-year change from another angle”, 2014, preprint).

SCR and Risk Margin definition

- The SCR is the capital required to cover the risk of a large increase of the technical provision from one year to the other.
  - The SCR for a non-life insurance portfolio, as defined in the Solvency II, is

$$\mathrm{SCR}_0 = \mathbf{VaR}_{99.5\%}(\mathrm{TP}_1 - \mathrm{TP}_0)$$

  where $\mathrm{TP}_n = \mathrm{BE}_n + \mathrm{RM}_n$ is the Technical Provision at year n, i.e. the Best Estimate of the ultimate loss plus the Risk Margin.

- The Risk Margin for the insurance liabilities quantifies their market value. It can be seen as the remuneration for the capital needed during the run-off of the portfolio.
  - The Risk Margin is defined by

$$\mathrm{RM}_n = \mathrm{CoC} \sum_{k=n}^{m-1} \frac{\mathbf{E}(\mathrm{SCR}_k \mid \mathrm{F}_n)}{(1 + r_{n,k-n+1})^{k-n+1}}$$

  where the Cost of Capital is assumed to be constant $\mathrm{CoC} = 6\%$ and the run-off lasts m years.
  - The SCR_n at year $n \geq 1$ is the random variable $\mathrm{SCR}_n = \mathbf{VaR}_{99.5\%}(\mathrm{TP}_{n+1} - \mathrm{TP}_n \mid \mathrm{F}_n)$, where F_n is the available information at year n.

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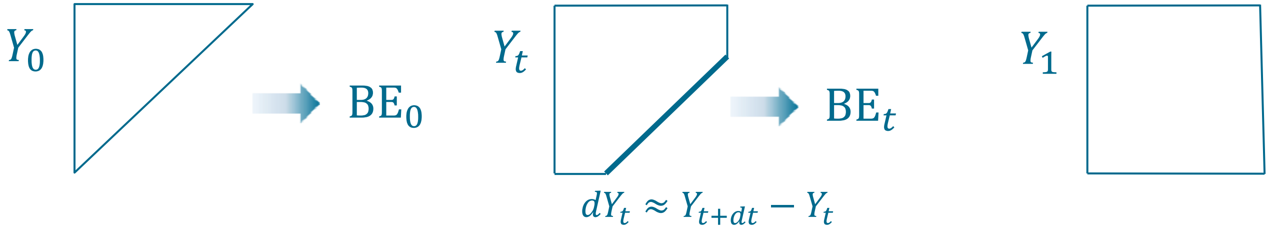
The Ideas behind our Model

- Let Y_1 be the ultimate attritional loss of a run-off portfolio. We want to model the dynamics which brings the losses, and thus the corresponding best estimates of the ultimate loss, from $\mathrm{t} = 0$ to $\mathrm{t} = 1$.



- At time $t > 0$ the realized losses Y_s, $s \leq t$, determine the estimation of BE_t.
  - For example, we may project Y_t to BE_t with the chain-ladder method.

- However, in reality we trust our estimations when things behave normally but we know that exceptionally things may happen which make our estimations wrong.
  - If Y_s, $s \leq t$, oscillate up and down around what we expected, then we are confident with our estimations and may even make occasional prudent reserves release.
  - If Y_s, $s \leq t$, are systematically above expectation over a certain period of time $[T^s, T^e]$, then we may distrust our estimations and thus make a material correction.
  - This change of regime marks the reserving cycle.

Our Model – The Losses over Time

- We assume that Y_1 has Log-Normal distribution.
  - This is appropriate because Y_1 is the attritional losses component.

- In order to model the two regimes of the reserving cycle we assume that the relative loss developments dY_t/Y_t have uncertainties around what expected which are:
  - uncorrelated and have normal distribution on $[0,1] \setminus [T^s, T^e]$, like in a Brownian motion,
  - positively correlated and have normal distribution on $[T^s, T^e]$, like in a fractional Brownian motion with dependency exponent h between 0.5 and 1.

- In mathematical terms,

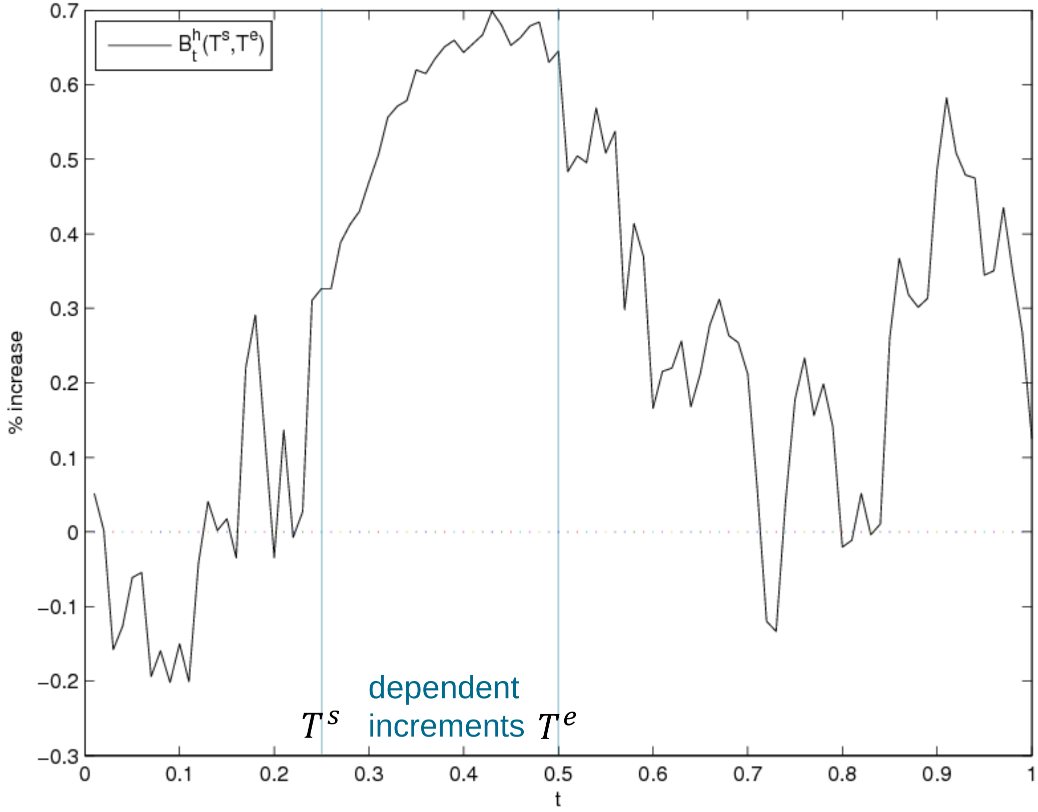
$$dY_t/Y_t = p_t dt + dB_t^h(T^s, T^e), \quad t \in [0,1],$$

  with initial loss $Y_0 > 0$, where $p_t dt$ is what expected and $dB_t\,(T^s, T^e)$ is the uncertainty.
  - The variance of the relative loss developments is assumed to be proportional to the expected incremental loss.

- The time T^e is when a sudden material reserves increase may occur as a result of a period $[T^s, T^e]$ of systematic under-estimation of the losses.

Our Model – The Losses over Time

$$dY_t/Y_t = p_t dt + dB_t^h(T^s, T^e)$$



Our Model – The Best Estimate of the Ultimate Loss over Time

- If γ is the relative size of a reserves jump, then we model the evolution of the best estimate of the ultimate loss over time by the stochastic differential equation

$$d\mathrm{BE}_t = d\mathbb{E}(Y_1|\mathfrak{F}_t) + \gamma(\mathrm{BE}_t - Y_t)dJ_t(T^e), \qquad \text{for } t \in [0,1],$$

  with initial value $\mathrm{BE}_0 = \mathbf{E}(Y_1)$, where:
  - F_t is the available information at time t,
  - $dJ_t(T^e)$ is approximately always null but in T^e where $dJ_{T^e}(T^e)$ may be 1, if a reserve strengthening occurs, otherwise is 0.

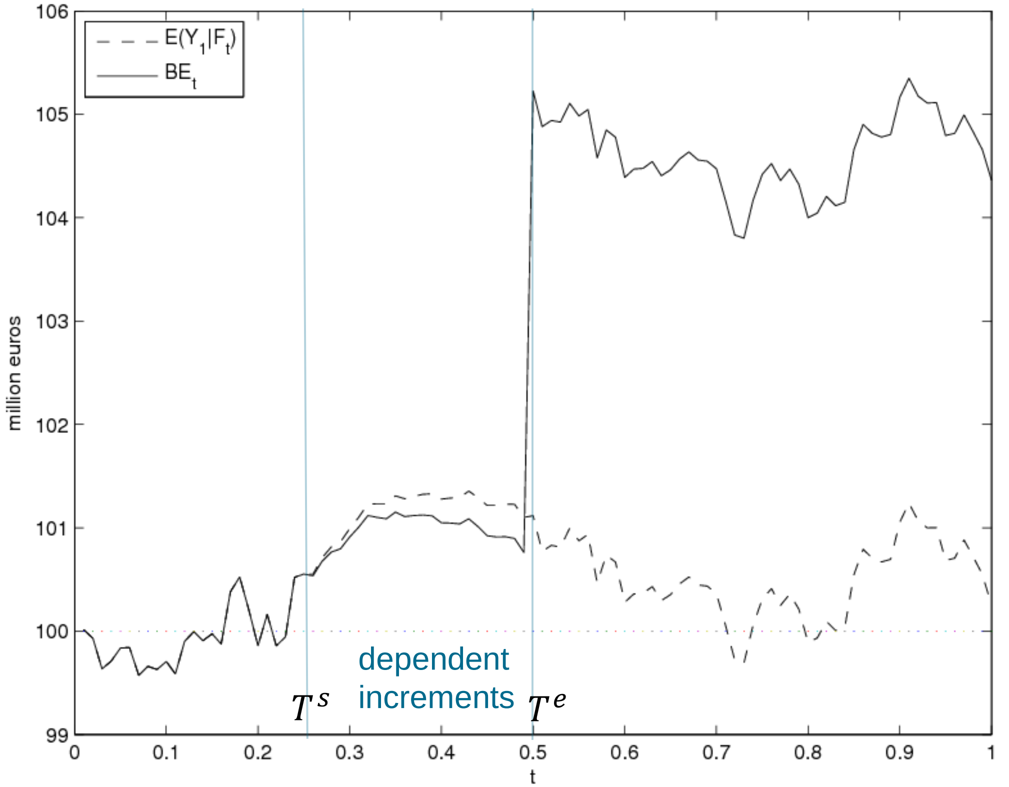
- A plausible reserving actuary criteria f_α triggering the reserve strengthening could be that, if the realized losses during $[T^s, T^e]$ exceed what expected by ξ_α-times the standard deviation, then the best estimate is increased by $\gamma(\mathrm{BE}_{T^e} - Y_{T^e})$.
  - Any such a criteria has an associated probability of occurrence α.

- $\{\mathrm{BE}_t\}$ is a martingale, as it should be, i.e. $\mathbf{E}(\mathrm{BE}_t|F_s) = \mathrm{BE}_s$, for $s \leq t$.

- The model is formulated in terms of stochastic differential equations. However it can be equivalently formulated in a simpler way which does not make use of stochastic differential equations.

Our Model – The Best Estimate of the Ultimate Loss over Time

$$d\mathrm{BE}_t = d\mathbb{E}(Y_1|\mathfrak{F}_t) + \gamma(\mathrm{BE}_t - Y_t)dJ_t(T^e), \qquad \text{for } t \in [0,1],$$



- In the figure, $Y_0 = 50$, $\mathrm{BE}_0 = 100$, $\sigma_0 := \mathrm{Std}(Y_1)/\mathrm{BE}_0 = 3\%$ and $\gamma = 18\%$, $\alpha = 0.05$, $m = 10$.

Our Model - Comments

- The quantity $\mathrm{BE}_t - Y_t$, which represents the reserves, tends to decrease over time, hence the reserves jump size $\gamma(\mathrm{BE}_t - Y_t)$ decreases too.

- As in reality we do not know a priori but only a posteriori if the loss developments have started to be dependent, T^s is not part of the available information F_t.

- The best estimate evolution is composed by two parts, a smooth part and a jump part.

$$d\mathrm{BE}_t = \underbrace{d\mathbb{E}(Y_1|\mathfrak{F}_t)}_{\text{smooth part}} + \underbrace{\gamma(\mathrm{BE}_t - Y_t)dJ_t(T^e)}_{\text{jump part}}, \qquad \text{for } t \in [0,1]$$

- Summarizing, our model describes a reserving cycle.
  - $B_t^h(T^s, T^e)$ captures the first phase of the cycle in which a systematic under-estimation of the losses may occur.
  - $J_t(T^e)$ captures the second phase of the cycle in which a sudden material deterioration of the reserves occurs as a result of the preceding systematic under-estimation.

Agenda

1	Objective and Motivation
2	Our Model
3	**SCR and Risk Margin approximation**
4	Conclusion

SCR and Risk Margin simplifications

- $\mathrm{TP}_{n+1} - \mathrm{TP}_n$ is replaced by $\mathrm{BE}_{n+1} - \mathrm{BE}_n$, which is justified by the fact that the Risk Margin is approximately constant from one year to the other.

- $\mathbf{VaR}_{99.5\%}$ is replaced by $\mathbf{tVaR}_{99\%}$ because $\mathbf{VaR}_{99.5\%}$ is not robust and not coherent.

- $\mathbf{E}[\mathbf{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathrm{F}_n)]$ is replaced by $\mathbf{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathrm{F}_0)$ in the Risk Margin at $t = 0$ because the first quantity is cumbersome.
  - The Risk Margin is a second order quantity with respect to the SCR, and however the proposed simplification is more prudent since $\mathbf{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathrm{F}_0) \geq \mathbf{E}[\mathbf{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n|\mathrm{F}_n)]$.

$$\mathrm{RM}_0 = \mathrm{CoC} \sum_{k=0}^{m-1} \frac{\mathbf{E}(\mathrm{SCR}_k \mid \mathrm{F}_0)}{(1 + r_{0,k+1})^{k+1}} \lesssim \mathrm{CoC} \sum_{k=0}^{m-1} \frac{\mathbf{tVaR}_{99\%}(\mathrm{BE}_{k+1} - \mathrm{BE}_k|\mathrm{F}_0)}{(1 + r_{0,k+1})^{k+1}}$$

SCR and Risk Margin approximation

- Our model has three parameters: the reserves jump size γ, the reserves jump probability α, the loss developments dependency exponent h.
  - The volatility parameter of the ultimate attritional loss is σ_0.

- If $\gamma\alpha, \sigma_0$ are small, then it can be showed that

$$\mathrm{BE}_t \simeq \begin{cases} \mathbb{E}(Y_1|\mathfrak{F}_{T^e}) + \gamma(\mathrm{BE}_{T^e} - Y_{T^e}) & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ \mathbb{E}(Y_1|\mathfrak{F}_t) & \text{otherwise.} \end{cases}$$

- Our model is formulated with continuous-time but it can be easily discretized.

- Suppose that α and γ are such that $\alpha/m < 1\%$ and $\gamma(\mathrm{BE}_0 - Y_0) \geq \mathbf{tVaR}_{99.5\%-|\alpha/m-0.5\%|}[Y_m - \mathbf{E}(Y_m)]$ and $\gamma(\mathrm{BE}_0 - Y_0) \leq \mathbf{tVaR}_{99.5\%+|\alpha/m-0.5\%|}[Y_m - \mathbf{E}(Y_m)]$. We can then show that, with $\lambda := \alpha/(m1\%)$ and $c_n := (e^{p_n} - 1)/(e^{p_m} - 1)$,

$$\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n) \simeq [(c_{n+1} - c_n)^h(1-\lambda) + (1-c_n)\lambda]\mathrm{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0).$$

one-year risk | smooth part | jump part | ultimate risk

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Our Model in practice

- To use our in model in practice we need the following inputs:
  - the ultimate loss model $\mathrm{BE_m}$, which could either include or not the large loss component,
  - the cumulative calendar year expected loss pattern $(c_1, \dots, c_m)$.

- The parameters of the model:
  - the reserves jump size γ; if the ultimate model includes the large losses component, then we can approximate γ by $\mathbf{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0)/(\mathrm{BE}_0 - Y_0)$, otherwise γ can be quantified by experts and its value should be around $\mathbf{tVaR}_{99.5\%}[Y_m - \mathbf{E}(Y_m)]/(\mathrm{BE}_0 - Y_0)$,
  - the reserves jump probability α,
  - the loss developments dependency exponent h.

$$\mathrm{SCR}_0 = [(c_1)^h(1-\lambda) + \lambda]\gamma(\mathrm{BE}_0 - Y_0)$$

$$\mathrm{RM}_0 = 6\% \left[\frac{(c_1)^h(1-\lambda) + \lambda}{1 + r_{0,1}} + \sum_{n=1}^{m-1} \frac{(c_{n+1} - c_n)^h(1-\lambda) + (1 - c_n)\lambda}{(1 + r_{0,n+1})^{n+1}} \right] \gamma(\mathrm{BE}_0 - Y_0)$$

Conclusion

- Behind our SCR and Risk Margin formulas we have the following assumptions:
  - $\gamma\alpha, \sigma_0$ small and γ around $\mathbf{tVaR}_{99.5\%}[Y_m - \mathbf{E}(Y_m)]/(\mathrm{BE}_0 - Y_0)$,
  - losses and the best estimates behave as described in the model.

- Our model addresses known limitations of the MW method:
  - captures the dependency between loss developments,
  - captures the reserving actuary behavior and the reserving cycle,
  - provides estimates for the SCR and the Risk Margin as opposed to the mean square error.

Conclusion

- Our method ensures consistency between ultimate and one-year risk.
  - The better understood ultimate risk can be maintained throughout the entire model.
  - The availability of consistent ultimate and one-year risk estimations enhances the potential use cases (solvency, pricing, capital allocation, planning and retro optimization, …).

- Our method has the practical advantage to be used for portfolios with limited credibility.
  - Given the ultimate attritional loss model and parameters γ, α and h, the SCR and Risk Margin can be estimated with our methodology.
  - However, while the calibration of the parameters γ, α can be elicited through expert judgment, the parameter h, i.e. the dependency between loss developments, would require credible data. It is noted though that such calibration can be benchmarked using data from similar portfolios.

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Appendix

Our Model – The Losses over Time

- We assume that Y_1 has Log-Normal distribution.
  - This is appropriate because Y_1 is the attritional losses component.

- We define the continuous-time dynamics which brings the losses from $\mathrm{t} = 0$ to $\mathrm{t} = 1$ with the stochastic process given by

$$Y_t = Y_0 e^{p_t + B_t^h(T^s, T^e)}, \quad \text{for } t \in [0,1],$$

  with initial loss $Y_0 > 0$, where
  - p_t is the expected loss development, which is an increasing concave function with $p_0 = 0$,
  - $B_t^h(T^s, T^e)$ is the uncertainty around p_t, which is a Brownian motion on $[0,1] \setminus [T^s, T^e]$ and a fractional Brownian motion on $[T^s, T^e]$ with dependency exponent h between 0.5 and 1, with mean such that $\mathbf{E}(Y_t) = Y_0 e^{p_t}$ and variance proportional to the expected outstanding loss.

- The random time T^e is when a sudden material reserves increase may occur as a result of a period $[T^s, T^e]$ of systematic under-estimation of the losses.
  - T^e is uniformly distributed on $[0,1]$, and T^s is a r.v. on $[T^e - 1, T^e]$ with exponential distribution, i.e. $\mathbf{P}(T^s \leq t) = a^{t - T^e}, a > 1$, so that times close to T^e are more probable.

Our Model – The Best Estimate of the Ultimate Loss over Time

- If γ is the relative size of a reserves jump, then we model the evolution of the best estimate of the ultimate loss over time by the Itô stochastic differential equation

$$dBE_t = d\mathbb{E}(Y_1|\mathfrak{F}_t) + \gamma(BE_t - Y_t)dJ_t(T^e), \qquad \text{for } t \in [0,1],$$

with initial value $BE_0 = \mathbf{E}(Y_1)$, where F_t is the σ-algebra generated by $\{Y_s|s \le t\}$ and $\{T^e \le s|s \le t\}$,

$$J_t(T^e) := -k_t + \begin{cases} 1, & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ 0, & \text{otherwise}, \end{cases}$$

with reserving actuary criteria

$$f_\alpha(T^e) := \begin{cases} 1, & \text{if } Y_{T^e}/(Y_{\bar{T}^s} e^{p_{T^e} - p_{\bar{T}^s}}) \ge 1 + \xi_\alpha \text{Std}[Y_{T^e}/(Y_{\bar{T}^s} e^{p_{T^e} - p_{\bar{T}^s}})], \\ 0, & \text{otherwise}, \end{cases}$$

$\bar{T}^s = [T^s]^+$, $\xi_\alpha \ge 0$ is such that $\mathbf{P}(f_\alpha(T^e) = 1|T^e = 1) = \alpha$, and $k_t := \int_0^{t \wedge T^e} \frac{\mathbb{P}(f_\alpha(T^e) = 1|T^e = s)}{1-s} ds.$

- The reserving actuary criteria means that, if the realized losses during $[T^s, T^e]$ exceed what expected by ξ_α-times the standard deviation, then the best estimate is increased by $\gamma(BE_{T^e} - Y_{T^e})$.

Our Model - Comments

- The model is formulated in terms of Itô's stochastic differential equation. However it can be equivalently formulated by the following simpler relation

$$\mathrm{BE}_t = \mathbb{E}(Y_1|\mathfrak{F}_t) - A_t + \begin{cases} \gamma[\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}) - A_{T^e}], & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $A_t := e^{-\gamma k_{t\wedge T^e}} \int_0^{t\wedge T^e} [\mathbb{E}(Y_1|\mathfrak{F}_s) - Y_s] de^{\gamma k_s}$.

- Note that BE_1 differs from Y_1. The reason being that Y_t represents the attritional losses only, whereas BE_t contains also the large losses behind the reserves jump.

- Summarizing, our model describes a reserving cycle.
  - $B_t^h(T^s, T^e)$ captures the first phase of the cycle in which a systematic under-estimation of the losses may occur.
  - $J_t(T^e)$ captures the second phase of the cycle in which a sudden material deterioration of the reserves occurs as a result of the preceding systematic under-estimation.

SCOR

Model Parameters

- Our model has three parameters for the ultimate-to-one-year relation:
  - the reserves jump size γ,
  - the reserves jump probability α,
  - the loss developments dependency exponent h.

- The volatility parameter of the ultimate attritional loss is σ_0.

- If $\gamma\alpha, \sigma_0$ are small, then the model is approximately equal to

$$\mathrm{BE}_t \simeq \begin{cases} \mathbb{E}(Y_1|\mathfrak{F}_{T^e}) + \gamma\mathbb{E}(Y_1 - Y_{T^e}|\mathfrak{F}_{T^e}), & \text{if } t \in (T^e, 1], f_\alpha(T^e) = 1, \\ \mathbb{E}(Y_1|\mathfrak{F}_t) & \text{otherwise.} \end{cases}$$

Indeed, A_t is small if $\gamma\alpha$ is small, and $\mathbf{E}(Y_1|\mathrm{F}_t) - \mathbf{E}(Y_1|\mathrm{F}_{T^e})$ is small if σ_0 is small.

SCR and Risk Margin approximation

- Our model is formulated with continuous-time but it can be easily discretized. We only need to restrict T^s, T^e to assume values on a discrete subset of equidistant points in [0,1].

- If $\gamma\alpha, \sigma_0$ are small, then

$$\mathrm{BE}_{n+1} - \mathrm{BE}_n \simeq \begin{cases} \gamma\mathbb{E}(Y_m - Y_n|\mathfrak{F}_n), & \text{if } t_{n+1} = T^e_*, f_\alpha(T^e_*) = 1, \\ \mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n), & \text{otherwise,} \end{cases}$$

- Suppose that α and γ are such that $\alpha/m < 1\%$ and $\gamma(\mathrm{BE}_0 - Y_0) \geq \mathbf{tVaR}_{99.5\%-|\alpha/m-0.5\%|}[Y_m - \mathbf{E}(Y_m)]$ and $\gamma(\mathrm{BE}_0 - Y_0) \leq \mathbf{tVaR}_{99.5\%+|\alpha/m-0.5\%|}[Y_m - \mathbf{E}(Y_m)]$. Then, with $\lambda := \alpha/(m1\%)$,

$$\begin{aligned} \mathrm{tVaR}_{99\%}(X) &\simeq \mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)](1-\lambda) + \gamma Y_0(e^{p_m} - 1)\lambda, \\ &= \mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)](1-\lambda) + \gamma(\mathrm{BE}_0 - Y_0)\lambda \\ &\simeq \gamma(\mathrm{BE}_0 - Y_0), \end{aligned}$$

and

$$\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n) \simeq \mathrm{tVaR}_{99\%+\alpha/m}[\mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n)](1-\lambda) + \gamma Y_0(e^{p_m} - e^{p_n})\lambda.$$

SCR and Risk Margin approximation

- In addition, with $c_n := (e^{p_n} - 1)/(e^{p_m} - 1)$,

$$\mathrm{tVaR}_{99\%+\alpha/m}[\mathbb{E}(Y_m|\mathfrak{F}_{n+1}) - \mathbb{E}(Y_m|\mathfrak{F}_n)] \simeq (c_{n+1} - c_n)^h \, \mathrm{tVaR}_{99\%+\alpha/m}[Y_m - \mathbb{E}(Y_m)].$$

  Indeed a fBM B_t^h with dependency exponent h is such that $B_{ct}^h \sim c^h B_t^h$, for any $c > 0$.

- We therefore obtain

$$\mathrm{tVaR}_{99\%}(\mathrm{BE}_{n+1} - \mathrm{BE}_n) \simeq [(c_{n+1} - c_n)^h(1-\lambda) + (1-c_n)\lambda]\mathrm{tVaR}_{99\%}(\mathrm{BE}_m - \mathrm{BE}_0)$$

one-year risk — smooth part contribution — jump part contribution — ultimate risk