TOWERS PERRIN

CAS Annual Meeting
ROC Solid Reserves

ROC/GIRO Working Party: Reserve Uncertainty

Mack Reserve Ranges

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Why might Mack's method understate the 99.5%-ile in real situations?

- Four suggestions:

1. Mack's formula is an approximation of the standard error of the unpaid claim estimate
2. Mack's heuristic for the standard error of the development in the last period may be too low
3. Student-T instead of Normal, Log-T instead of Lognormal, or other distribution
4. Regression tends to "over fit" the data for prediction purposes

But first, Mack's model

- These three assumptions comprise "the model" which forms the basis of Mack's formulas (see his 1993 paper)

  (CL1) $E(C_{i,k+1} \mid \text{the triangle}) = C_{ik} f_k$

  (CL2) $\boldsymbol{Var}(\boldsymbol{C}_{i,k+1} \mid \text{the triangle}) = \boldsymbol{C}_{ik} \boldsymbol{\sigma}_k^2$ for unknown parameters $\boldsymbol{\sigma}_k^2$

  (CL3) accident years are independent

- From those assumptions, he derives that

$$m\hat{s}e(\hat{C}_{iI}) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

where

$$\hat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{ik} \left(\frac{C_{i,k+1}}{C_{ik}} - \hat{f}_k \right)^2$$

the "f-hats" are the weighted average link ratios, and the "C-hats" are the chain ladder estimates of ultimate loss for accident yr I.

Mack's formula is an approximation

1. Mack’s Missing Term

- Mack’s follow-up 1999 paper gave a recursive version of his closed form formula above
- A re-derivation of his recursive formulas from first principals yields a formula that has an extra cross-product term
  - See Buchwalder, Bühlmann, et al, “The Mean Square Error of Prediction in the Chain Ladder Reserving Method (Mack and Murphy Revisited),” Astin 2006
  - Murphy, “Chain Ladder Reserve Risk Estimators,” CAS eForum August 2007
- Impact of missing term is inconsequential for well-behaved triangles (small CVs)
- An example

Mack vs. Murphy Formulas
Taylor & Ashe data analyzed by Mack (1993)

AY/DY	1	2	3	4	5	6	7	8	9	10
i/k	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10
i=1	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
i=2	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	5,433,719
i=3	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315	5,285,148	5,378,826
i=4	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268	4,835,458	5,205,637	5,297,906
i=5	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311	4,207,459	4,434,133	4,773,589	4,858,200
i=6	396,132	1,333,217	2,180,715	2,985,752	3,691,712	4,074,999	4,426,546	4,665,023	5,022,155	5,111,171
i=7	440,832	1,288,463	2,419,861	3,483,130	4,088,678	4,513,179	4,902,528	5,166,649	5,562,182	5,660,771
i=8	359,480	1,421,128	2,864,498	4,174,756	4,900,545	5,409,337	5,875,997	6,192,562	6,666,635	6,784,799
i=9	376,686	1,363,294	2,382,128	3,471,744	4,075,313	4,498,426	4,886,502	5,149,760	5,544,000	5,642,266
i=10	344,014	1,200,818	2,098,228	3,057,984	3,589,620	3,962,307	4,304,132	4,536,015	4,883,270	4,969,825
LDFs	3.491	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018	1.000
CDFs	14.447	4.139	2.369	1.625	1.384	1.254	1.155	1.096	1.018	1.000
σ_k^2	160,280	37,737	41,965	15,183	13,731	8,186	447	1,147	447	

Mack vs. Murphy Formulas
Taylor & Ashe data analyzed by Mack (1993)

	Chain Ladder Method: Standard Error of Estimated Liability					
Origination Year	Mack Formula			Murphy Formula		
	Process	Parameter	Total	Process	Parameter	Total
i=2	48,832	57,628	75,535	48,832	57,628	75,535
i=3	90,524	81,338	121,699	90,524	81,340	121,700
i=4	102,622	85,464	133,549	102,622	85,467	133,551
i=5	227,880	128,078	261,406	227,880	128,091	261,412
i=6	366,582	185,867	411,010	366,582	185,907	411,028
i=7	500,202	248,023	558,317	500,202	248,110	558,356
i=8	785,741	385,759	875,328	785,741	385,991	875,430
i=9	895,570	375,893	971,258	895,570	376,222	971,385
i=10	1,284,882	455,270	1,363,155	1,284,882	455,957	1,363,385
Total:	1,878,292	1,568,532	2,447,095	1,878,292	1,569,349	2,447,618

- Mack and Murphy process risk estimates are identical
- Differences in parameter risk occur, at most, only in the 3rd or 4th significant digit
- Mack and Murphy overall CV’s are virtually identical – 13.1%
- See http://www.casact.org/pubs/forum/07sforum/07s-murphy.pdf

Mack's heuristic for the last development period's variance

2. Heuristic for σ² of last development period may be too restrictive

- For the last link ratio where there is only one observation, Mack suggests a heuristic for imputing variability from its neighbors

AY/DY	6	7	8	9	10
i/k	k=6	k=7	k=8	k=9	k=10
i=1	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463
i=2	4,120,063	4,647,867	4,914,039	5,339,085	5,433,719
i=3	4,132,918	4,628,910	4,909,315	5,285,148	5,378,826
i=4	4,381,982	4,588,268	4,835,458	5,205,637	5,297,906
i=5	3,873,311	4,207,459	4,434,133	4,773,589	4,858,200
LDFs	1.086	1.054	1.077	1.018	1.000
CDFs	1.254	1.155	1.096	1.018	1.000
σ_k^2	8,186	447	1,147	447	

$$\sigma_9^2 = \min\left(1447^2 / 447, \min(447,1147)\right) = 447$$

- If σ_9^2=1147, Mack's CV would increase by 4% (to 13.6%)
- By the way, Bootstrap allows the residuals throughout the triangle to be sampled in the last position, potentially explaining Bootstrap's CVs higher than Mack's

Replace lognormal with another distribution

3. Lognormal / Normal Are Inappropriate Distributions

- It is an elementary statistics principal that when the standard deviation is unknown but is estimated from the data, then the Normal distribution understates the width of estimated confidence intervals
  - The Student-T is the appropriate distribution to use
- The Log-T is the analogous distribution when the underlying distribution is Lognormal (see Michael Wacek's papers on the CAS site)
- Different, more skewed distributions may be utilized
  - By matching higher moments?
- An example

Small Liability Triangle

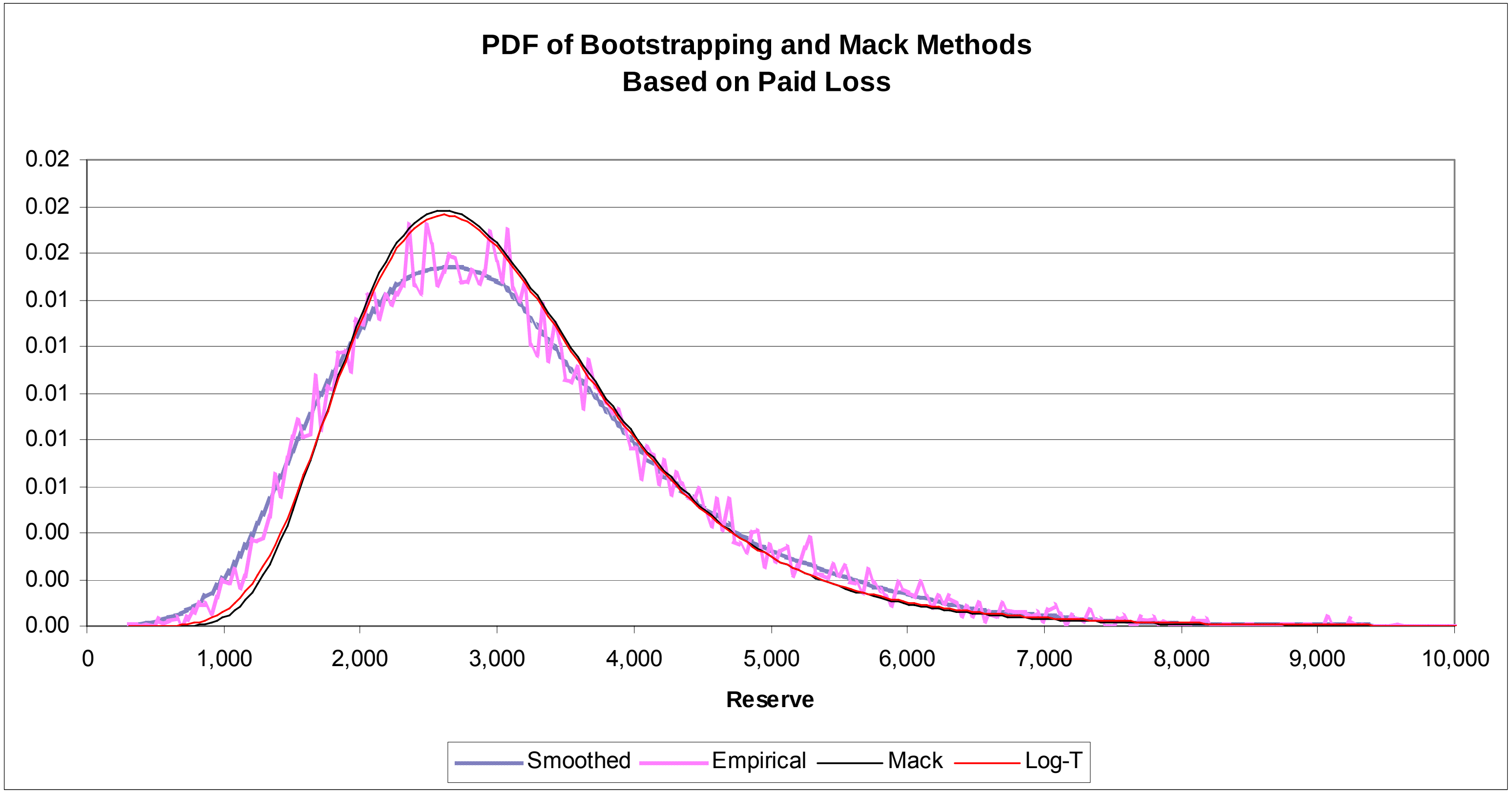
Paid Loss

Accident Period	Evaluation Age in Months								
	12	24	36	48	60	72	84	96	108
1998	460	1,412	1,955	2,092	2,102	2,592	2,649	2,861	3,004
1999	655	1,184	1,624	1,675	1,800	1,831	1,838	1,985	
2000	228	680	780	786	862	863	863		
2001	680	968	1,594	1,656	1,673	1,648			
2002	184	1,081	1,210	1,235	1,260				
2003	263	681	998	1,256					
2004	82	194	506						
2005	175	602							
2006	357								

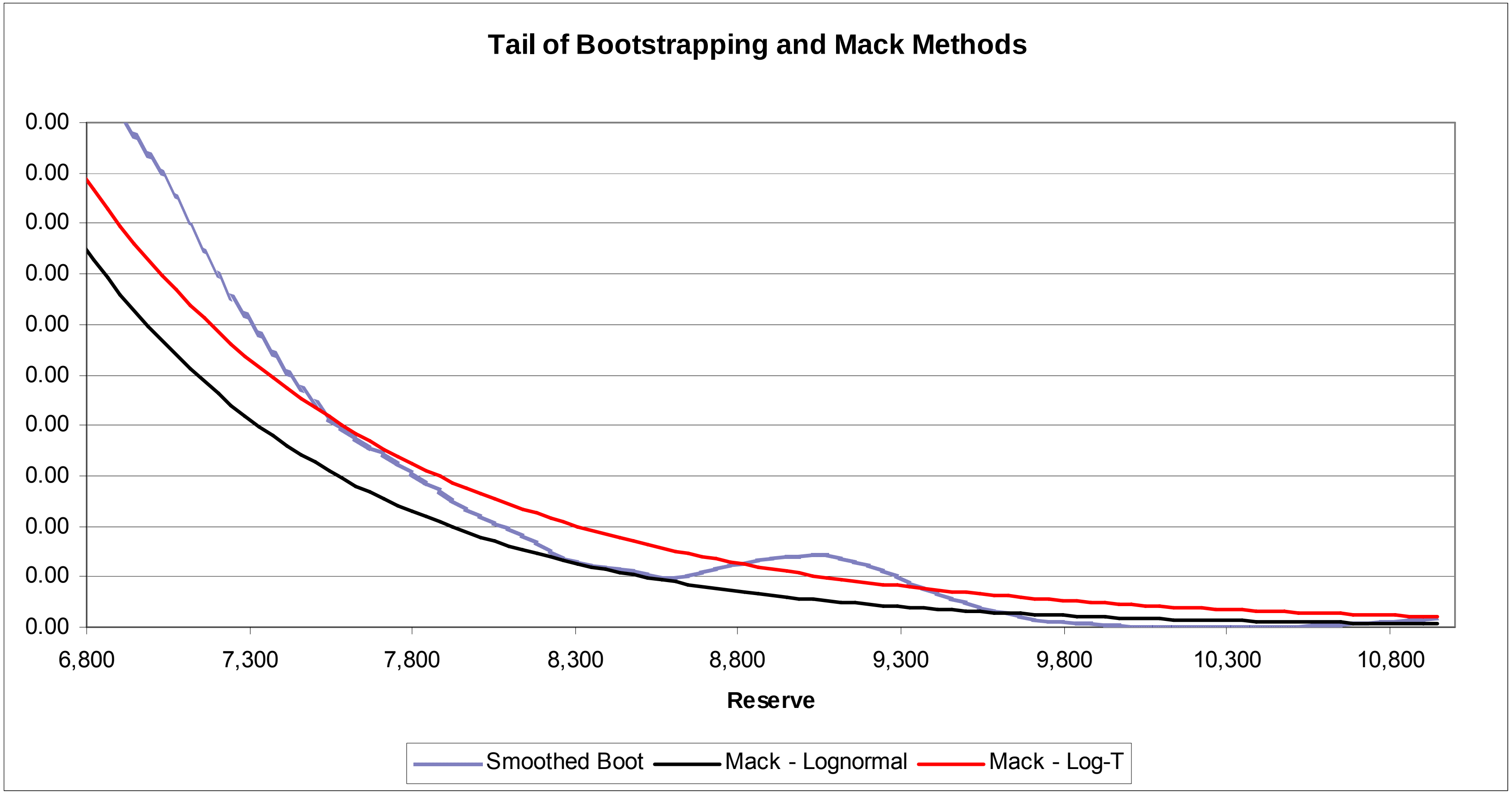
Accident Period	Age Interval in Months								
	12 - 24	24 - 36	36 - 48	48 - 60	60 - 72	72 - 84	84 - 96	96 - 108	108 - Ult
1998	3.287	3.070	1.070	1.005	1.233	1.022	1.080	1.050	
1999	1.808	1.372	1.031	1.075	1.017	1.004	1.080		
2000	2.982	1.147	1.008	1.097	1.001	1.000			
2001	1.865	1.257	1.039	1.010	.985				
2002	5.875	1.119	1.021	1.020					
2003	2.589	1.465	1.259						
2004	2.366	2.608							
2005	3.440								
2006									
Volume Weighted Average All Values									
	2.641	1.313	1.066	1.034	1.077	1.012	1.080	1.050	1.000

Small Liability Triangle

Estimated Unpaid Claims	Mean	SE	Percentile 0.90	0.99	0.995
(1) Paid Bootstrap Method on Weighted Average Factors	3,095	1,230	4,731	6,826	7,318
(2) Paid Mack Method on Weighted Average Factors	3,095	1,107	4,546	6,531	7,121
(3) Paid Mack Method using Log-T distribution	3,095	1,107	4,566	6,820	7,585



Small Triangle –
Tail of the Estimated Liabilities



- For degrees of freedom: there are n=36 observed link ratios and p=8 estimated parameters, so df=n-p=28

Mack's model is equivalent to regression, so can understate projection variability

4. Mack's model and regression

- Mack's model can be written $Y = f \cdot X + e$ where $Var(e)=X\sigma^2$
- Divide both sides by the square root of X and you have a linear regression model with intercept through the origin
  - Solution for the slope *f* of the regression line is the weighted average link ratio
  - Also produces an estimate of the parameter risk of *f* and of the process risk σ^2, both of which agree with Mack's formulas
- Point is: since Mack's method is equivalent to a regression model, it is subject to all the vagaries of regression
  - One of the recognized problems with doing *projections* from linear regressions is that the regression solution can understate *projection* variability
  - A potential work-around: Cross-Validation
    (cf. http://www.autonlab.org/tutorials/overfit.pdf)
- An example

Mack and regression – an example

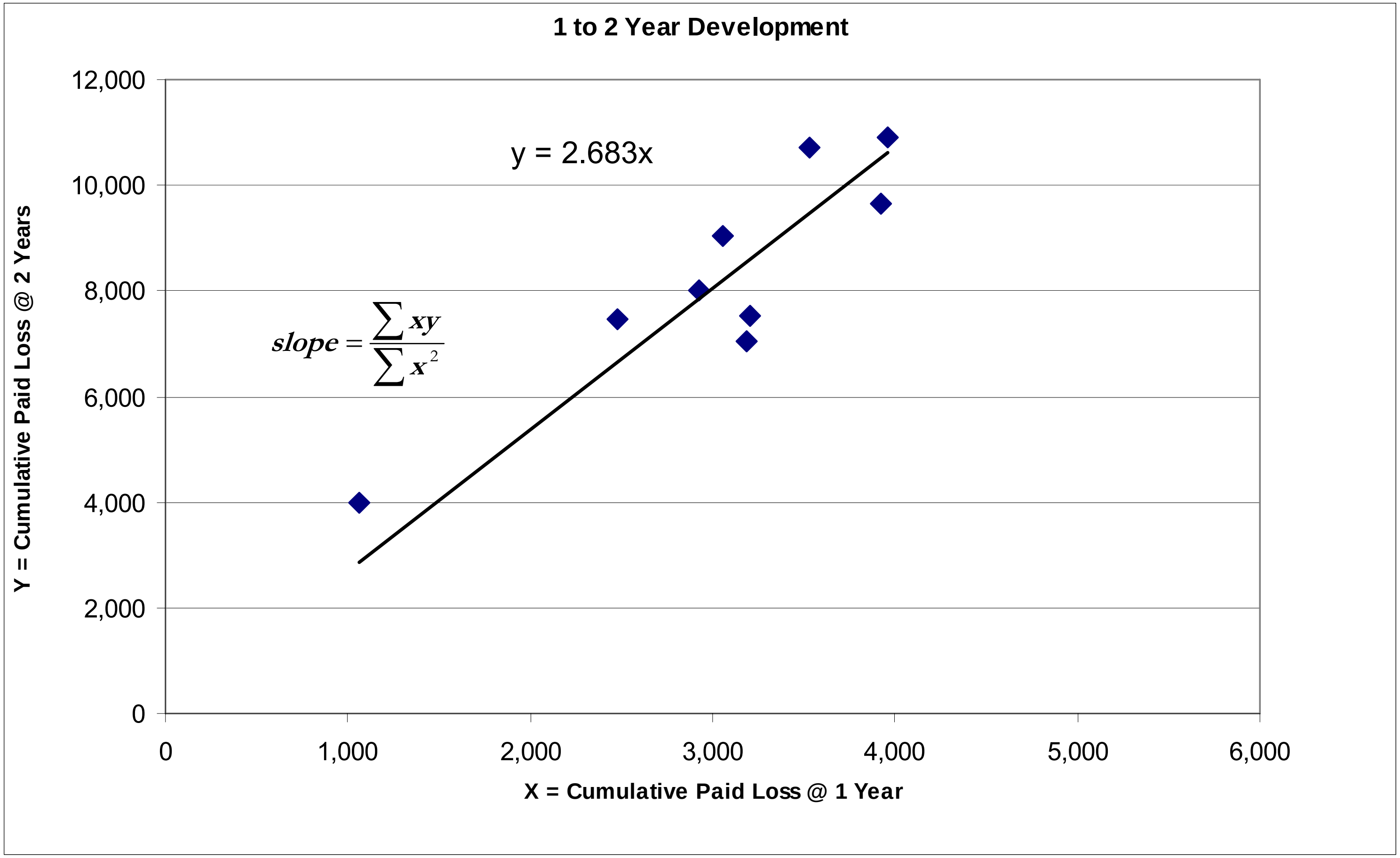
Industry Workers Compensation Paid Net Loss and DCC

Accident Year	Age (years) 1	2	3	4	5	6	7	8	9	10
1996	1,062	3,978	12,068	13,791	14,975	15,710	16,294	16,681	17,007	17,315
1997	3,920	9,667	12,813	14,662	15,786	16,624	17,207	17,736	18,109	
1998	3,960	10,896	13,874	15,657	17,018	17,957	18,672	19,233		
1999	3,202	7,515	13,988	16,143	17,361	18,127	18,798			
2000	2,927	7,999	15,214	17,426	18,850	19,705				
2001	3,056	9,027	16,249	18,746	19,823					
2002	2,479	7,455	15,628	17,943						
2003	3,528	10,723	15,371							
2004	3,191	7,030								
2005	3,274									

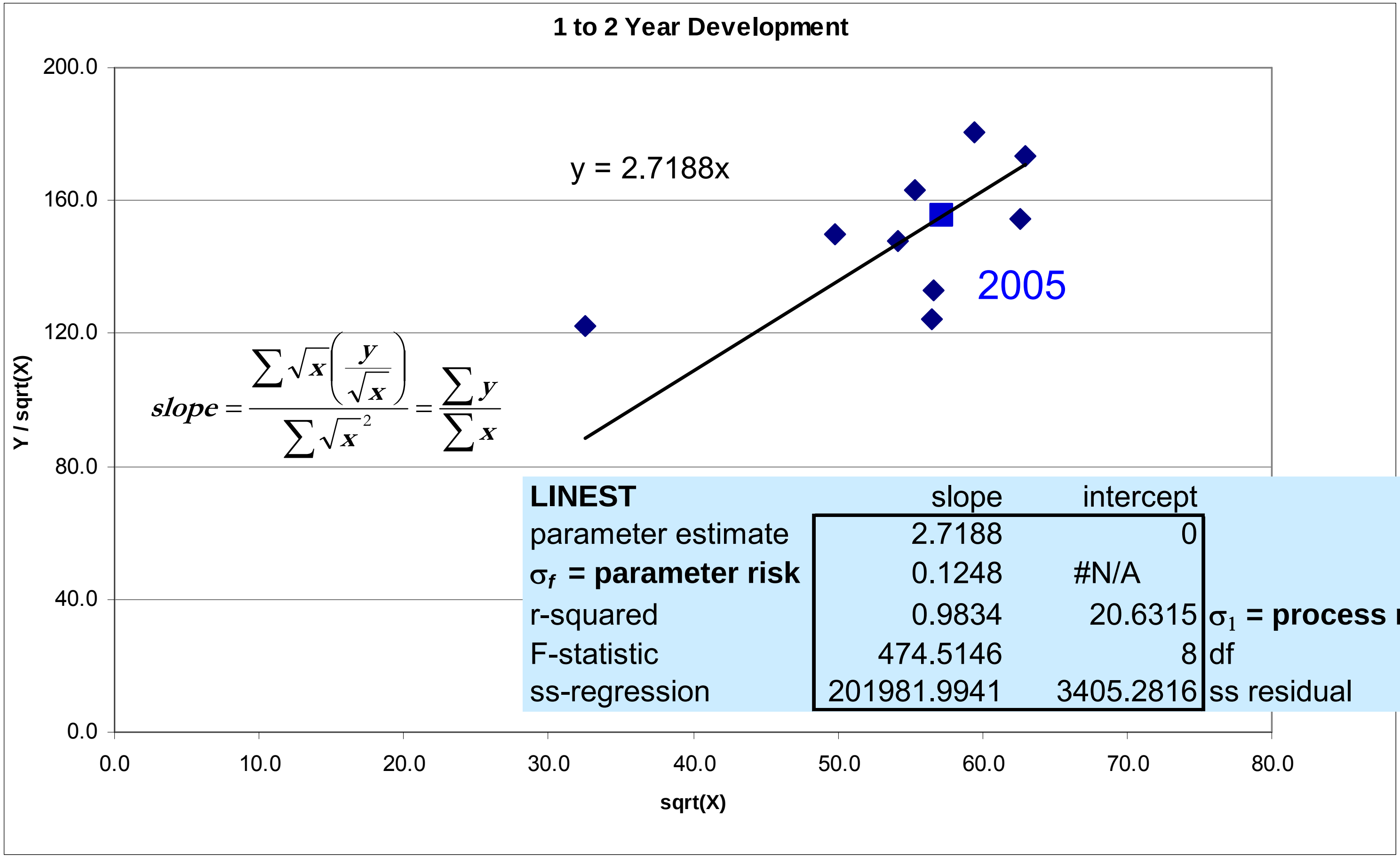
Link Ratios

Accident Year	Development Periods 1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6 to 7	7 to 8	8 to 9	9 to 10
1996	3.746	3.033	1.143	1.086	1.049	1.037	1.024	1.020	1.018
1997	2.466	1.325	1.144	1.077	1.053	1.035	1.031	1.021	
1998	2.752	1.273	1.129	1.087	1.055	1.040	1.030		
1999	2.347	1.861	1.154	1.075	1.044	1.037			
2000	2.733	1.902	1.145	1.082	1.045				
2001	2.954	1.800	1.154	1.057					
2002	3.007	2.096	1.148						
2003	3.039	1.434							
2004	2.203								
Weighted Average	2.7188	1.7128	1.1456	1.0766	1.0492	1.0373	1.0283	1.0203	1.0181

Mack and regression – an example
12 – 24 months of development



Mack and regression – an example
With transformed data, slope = wtd avg



$$slope = \frac{\sum \sqrt{x}\left(\frac{y}{\sqrt{x}}\right)}{\sum \sqrt{x}^{2}} = \frac{\sum y}{\sum x}$$

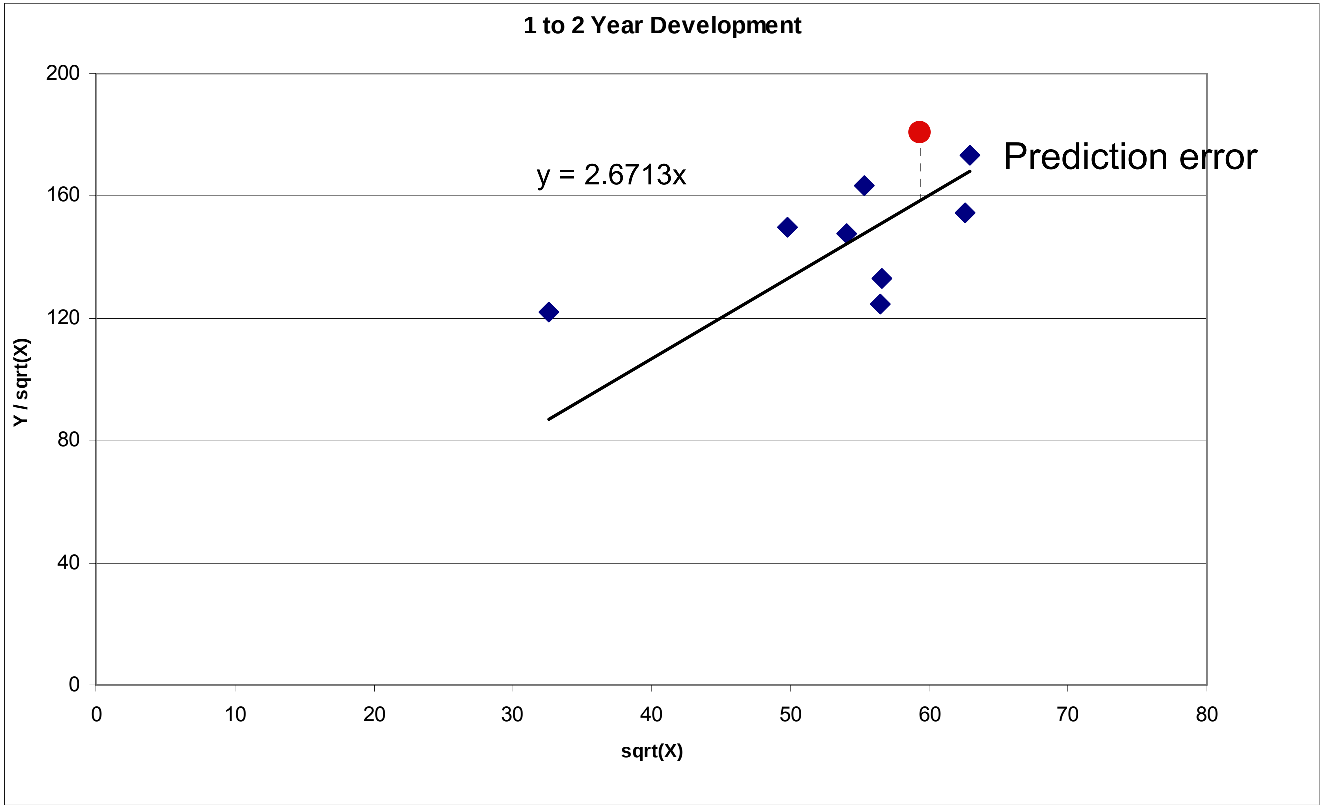
LINEST	slope	intercept	
parameter estimate	2.7188	0	
σ_f = parameter risk	0.1248	#N/A	
r-squared	0.9834	20.6315	**σ_1 = process risk**
F-statistic	474.5146	8	df
ss-regression	201981.9941	3405.2816	ss residual

- Question is: How good is that fit for the projection of accident year 2005?

Mack and regression – an example

Error of fitted data understates prediction fit

- Leave-One-Out Cross-Validation
  - Exclude data point, fit regression to remaining points
  - Measure the prediction error of omitted point
  - Repeat



ROC Solid Reserves: Mack Method Wrap-up

- Mack's analytic method is as basic as measuring the standard deviation of a sample
- Mack's model is equivalent to regression
  - A useful analytical tool in a wide variety of applications
  - Subject to potentially understated prediction errors
- Actuaries may want to fine-tune their Mack implementations to incorporate some basic statistical principals
  - Student-T, Log-T
  - Utilize other distributions
  - Borrow statistical methodologies from other sciences (e.g., cross-validation)
- Caution against extrapolating beyond the data