CAS 2007 Annual Meeting

What Color is Your Copula: The Language of Uncertainty

Terminology Surrounding Loss Reserve Variability/Ranges

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Why analyze the variability of claim liabilities?

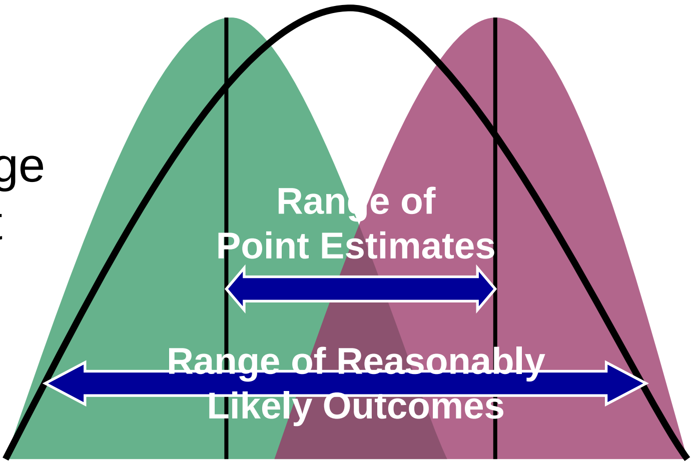
ASOP 43 effective Sep. 1, 2007	■ "The actuary should consider the uncertainty associated with the unpaid claim estimate." ■ Actuary not required to, nor prohibited from, measuring uncertainty ■ "The actuary should consider the types and sources of uncertainty." ■ "May include model risk, parameter risk, and process risk."
NAIC	■ Actuarial opinions are produced on a "reasonableness" standard ■ Variation from the "best estimate" is the issue ■ Actuarial Opinion Summary (AOS) includes focus on ranges
SEC	■ Require public companies to discuss reserve uncertainty in 10-K filings ■ Increasing pressure…hand-waving rationale will soon be inadequate
Rating Agencies	■ Capital adequacy analyses usually assume reserve shortfalls ■ Management is expected to consider more than just the best estimate
Fiduciary Duty	■ It is prudent business practice to recognize your business risks to the best of your ability ■ Range analysis can provide strategic operational and financial insights

ASOP 43 includes various definitions of "estimate"

- ***Unpaid Claim Estimate*** – "The actuary's estimate of the obligation for future payment resulting from claims due to past events"
- ***Scope of the Unpaid Claim Estimate*** should identify its *intended measure*, examples of which include
  - Mean, median, mode, or specific percentile
  - High estimate, low estimate
  - **Actuarial Central Estimate** – "An estimate that represents an expected value of the range of reasonably possible outcomes."
    - May not include all conceivable outcomes, e.g., "extreme events where the contribution of such events to an expected value is not reliably estimable."
    - May or may not be the result of a probabilistic/statistical analysis
- ASOP 43 deems the terms *best estimate* and *actuarial estimate* as insufficient identifiers of the unpaid claim estimate's intended measure

What is a probabilistic/statistical “estimate”?

- A probabilistic *point estimate* of the ultimate value of future claim obligations is a prediction of the mean (or another “central tendency”: median, mode) of that random variable from a given algorithm
- The *prediction error* of that point estimate can arise from three basic sources
  - Your estimate will differ from the average value of all the potential estimates that your algorithm might produce (“parameter risk”)
  - The average value of all the potential estimates from your algorithm might not coincide with the “true mean” of the random variable being estimated (“model risk” or “bias”)
  - Ultimate future obligations will (probably) differ from their own true mean (“process risk”)



A “real life” analogy

- Rather than estimating the value of future payments going out of the claims department, suppose we want to estimate the IQ of the next person to walk in the back door
- If we knew the average IQ of all the Annual Meeting registrants, that would probably be our guess
- We can estimate that average by measuring the IQ of everyone in Room ___X___
- When the next person, Joe, walks in, by what amount could our guess be wrong?
  - The average IQ in the room could be different from the true Annual Meeting mean IQ
  - Joe’s IQ could be different from the true mean

Several distinct types of risks are inherent in the estimation of claim liabilities

Total Risk

<u>Estimation error</u>
Difference between estimated mean and true mean

<u>Parameter Risk</u>
Variability due to fact that our model's parameters are estimates

Our room's average IQ today could be different from the average of all such samples from Room __X__

<u>Model Risk</u>
Variability due to fact that our model imperfectly represents reality

Room X's population may not be representative of the entire meeting population

<u>Process Risk</u>
Difference between actual costs and true mean

The IQ of Joe walking through the door will differ from the true Annual Meeting mean IQ

Estimate of Expected Outcome

True Mean Outcome

Eventual Outcome

Total Risk, aka Mean Square Error, is the statistical equivalent of the Pythagorean Theorem

Estimate of Accident Year i's Outstanding Liability given today's information

$\hat{C}_i$

Liability's True Mean

bias

$E(C_i) = E(\hat{C}_i)$

(if unbiased)

Parameter Risk

"Explained Deviation"

$$Var(\hat{C}_i)$$

Process Risk

"Unexplained Deviation"

$$Var(C_i)$$

C_i

Accident Year i's Eventual Outstanding Liability

Total Risk

aka, Mean Square Error (mse)

$$mse(\hat{C}_i) \equiv E(\hat{C}_i - C_i)^2$$

$$= Var(\hat{C}_i) + Var(C_i)$$

excluding bias

One last page of introductory comments

- Sometimes the term “risk” refers to variance and sometimes to standard deviation, depending on the context
- Most of the time, mse and “total variance” are used interchangeably
  - When the estimate is unbiased, mse and total variance are identical
- Bias occurs when the expected value of your estimate does not coincide with the true mean value of the target quantity for some systematic reason

$$\textbf{\textit{Estimation Error}} \equiv \boldsymbol{E}(\hat{\boldsymbol{C}}_i - \mu_{C_i})^2 = \textbf{\textit{Parameter Variance}} + \textbf{\textit{bias}}^2$$

- Coefficient of Variation, or CV
  - CV(X) = StdDev(X) ÷ Mean(X)
  - StdDev(Ultimate Loss) = StdDev(Outstanding Loss)
    because ultimate and outstanding only differ by a scalar (paid loss)
  - The CV’s of ultimate and outstanding loss will differ however because, although their numerators are the same, their denominators are different
  - Most of the popular stochastic methods derive StdDev(Ultimate Loss) directly because that’s what our actuarial methods usually do

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Mack Method: Overview

- Mack Method derives formulas for the standard error of the liability projected by the chain ladder method
- Tillinghast recommends using the recursive formulas from Murphy's 1994 paper "Unbiased Loss Development Factors"
- The formulas provide for process and parameter risk, separately and in total
- The method can be extended to incorporate age-to-age factors other than the volume weighted average
- Mack recommends fitting a normal or lognormal distribution to the mean and variance of the liability to yield a distribution of liabilities
- The variability of the tail beyond the triangle can be incorporated in various ways

Mack: Summary

Advantages	Disadvantages
■ Widely regarded in the industry ■ Founded in statistical theory ■ Works with chain-ladder eligible triangles ■ Can reflect tail variability	■ Data outliers can have a leveraged effect on the results ■ May over-parameterize the risk ■ A 10x10 triangle will estimate 9 link ratios from 36 observations ■ Prediction errors may be understated because it is essentially a regression method

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Bootstrap Method: Overview

- Bootstrapping is a simulation technique that generates empirical probability distributions of complex functions
- A triangle of cumulative fitted values for the past triangle is obtained by backwards recursion on the most recent diagonal using standard chain ladder link ratios
- A set of Pearson residuals is calculated from the fitted and actual data
- Bootstrapping utilizes the sampling-with-replacement technique on the residuals of the historical data
- Each simulated sampling scenario produces a new “realization” of triangular data that has the same statistical characteristics as the actual data
- Our model calculates both parameter and total risk
- Our bootstrapping implementation can calculate tail volatility by employing curve fitting to each realization of average loss development factors
- Our Bootstrapping implementation includes a B-F option for the new data “realization”
- Outlier observations can be restricted
- The sampling of residuals can be restricted for the first development period

Bootstrapping: Summary

Advantages	Disadvantages
■ Easy to understand and explain ■ Commonly used in industry ■ Accommodates BF method ■ Facilitates the calculation of tail volatility	■ Data outliers can have a leveraged effect on the results ■ Method does not work well with negative loss development (due to underlying theoretical model)

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Practical Method: Basic Theory and assumptions

- The Practical Method uses Monte Carlo simulation to estimate liability distributions based on the three most popular deterministic methods – Chain Ladder, Loss Ratio, and Bornhuetter-Ferguson
- Practical simulates age-to-age (ATA) factors and loss ratios as normal or lognormal random variables
  - Means and variances of those distributions are selected inputs
  - For BF method, LDFs can be “fixed” based on the ATA means, or “variable” based on the ATA simulations
  - The variability of the tail factor is a manual entry or, if left blank, can be modelled by assuming that the standard error of the last age-to-age factor is repeated for as many years as the user selects
- Explicitly reflects process risk only
  - Parameter can be incorporated with some additional analysis

Practical: Summary

Advantages	Disadvantages
■ Easy to understand and explain ■ Accommodates the three most popular actuarial deterministic methods ■ Can incorporate tail variability	■ Not as well known in the actuarial community ■ Does not explicitly measure parameter risk

Flexibility allows user to obtain wide range of results.

Relevant sources of variability depend on the question at hand

- When is it important to analyze total risk?
  - Financial solvency/economic capital context
  - When solvency is the issue ➡ all sources of risk are relevant
- When is it important to analyze only parameter risk?
  - “Reasonable range of estimates”
  - When the reasonability of the estimate is the issue ➡ only estimation error (= parameter risk in the absence of bias) is relevant
- Ability to estimate those separate sources of risk varies by stochastic method
  - Mack ➡ both parameter and process risk estimation are built-in
  - Bootstrap ➡ measures parameter risk; process risk measurement can be an “overlay”
  - Monte Carlo methods ➡ e.g., the “process” of loss development factors is simulated around a selected factor; the variability of the factor – the parameter – requires additional analysis

Tail variability

- Many of the popular stochastic methods only measure risk to the edge of the triangle
- Variability for development beyond the triangle – so called “tail variability” – must be measured and incorporated separately
- Mack
  - Heuristic approach to tail variability in his 1999 paper
- Bootstrap
  - England and Verrall (1998) only measure risk to the edge of the triangle
- Practical
  - Assume you incorporate a tail in your deterministic analysis
  - For a stochastic simulation you will need to have some idea of the variability of that tail factor

Example: Tail variability can be reflected with the Mack Method using the heuristic in his 1999 paper …

Mack Method **Taylor-Ashe Data**

Taylor, G., and F. Ashe, "Second Moments of Estimates of Outstanding Claims," *Journal of Econometrics*, **1983**, 23, pp. 37-61.

AY/DY	1	2	3	4	5	6	7	8	9	10	Ultimate
i/k	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=∞
i=1	357,848	1,124,788	1,735,330	2,218,270	2,745,596	3,319,994	3,466,336	3,606,286	3,833,515	3,901,463	4,291,609
i=2	352,118	1,236,139	2,170,033	3,353,322	3,799,067	4,120,063	4,647,867	4,914,039	5,339,085	5,433,719	5,977,091
i=3	290,507	1,292,306	2,218,525	3,235,179	3,985,995	4,132,918	4,628,910	4,909,315	5,285,148	5,378,826	5,916,709
i=4	310,608	1,418,858	2,195,047	3,757,447	4,029,929	4,381,982	4,588,268	4,835,458	5,205,637	5,297,906	5,827,696
i=5	443,160	1,136,350	2,128,333	2,897,821	3,402,672	3,873,311	4,207,459	4,434,133	4,773,589	4,858,200	5,344,020
i=6	396,132	1,333,217	2,180,715	2,985,752	3,691,712	4,074,999	4,426,546	4,665,023	5,022,155	5,111,171	5,622,289
i=7	440,832	1,288,463	2,419,861	3,483,130	4,088,678	4,513,179	4,902,528	5,166,649	5,562,182	5,660,771	6,226,848
i=8	359,480	1,421,128	2,864,498	4,174,756	4,900,545	5,409,337	5,875,997	6,192,562	6,666,635	6,784,799	7,463,279
i=9	376,686	1,363,294	2,382,128	3,471,744	4,075,313	4,498,426	4,886,502	5,149,760	5,544,000	5,642,266	6,206,493
i=10	344,014	1,200,818	2,098,228	3,057,984	3,589,620	3,962,307	4,304,132	4,536,015	4,883,270	4,969,825	5,466,807
sum below diagonal	0	1,200,818	4,480,356	10,704,484	16,654,155	22,458,247	28,603,165	34,979,600	42,942,618	49,137,483	58,342,840
										selected	
f_i = ATAs	3.491	1.747	1.457	1.174	1.104	1.086	1.054	1.077	1.018	1.100	
CDFs	15.891	4.553	2.605	1.788	1.523	1.380	1.270	1.205	1.119	1.100	
										heuristic	
σ_i^2	160,280	37,737	41,965	15,183	13,731	8,186	447	1,147	447	13,731	
σ_f^2	0.04817	0.00368	0.00279	0.00082	0.00076	0.00051	0.00004	0.00013	0.00012	0.00076	

… Variance estimates are completed for future development periods using the Murphy formulas …

Mack Method **Taylor-Ashe Data**

Total Variance of Chain Ladder Projection

i/k	k=1	k=2	k=3	k=4	k=5	k=6	k=7	k=8	k=9	k=10	k=∞	Standard Error
i=1											6.52E+10	255,358
i=2										5.71E+09	1.04E+11	322,627
i=3									8.88E+09	1.48E+10	1.14E+11	337,491
i=4								2.79E+09	1.19E+10	1.78E+10	1.16E+11	340,279
i=5							3.94E+10	4.63E+10	6.14E+10	6.83E+10	1.67E+11	409,208
i=6						6.11E+10	1.14E+11	1.29E+11	1.58E+11	1.69E+11	2.95E+11	542,772
i=7					6.29E+10	1.46E+11	2.19E+11	2.46E+11	2.95E+11	3.12E+11	4.80E+11	692,462
i=8				1.43E+11	2.75E+11	4.21E+11	5.56E+11	6.21E+11	7.32E+11	7.66E+11	1.06E+12	1,027,513
i=9			5.83E+10	2.40E+11	3.93E+11	5.47E+11	6.93E+11	7.73E+11	9.05E+11	9.44E+11	1.24E+12	1,115,196
i=10		6.08E+10	2.36E+11	6.03E+11	8.84E+11	1.14E+12	1.38E+12	1.54E+12	1.79E+12	1.86E+12	2.34E+12	1,528,545
Sum		6.08E+10	3.07E+11	1.11E+12	1.91E+12	2.92E+12	4.02E+12	4.52E+12	5.50E+12	5.99E+12	1.03E+13	3,202,424
Standard Error		246,656	553,840	1,053,717	1,382,465	1,710,220	2,005,746	2,126,418	2,345,361	2,447,618	3,202,424	

… The 10% tail increased total CV by only 1.5%

Mack Method		Taylor-Ashe Data						
			Standard Errors w/ Murphy Recursive Formulas			CV		
AY	Est'd Ultimate	Liability	Process risk	Parameter risk	Total risk	Process risk	Parameter risk	Total risk
1	4,291,609	390,146	231,457	107,868	255,358	59.3%	27.6%	65.5%
2	5,977,091	638,006	278,384	163,066	322,627	43.6%	25.6%	50.6%
3	5,916,709	1,007,394	289,437	173,570	337,491	28.7%	17.2%	33.5%
4	5,827,696	1,239,428	292,387	174,067	340,279	23.6%	14.0%	27.5%
5	5,344,020	1,470,709	359,922	194,697	409,208	24.5%	13.2%	27.8%
6	5,622,289	1,930,577	482,479	248,627	542,772	25.0%	12.9%	28.1%
7	6,226,848	2,743,718	616,826	314,688	692,462	22.5%	11.5%	25.2%
8	7,463,279	4,598,781	916,626	464,305	1,027,513	19.9%	10.1%	22.3%
9	6,206,493	4,843,199	1,023,695	442,392	1,115,196	21.1%	9.1%	23.0%
10	5,466,807	5,122,793	1,437,309	520,187	1,528,545	28.1%	10.2%	29.8%
Total:	58,342,840	23,594,604	2,235,431	2,293,113	3,202,424	9.5%	9.7%	13.6%
With no tail								
Total:	53,038,946	18,290,709	1,878,292	1,569,349	2,447,618	10.3%	8.6%	13.4%
					% increase in CV with tail variability			1.4%

Aggregation: combining lines

- Means aggregate without much fuss:
  - $E(X+Y) = E(X) + E(Y)$
  - I.e., to get the aggregate mean, just aggregate the marginals
- Variances aggregate without much fuss when the lines are independent (more precisely, uncorrelated)
  - $Var(X+Y) = Var(X) + Var(Y)$
  - I.e., to get aggregate variance, just aggregate the marginals, but only when the lines are uncorrelated
  - When the lines are correlated, there is an extra covariance term
    - $Var(X+Y) = Var(X) + 2Cov(X,Y) + Var(Y)$
    - Covariance is to the formula for the variance of the sum of two random variables as the cross product term is to the square of a binomial
- Entire distributions aggregate without much fuss when the random variable pairs are joint normally distributed
  - Otherwise, more advanced techniques are required

Aggregation continued: correlation

- Correlation scales the covariance of two lines by dividing by their standard deviations

$$\text{Correlation Coefficient}: \quad \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

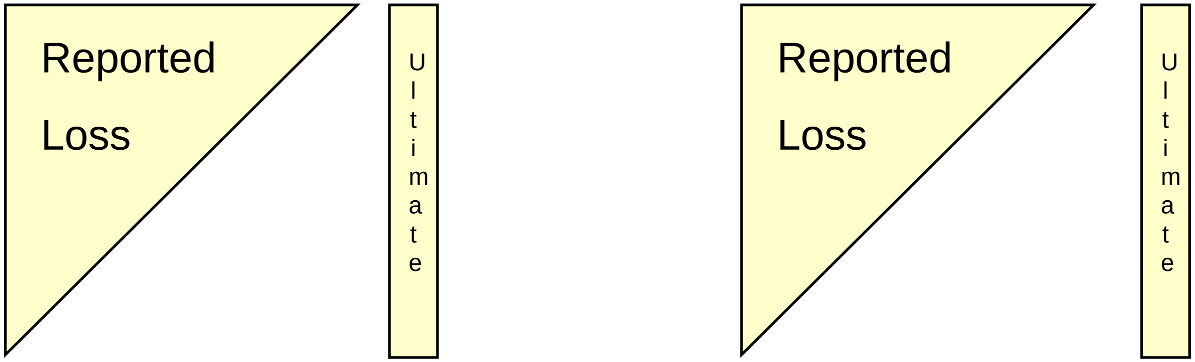
  - Correlation is "standardized" covariance
  - Allows comparison of two lines of difference sizes

- Such relationships between N lines of business are encapsulated in the covariance matrix and the correlation matrix

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_N) \\ Cov(X_2,X_1) & Var(X_2) & \cdots & Cov(X_2,X_N) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_N,X_1) & Cov(X_N,X_2) & \cdots & Var(X_N) \end{bmatrix} \quad Corr = \begin{bmatrix} 1 & corr(X_1,X_2) & \cdots & corr(X_1,X_N) \\ corr(X_2,X_1) & 1 & \cdots & corr(X_2,X_N) \\ \vdots & \vdots & \ddots & \vdots \\ corr(X_N,X_1) & corr(X_N,X_2) & \cdots & 1 \end{bmatrix}$$

  - If you don't make a mistake in building these matrices, they are always positive-semidefinite (you can take their "square root," as standard deviation is the square root of variance)
  - Can be inverted only if positive-definite (cannot be "zero")

Correlation between two quantities measures the degree to which deviations from the mean move – or don't move – in conjunction with each other



Auto BI

GL

$$C_{ABI}(i,Ult) - \overline{C}_{ABI}(i,Ult) \qquad C_{GL}(i,Ult) - \overline{C}_{GL}(i,Ult)$$

- Given pairwise estimates of ultimates from two lines for I accident years, the strength to which the estimated ultimates "co-vary" can be measured by the sample correlation coefficient

$$\frac{1}{I}\sum_{i=1}^{I}\left(C_{ABI}(i,Ult)-\overline{C}_{ABI}(i,Ult)\right)\left(C_{GL}(i,Ult)-\overline{C}_{GL}(i,Ult)\right) \Big/ s_{ABI}\,s_{GL}$$

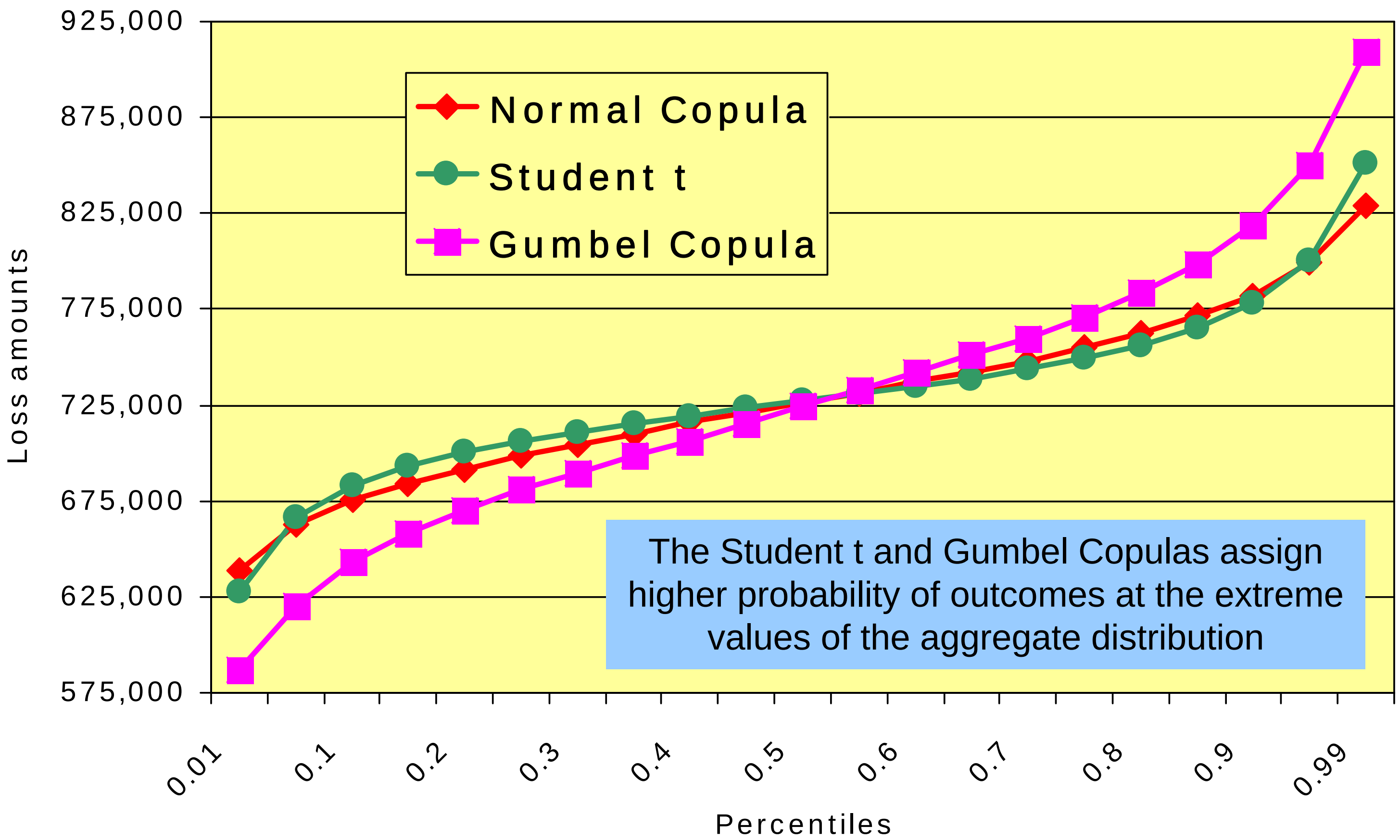
Aggregation continued: combining entire distributions

- Correlation measures the <u>average</u> strength of the relationship between lines over the entire distribution
- When is the correlation coefficient not enough?
  - When the strength of the relationship between two lines changes in different parts of their distributions
  - Example: Correlation between property lines might be higher in the tails of their distributions, which could be important to an actuary parameterizing a CAT contract
- Ideally, a company writing N lines of business one would like the complete joint distribution of all N lines
- It turns out that every joint distribution of N lines of business can be decomposed into N marginal distributions by virtue of an amalgamating function called a "copula"
- Vice versa, given the marginal distributions of N lines of business, the joint distribution can be calculated with the help of an appropriate copula

Copulas provide a convenient way to express the aggregate distribution of several lines

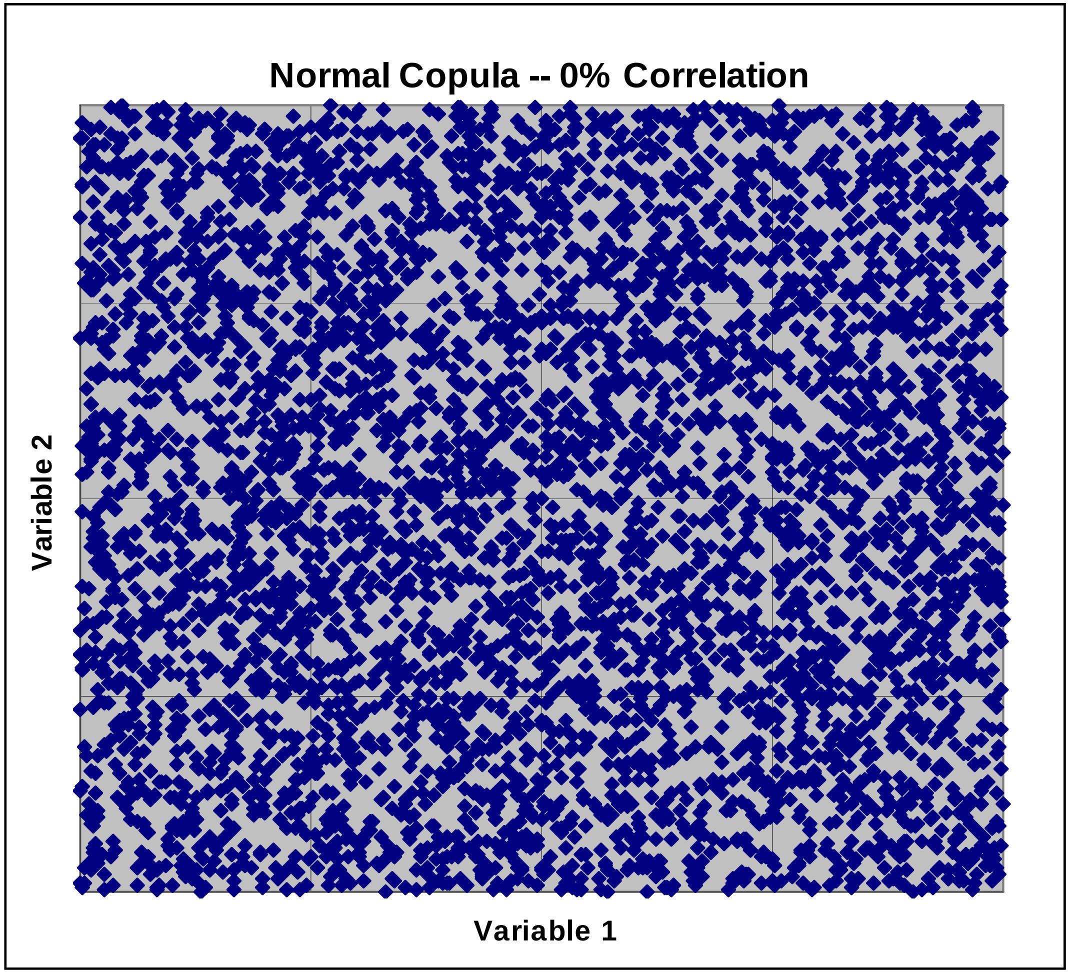
- Three popular copulas in actuarial use today are
  - The Normal copula
  - The Student-t copula
  - The Gumbel copula
- Copula required components (with the exception of Gumbel):
  - The marginal distributions of the individual lines
  - Correlations among these lines
- The Gumbel copula is different from the Normal and Student-t
  - It does not need a complete correlation matrix
  - Association is expressed by a single parameter applying to all lines
  - Upper tail dependence is strong while lower tail dependence always equals 0

The choice of the appropriate copula is a matter of judgment

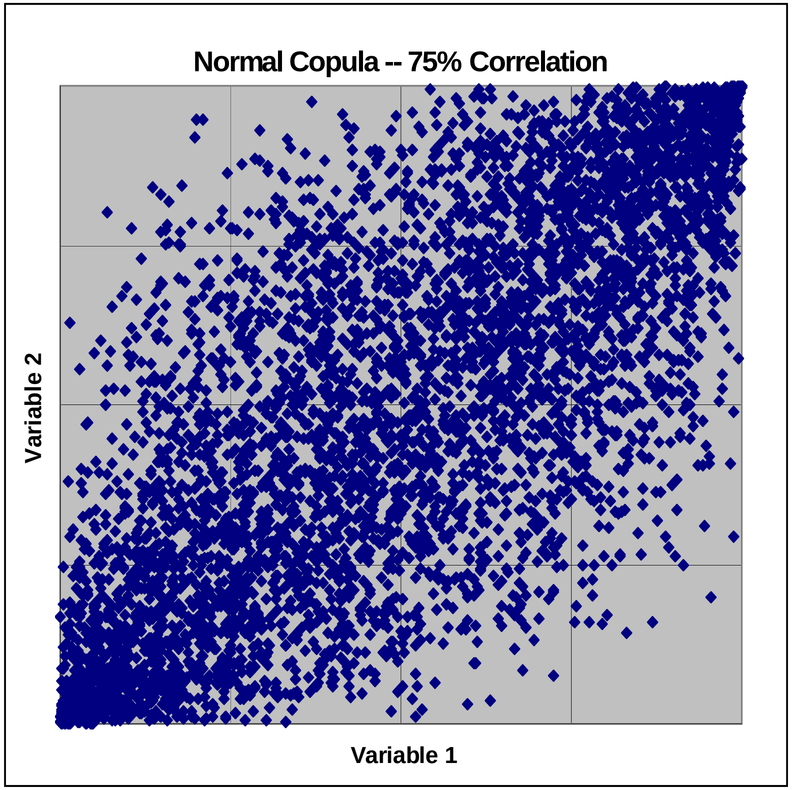


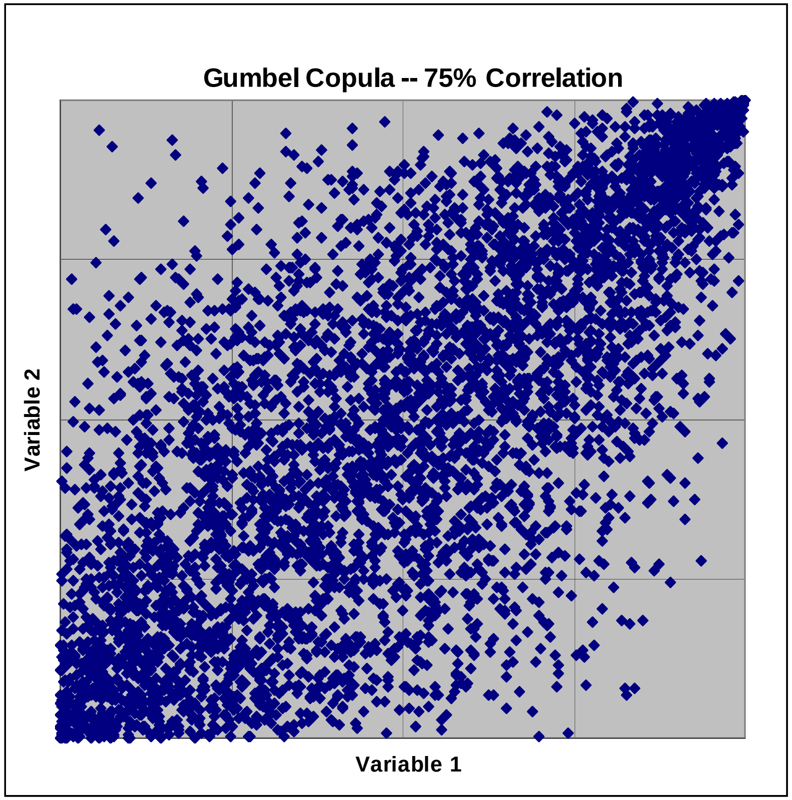
- The portfolio of liabilities can be stress-tested under varying copula assumptions

With independent variables results are not correlated

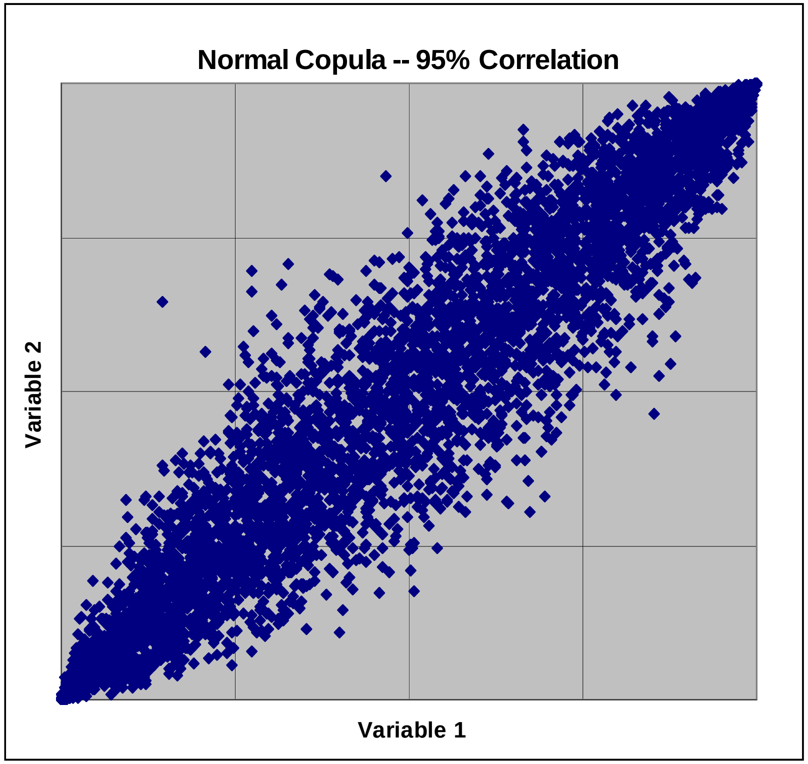


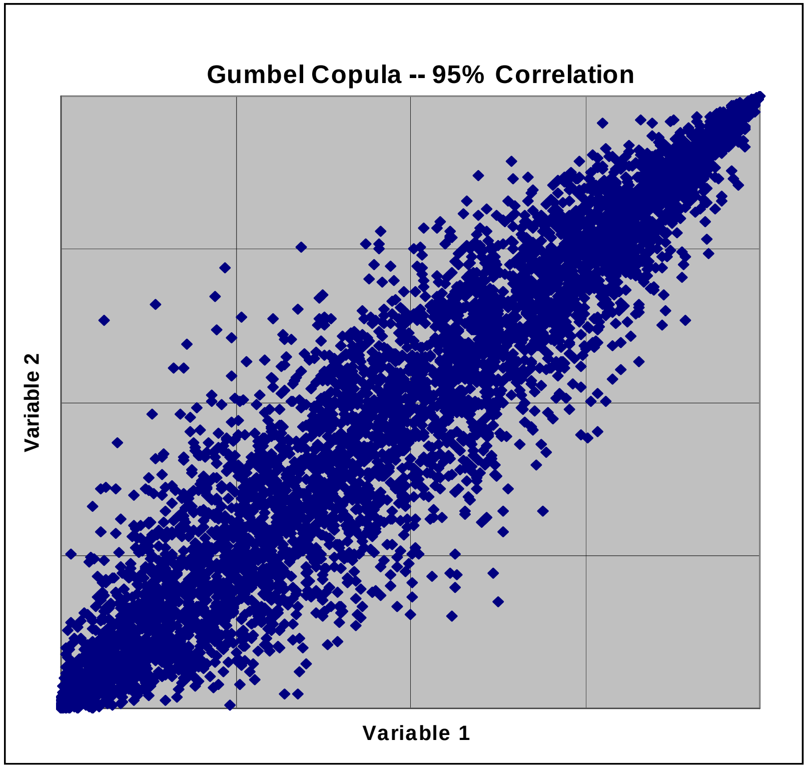
75% correlation: bad results in one line make it more likely to have bad results in the second line





The relationship is even more pronounced with 95% correlation





Questions?

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