Generalized Linear Mixed Models for Ratemaking

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Agenda

- Statement of problem.
- The classical Linear Mixed Effects (LME) model.
- How Buhlmann credibility emerges from the LME model in a simple example.
- A generalization: Generalized Linear Mixed Models (GLMMs) and their relationship to credibility.
- A case study for class group relativities.

The Problem

- In Generalized Linear Models (GLMs), independent variables with many levels (e.g. territory, class) aren't credibility adjusted.
- That is, coefficients/relativities of levels with little exposure are squarely in the middle of wide confidence intervals driven by large standard errors.

The Problem

- Therefore, to deploy the model, actuaries feel obliged to apply an ad hoc credibility adjustment to the coefficients/relativities.
- Wouldn't it be better to incorporate the credibility adjustment within the modeling process, so that the model and credibility adjustment are optimized as a unit, instead of separately?

A Solution

- Generalized Linear Mixed Models (GLMMs) accomplish this credibility adjustment.
- The following case study will demonstrate this.

Linear Mixed Effects (LME) Models

- LMEs are linear regressions with:
  - Normally distributed errors.
  - (Identity link.)
  - Both fixed and random effects.

LMEs: The Traditional Motivation

- The traditional motivation: designed experiments.
- Consider a drug trial at several medical centers. Drug effects are fixed. Center effects are random.
- If you treat centers as fixed effects, you draw inferences for drugs valid for those centers, but they are not extendible to other centers.
- If you treat centers as random effects, you recognize additional source of randomness and render drug inferences valid for other centers as well. The confidence intervals on fixed effects are wider, but they allow for the possibility that you might use these drugs at other medical centers.

LMEs:
The Actuarial Motivation

- If we treat territory or class as a random effect, it doesn't mean that we believe these effects to be randomly assigned, nor do we seek inferences for the other rating variables that apply to additional territories or classes.
- Rather, we seek to exploit the fact that the Best Linear Unbiased Predictor (BLUP) for random effects (territories, classes) includes a credibility-like shrinkage towards the mean that would not occur if we treated these variables as fixed.

LME Structure, Expectation and Variance

- LME structure is $Y=X\beta+Zu+e$.
- X the fixed effects design matrix.
- β the vector of fixed effects regression coefficients.
- Z the random effects design matrix.
- u and e normally distributed random variables (vectors) with:
    - $E[u]=E[e]=0$
    - $Var[u]=G$, $Var[e]=R$, $Cov[u,e]=0$

LME Structure, Expectation and Variance

- Two relevant distributions are:
  - Conditional: $Y|u$
    - Expectation: $X\beta + Zu$
    - Variance: R
  - Marginal: Y
    - Expectation: $X\beta$
    - Variance: $V=ZGZ' + R$

LME Solution: Fixed Effects

- Closed-form solutions exist for LMEs using generalized least squares.
- Fixed effect estimators are Best Linear Unbiased Estimators (BLUE, actually EBLUE)
- The solution is:

$$\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}Y$$

LME Solution: Random Effects

- Predictors for random effects are Best Linear Unbiased Predictors (BLUP, actually EBLUP)
- The solution is:

$$E[u \mid Y] = E[u] + Cov[u, Y]Var[Y]^{-1}(Y - E[Y])$$

$$= GZ'V^{-1}(Y - X\hat{\beta})$$

A Simple Example

- Let's assume that class is the only independent variable. We assume that the grand mean is a fixed effect, and class is a random effect.
- Data has been aggregated, so that there is only one observation per class, the class average response, y_i. Y is the vector of y_i.
- Exposures for class i are e_i.
- X is the design matrix appropriate for an intercept only model: a single column of 1s.

A Simple Example

Therefore:

- Z is an identity matrix.
- R, the variance matrix of e, is diagonal with "within" variances σ_w^2/e_i. That is, the stability of the class estimates are directly proportional to their exposures.
- G, the variance matrix of u, is diagonal with "between" variance σ_b^2.

Solution

Recall that $V = ZGZ' + R$

Therefore, V is diagonal with ith element :

$$V_i = \sigma_b^2 + \frac{\sigma_w^2}{e_i}$$

Solution

Recall that the grand mean $\beta = (X'V^{-1}X)^{-1}X'V^{-1}Y$

$$= \left(\sum_i \frac{1}{\sigma_b^2 + \frac{\sigma_w^2}{e_i}} \right)^{-1} \sum_i \frac{y_i}{\sigma_b^2 + \frac{\sigma_w^2}{e_i}}$$

Letting credibility $= z_i = \dfrac{e_i}{e_i + \frac{\sigma_w^2}{\sigma_b^2}}$

$\hat{\beta} = \sum_i z_i y_i \Big/ \sum_i z_i =$ credibility - weighted average of the observations

Solution

- To solve for the random effects, note that E[u]=0, and Cov[u,Y] and Var[Y] are both diagonal (see below), so the expression for E[u|Y] simplifies greatly. Furthermore, note that:

$$Cov[u_i, y_i] = Cov[u_i, \beta + u_i + e_i] = Var[u_i] = \sigma_b^2$$

$$Cov[u_i, y_j] = Cov[u_i, \beta + u_j + e_j] = Cov[u_i, u_j] = 0$$

Solution

So :

$$E[u \mid Y] = E[u] + Cov[u, Y]Var[Y]^{-1}(Y - E[Y])$$

$$E[u_i \mid Y] = E[u_i \mid y_i] = Cov[u_i, y_i]Var[y_i]^{-1}(y_i - \hat{\beta})$$

$$= \frac{\sigma_b^2}{\sigma_b^2 + \dfrac{\sigma_w^2}{e_i}}(y_i - \hat{\beta})$$

$$= z_i(y_i - \hat{\beta})$$

Solution

- This is just Buhlmann credibility, predicated, admittedly, on assumption of normally distributed errors and identity link.

A Generalization of LME: GLMM

- GLMM=Generalized Linear Mixed Models
- In the previous notation, the conditional distribution Y|u is in the exponential family.
- Link function g such that $g(E[Y|u])=X\beta+Zu$
- u is still multivariate normal with mean 0 and variance matrix G. And uncorrelated with the random error Y-E[Y|u]
- Note that the marginal distribution Y, which is Y|u integrated over u, may very well not be in the exponential family.

Solution

- In the Generalized Linear Mixed Model setting, equations are not solvable in closed form, so they must be iterated. So no closed-form algebra produces credibilities as we did above.
- But, motivated by the above result in a special case, we can “back into” inferred credibilities, and compare them to exposure.

Implied Credibility: GLMM Setting

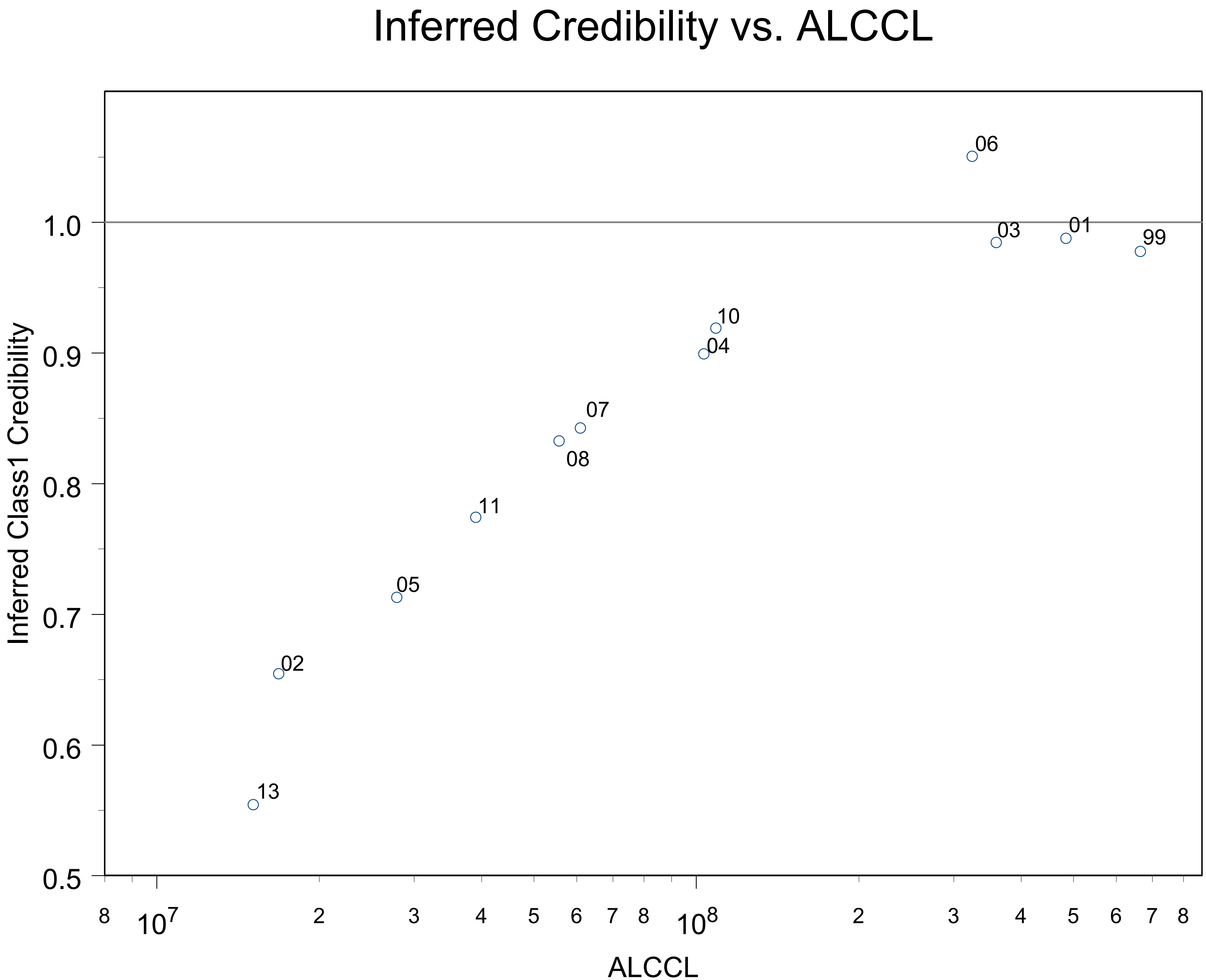
- Fit two models:
  - Generalized Linear Mixed Model
  - GLM in which random effects are treated as fixed.
- Comparing output of these two models, we can infer the implied credibility on the random effects.
- Plot the implied credibility against measures of volume. Do credibilities seem reasonable?

Inferred Credibility

		Fixed Effects			Random Effects			(9)
		(3)		(5)	(6)		(8)	Inferred
(1)	(2)	Class1	(4)	Class1	Class1	(7)	Class1	Class1
Class1	ALCCL	Effect	exp(effect)	Relativity	Effect	exp(effect)	Relativity	Credibility
01	484,185,185	0.3346	1.3974	1.1417	0.1241	1.1321	1.1399	0.9877
02	16,832,999	0.2585	1.2950	1.0580	0.0304	1.0309	1.0380	0.6545
03	359,748,011	0.3056	1.3574	1.1090	0.0951	1.0998	1.1073	0.9845
04	103,293,336	-0.1181	0.8886	0.7260	-0.2898	0.7484	0.7536	0.8994
05	27,864,645	0.4388	1.5508	1.2671	0.1674	1.1822	1.1904	0.7129
06	324,592,379	0.2196	1.2456	1.0176	0.0115	1.0116	1.0185	1.0506
07	60,941,612	0.4695	1.5992	1.3066	0.2229	1.2497	1.2583	0.8426
08	55,682,170	0.4268	1.5323	1.2519	0.1836	1.2015	1.2098	0.8328
10	108,633,028	0.2978	1.3469	1.1004	0.0814	1.0848	1.0923	0.9190
11	39,019,053	-0.1779	0.8370	0.6839	-0.2876	0.7501	0.7552	0.7742
13	15,101,361	-0.0423	0.9586	0.7832	-0.1349	0.8738	0.8798	0.5542
99	664,914,612	0.0000	1.0000	0.8170	-0.2040	0.8155	0.8211	0.9777
	2,260,808,391		1.2240	1.0000		0.9932	1.0000	

Inferred Credibility

$$\text{Inferred credibility} = \frac{\text{Random effect relativities} - 1}{\text{Fixed effect relativities} - 1}$$


Inferred Credibility vs. ALCCL
Inferred Class1 Credibility
1.0
0.9
0.8
0.7
0.6
0.5
06
03
01
99
10
04
07
08
11
05
02
13
8
10^7
2
3
4
5
6
7
8
10^8
2
3
4
5
6
7
8
ALCCL

Inferred Credibilities in Buhlmann Form

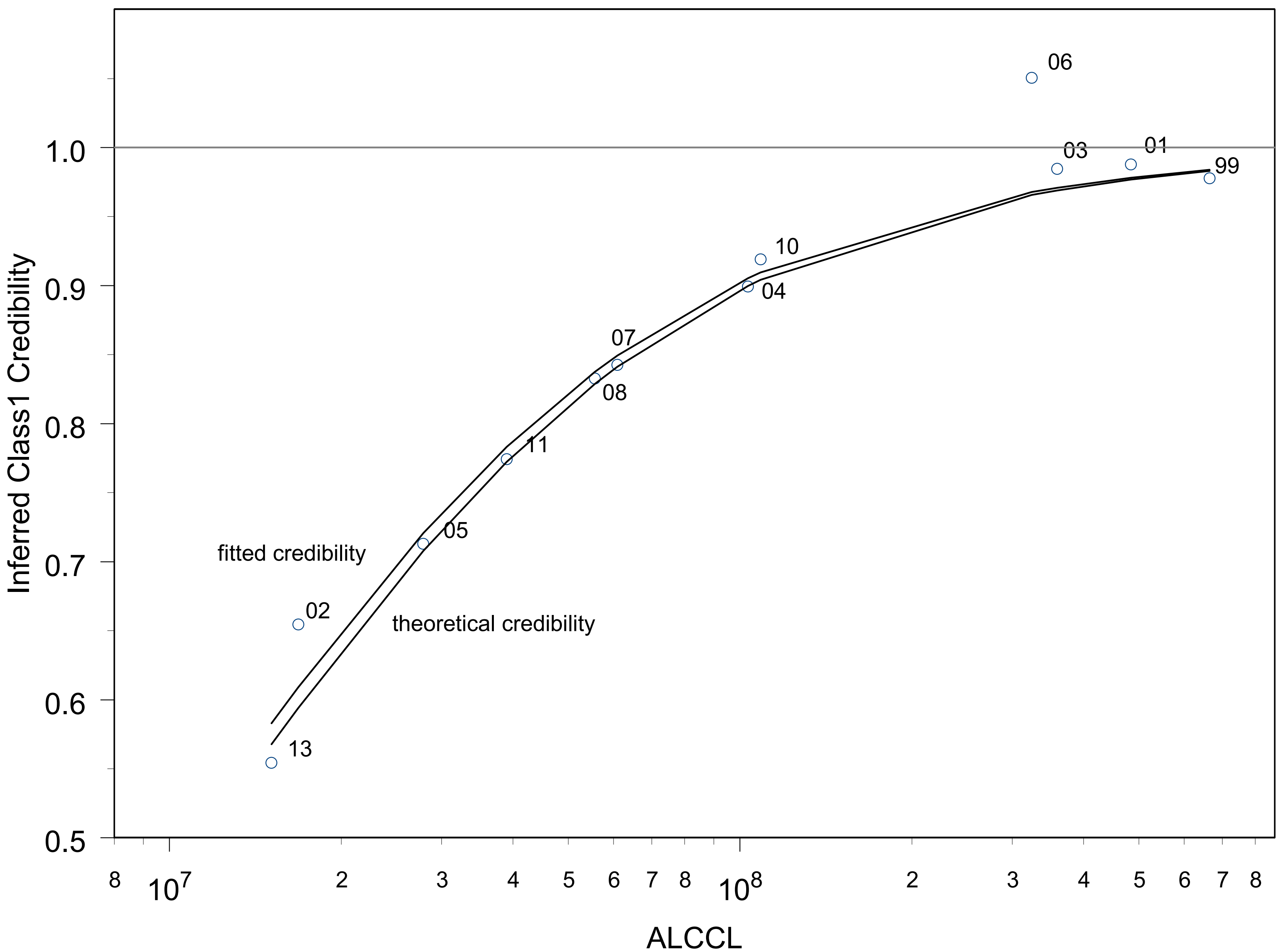
$$c = \frac{e}{e + k}$$

- Can be reworked into the following form, suggesting we estimate k via a regression through the origin with explanatory variable 1/e.

$$\frac{1}{c} - 1 = \frac{k}{e}$$

- Can also approximate k from GLMM package output.

Inferred Credibility vs. ALCCL



For Further Detail

- Klinker, Fred, “Generalized Linear Mixed Models for Ratemaking: A Means of Introducing Credibility into a Generalized Linear Model Setting,” *Casualty Actuarial Society E-Forum*, Winter 2011-Volume 2.