Introduction to Reserve Range Theory and Practical Model Application

CAS Annual Meeting

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Agenda

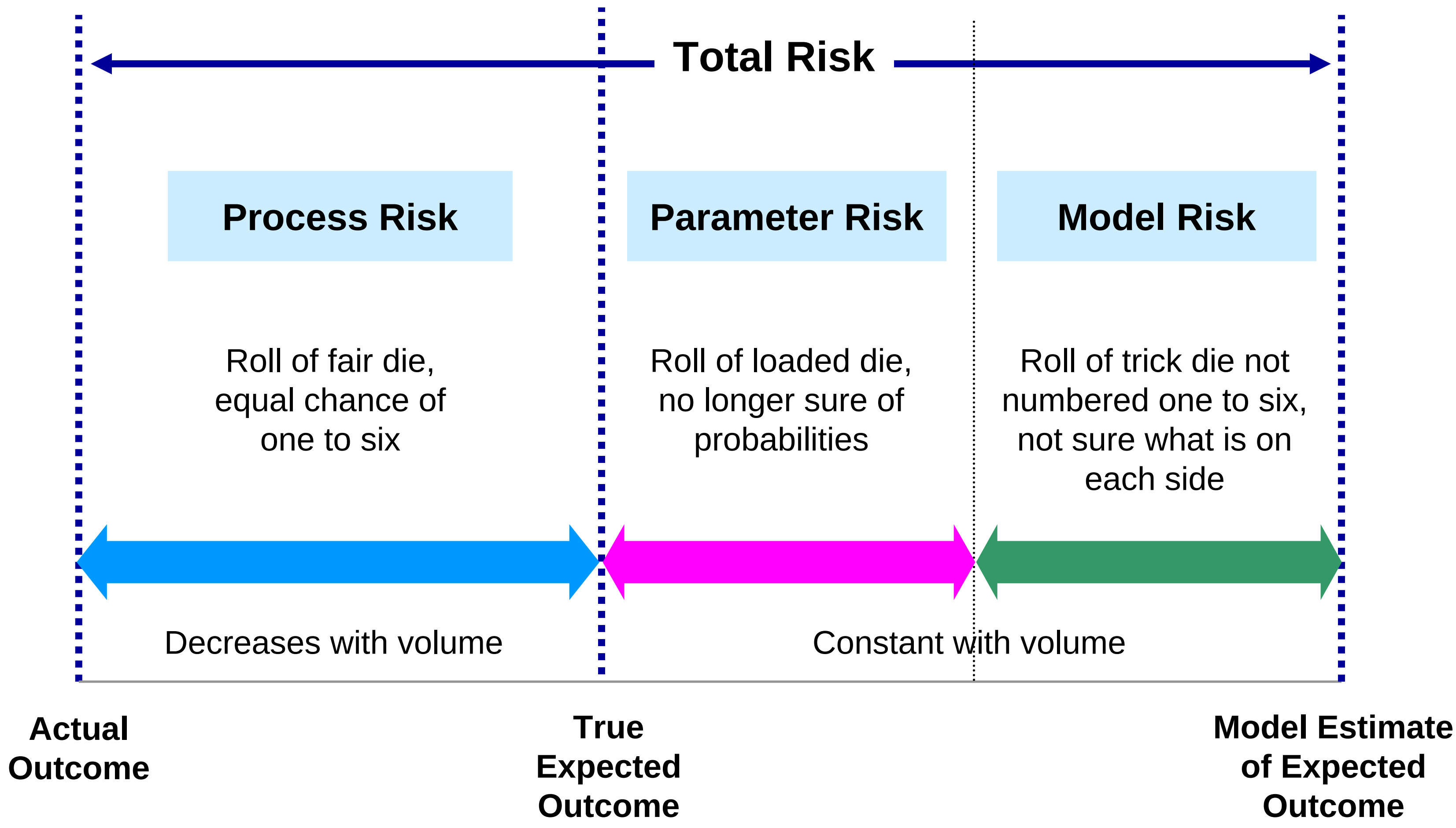
- Terminology
- Popular stochastic techniques
  - Focus on Mack and Bootstrapping
- Aggregation of liabilities

Terminology

Method vs. Model

Method	Model
■ Mathematical algorithm for estimating unpaid amounts ■ Parameters are selected ■ Selections assumed appropriate based on judgment ■ Chain Ladder algorithm	■ Mathematical description of the world ■ "Best-Fitted" Parameters ■ Selections can be tested ■ Mack, Bootstrapping models

Several distinct types of risks are inherent in the measurement of claim liabilities — the actuary and the audience need to be clear about which are relevant to a particular application



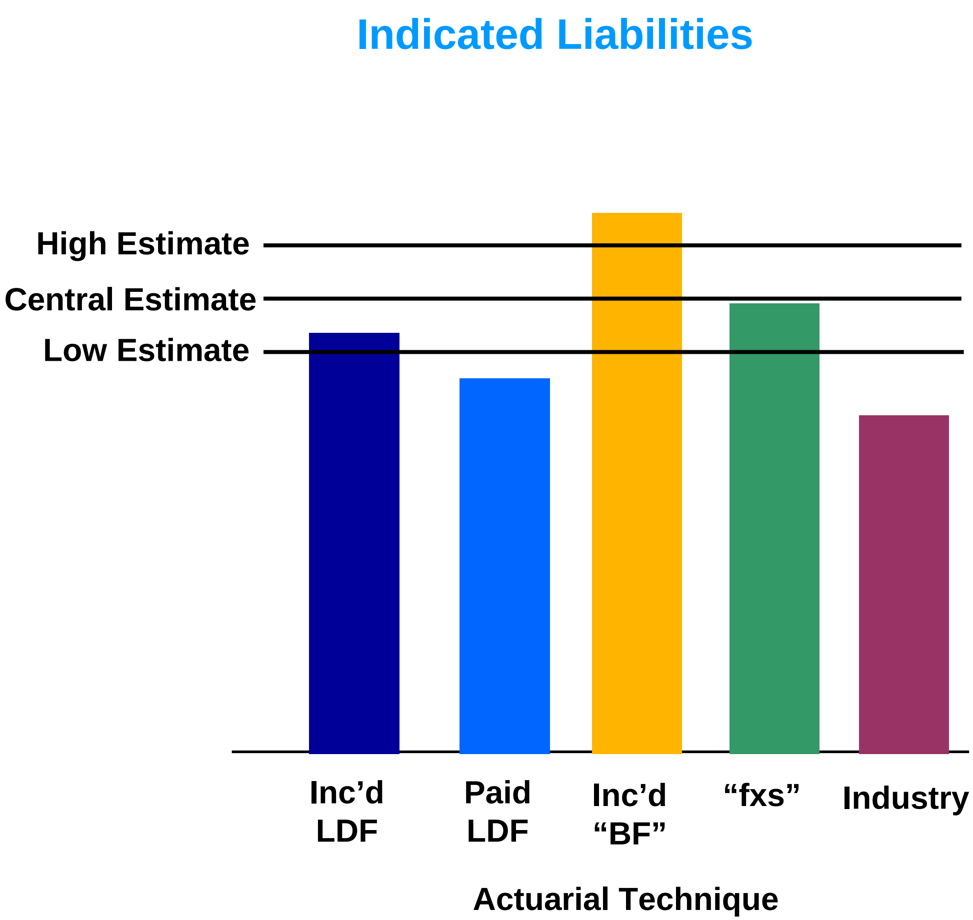
Several distinct types of risks are inherent in the measurement of claim liabilities — the relevant risks depend on the intended use of the analysis

- Financial Solvency/Capital adequacy context
  - “Stress testing” the balance sheet
  - Variation of actual outcome around the true expected outcome
  - All types of risk are relevant here
- Reserve variability context
  - Comparing two actuarial estimates
    - — Variation around the true expected outcome
    - — Parameter and model risk are relevant here

What “risk” do stochastic methods measure?

- Risk could mean different things to different audiences
- Actuaries usually think of risk in terms of “variance” and “standard deviation”
  - “coefficient of variation” (CV) is “scaled” by the mean and measures “relative” risk
- Other definitions
  - (VAR) - Value at Risk: a percentile (i.e. losses at the 75^{th})
  - (TVar) – Tail value at Risk: expected losses in excess of a given percentile

Deterministic: What range of estimates is implied by the actuarial techniques used?



General Approach — Deterministic

- Estimate range of claim liabilities based on the results of several projections
- Applied to current data evaluation

Advantages

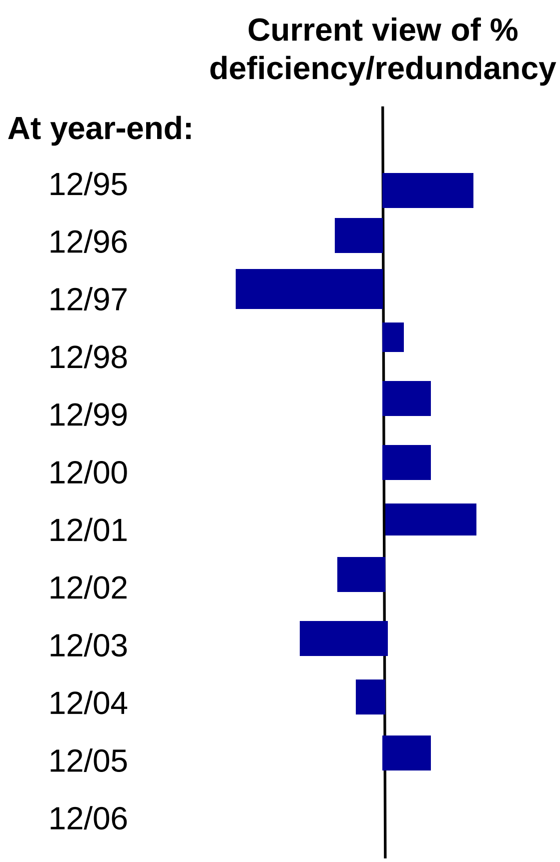
- Easy to understand and apply
- Based on liability estimates of traditional actuarial methods
- No extra work needed

Disadvantages

- Does not include process risk
- Does not separate model and parameter risk
- Does not produce confidence interval estimates
- Highly judgmental
- Simplistic

Performance Test: How accurate have the past estimates proven to be?

Actuarial Scorecard for Method X



General Approach — Hindcast Test

- Retrospective test of a consistently applied methodology
- Uses current view of claim liabilities versus historical estimates
- Quantifies the degree of departure that has occurred around the results that would have been indicated by that methodology

Advantages

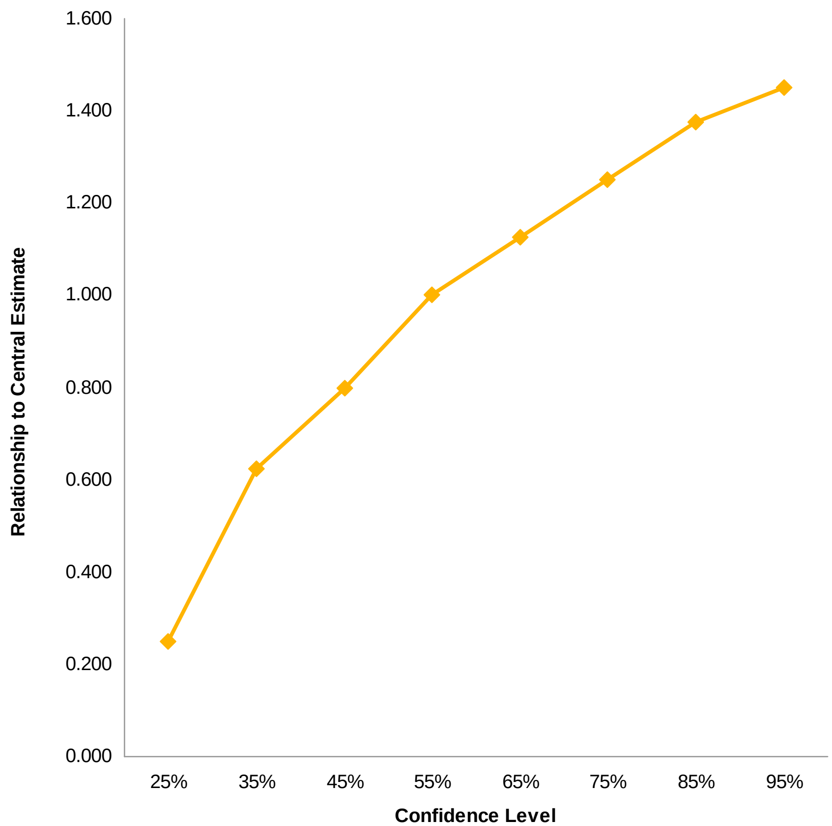
- Easy to understand and apply
- Few assumptions needed for each model being tested
- Should do this test anyway in arriving at central estimate

Disadvantages

- Does not separate model, parameter and process risk
- Does not produce confidence interval estimates
- The actual “model” used is likely a combination of methods

Stochastic: What claim liability outcomes are reasonably likely?

Indicated Unpaid Claim Liabilities as of December 31, 2008



General Approach — Stochastic Methods

- Estimate probability distribution
- Based on statistical methods
- Applied to historical development data

Advantages

- Produces estimates of confidence intervals
- Can approximately separate parameter and process risk
- More complete description of loss generating process
- Feeds other analyses (ERM)

Disadvantages

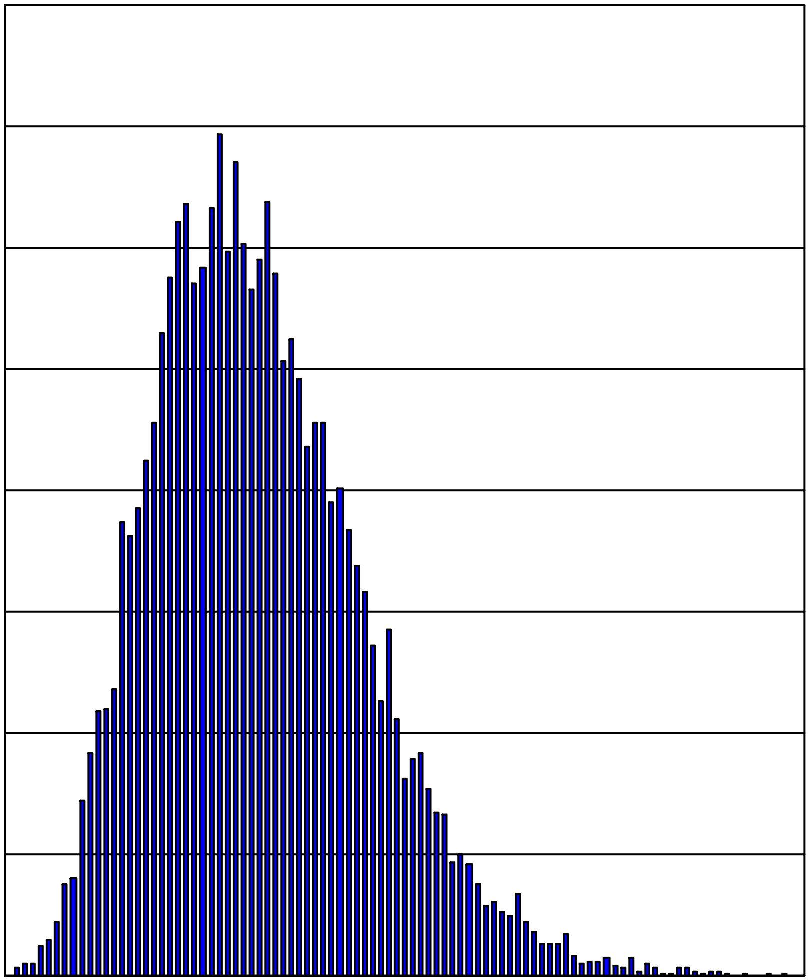
- Involves relatively complex statistical analysis
- An emerging practice within P/C actuarial field
- Lack of general agreement among actuaries on the best approach
- Some exposures not amenable to this approach (A&E)

Popular Stochastic Methods

Simulation approach: Monte-Carlo

- Simulation techniques help model the complex loss generating process
- Simulation methods assume that the simulated data has the same statistical characteristics as the actual data
- Simulation works as follows:
  - Start with a deterministic method that generates ultimate loss outcomes (i.e. chain ladder)
  - Makes assumptions about the method parameters
    - i.e. the mean and variance of the link ratios
      - Parameter risk needs to handled separately
    - Randomly generate input values
    - Calculate and save ultimate outcomes
    - Repeat many times

Output simulated distribution



- The simulated "empirical" distribution provides an estimate of the "theoretical" claim liabilities distribution
- A number of statistics are produced (i.e. mean, variance, skewness, etc.)
- Simulated distribution "smooths" with a larger number of simulations

Monte Carlo simulations: Pros and Cons

Pros	Cons
■ Popular method in many sciences ■ Produces an empirical distribution of the reserves ■ Method can be applied to incomplete data triangles (i.e. trapezoids) ■ It explicitly calculates tail volatility	■ Data outliers can have a leveraged effect on the results ■ Needs additional complexity to measure parameter risk

Bootstrapping is a “second generation” simulation technique

- Monte Carlo techniques simulate the parameter inputs of a method
- Bootstrapping simulates the actual data employed by these methods
  - If the distribution of the data is known, then we sample from that distribution
    - — Parameters are estimated
    - — This is called **Parametric Bootstrapping**
  - If we do not know the distribution of the data, then we simulate from the actual data
    - — This is called **Nonparametric Bootstrapping**
    - — The process “resample” the residuals with “replacement”
- References
  - “Analytic and bootstrap estimates of prediction errors in claims reserving”

  by England & Verall, “Insurance Mathematics and Economics” (1999) pg. 281-293
  - “”Addendum to Analytic and Bootstrap Estimates of Prediction Errors in Claims Reserving”:

  by England, “City University London, School of Mathematics” (2001)
  - “Bootstrap methodology in claim reserving”

  by Pinheiro, Andrade e Silva & de Lourdes Centeno, “Journal of Risk and Insurance” (2003) pg. 701-714

- Based on a generalized linear "over-dispersed Poisson" (GLM) model where C_{ij} are the incremental payments:
  - $E(C_{ij}) = m_{ij}$ and $Var(C_{ij}) = \phi * m_{ij}$

- C_{ij} have a probability density function f:
- $C_{ij} = f(c; m_{ij}, \phi)$:
  - f(.) belongs to the exponential family;
  - The mean is linked to the linear predictor through a logarithmic link function:
  - $\log(m_{ij}) = n_{ij}$, where:
  - $n_{ij} = c + a_i + b_j$; with $a_1=b_1=0$ to avoid overparameterization; and
  - ϕ is the "scale" parameter
- The model described above gives exactly the same reserve estimates as the deterministic chain ladder method
- The fitting process that calculates the optimal parameters of the model is complicated
- The bootstrapping technique replaces the analytical calculation of the parameter and process risk with a simulation approach

Step-by-step description of the Bootstrapping algorithm

Actual Cumulative Historical Losses:

Accident Year	Development Age 12	24	36	48	60
1	1,000	1,500	1,800	2,000	2,100
2	1,200	1,900	2,150	2,300	
3	1,700	2,400	3,000		
4	2,000	2,900			
5	2,100				

	12-24	24-36	36-48	48-60	60-Ult
Selected RTRs	1.475	1.198	1.089	1.050	1.000

Expected Cumulative Paid Losses:

Accident Year	Development Age 12	24	36	48	60
1	1,040	1,533	1,837	2,000	2,100
2	1,196	1,763	2,113	2,300	
3	1,698	2,504	3,000		
4	1,967	2,900			
5	2,100				

For Example:
2,504 = 3,000 / 1.198

- Keep current diagonal intact
- Employ selected RTR factors to calculate expected cumulative payments

The Bootstrapping technique calculates residuals based on incremental losses

Actual Incremental Historical Losses:

Accident Year	Development Age 12	24	36	48	60
1	1,000	500	300	200	100
2	1,200	700	250	150	
3	1,700	700	600		
4	2,000	900			
5	2,100				

Expected Incremental Paid Losses:

Accident Year	Development Age 12	24	36	48	60
1	1,040	493	304	163	100
2	1,196	567	350	187	
3	1,698	806	496		
4	1,967	933			
5	2,100				

"Unscaled" Pearson Residuals: $r_{P_{ij}}$

Accident Year	Development Age 12	24	36	48	60
1	-1.233	0.295	-0.229	2.916	0.000
2	0.124	5.564	-5.327	-2.719	
3	0.052	-3.726	4.650		
4	0.752	-1.091			
5	0.000				

For Example:

$$-1.091 = \frac{900 - 933}{\sqrt{933}}$$

- The "unscaled" Pearson residuals are defined as:

$$r_{P_{ij}} = \frac{C_{ij} - m_{ij}}{\sqrt{m_{ij}}}$$

- The denominator represents the standard error of the incremental losses
- The Pearson residuals are "unscaled" in the sense they exclude ϕ which is needed only when considering the process error
- The (5,12) and (1,60) residuals will be zero. They could be excluded from the remainder of the analysis

The "unscaled" Pearson residuals need to be adjusted for the "degrees of freedom"

"Unscaled" Pearson Residuals: $r_{P_{ij}}$

Accident Year	Development Age 12	24	36	48	60
1	-1.233	0.295	-0.229	2.916	0.000
2	0.124	5.564	-5.327	-2.719	
3	0.052	-3.726	4.650		
4	0.752	-1.091			
5	0.000				

Degrees of Freedom adjustment factor: 1.581 $= \sqrt{\dfrac{15}{15 - 9}}$

"Adjusted" Pearson Residuals:

Accident Year	Development Age 12	24	36	48	60
1	-1.950	0.466	-0.363	4.611	0.000
2	0.195	8.797	-8.422	-4.300	
3	0.083	-5.891	7.352		
4	1.188	-1.725			
5	0.000				

For Example:
-1.725 = -1.091 x 1.581

- The "Unscaled" Pearson residuals need to be adjusted for the difference in the degrees of freedom between the analytical model and the bootstrapping technique
- The adjustment is equal to:

$$\sqrt{\frac{n}{n - p}}$$

Where n = 15 is the number of data points and

p = 9 are the parameters that need to be estimated

- In general, n-p represent the degrees of freedom of a model

Simulation of "pseudo" incremental loss data

"Adjusted" Pearson Residuals:

Accident Year	Development Age 12	24	36	48	60
1	-1.950	0.466	-0.363	4.611	0.000
2	0.195	8.797	-8.422	-4.300	
3	0.083	-5.891	7.352		
4	1.188	-1.725			
5	0.000				

Sampling with replacement of the Pearson Residuals: $r^*_{P_{ij}}$

Accident Year	Development Age 12	24	36	48	60
1	-5.891	7.352	-0.363	-0.363	4.611
2	8.797	8.797	-5.891	7.352	
3	4.611	-1.950	-4.300		
4	-8.422	1.188			
5	-8.422				

C^*_{ij}

"Pseudo" incremental loss data:

Accident Year	Development Age 12	24	36	48	60
1	850	657	298	158	146
2	1,500	777	239	288	
3	1,888	750	401		
4	1,593	970			
5	1,714				

For Example:

$$970 = 1.188 * \sqrt{933} + 933$$

- The resampling of the "Adjusted" Pearson residuals is based on the assumption that the residuals are independent and identically distributed
- The sampling with replacement could cause the sampled residuals to appear more than once
- The "pseudo" incremental loss data is created by solving the Pearson residual equation

$$C^*_{ij} = r^*_{P_{ij}} \sqrt{m_{ij}} + m_{ij}$$

Incorporation of process risk

Cumulative "pseudo" loss data and "squaring" of the triangle

Accident Year	12	24	36	48	60
	Development Age				
1	850	1,507	1,804	1,962	2,109
2	1,500	2,277	2,516	2,804	3,013
3	1,888	2,638	3,039	3,353	3,602
4	1,593	2,563	2,937	3,240	3,481
5	1,714	2,641	3,027	3,339	3,588

	12-24	24-36	36-48	48-60	60-Ult
Simulated RTRs	1.541	1.146	1.103	1.074	1.000

Incremental future loss data:

Accident Year	12	24	36	48	60
	Development Age				
1					
2					209
3				314	250
4			374	303	241
5		927	386	312	249

Simulate Incremental payments from a Gamma distribution with parameters α = mean / ϕ, and $\beta = \phi$

Accident Year	12	24	36	48	60	Estimated Reserves:
	Development Age					
1						0
2					299	299
3				229	335	564
4			349	464	225	1,038
5		822	300	214	129	1,466
					Total:	3,367

- The bootstrapping technique, up to now, has considered parameter risk only
- The scale parameter can be estimated as the Chi-square statistic divided by the degrees of freedom

$$\phi = \frac{\sum r_{ij}^2}{n - p} = 19.027$$

- We simulate from a Gamma distribution with the appropriate parameter's transformation. Advantages:
  - a) Simulate from the continuous Gamma distribution, and
  - b) avoid simulating values that are a multiple of ϕ from the overdispersed Poisson distribution

Standardized “unscaled” Pearson residuals - Theory

The "Unscaled" Pearson Residuals are not necessarily identically distributed

they can be "standardized" with the help of the Hat Matrix

For a regression model: Y = X*β + ε, we have:

$$\hat{y} = H y$$

H is the Hat matrix:

$$H = X(X^T W X)^{-1} X^T W$$

where:

X is the Design Matrix and W is the "Weight" Matrix

H has the good property that the Variance/Covariance matrix of the residuals is:

$$\hat{\sigma}_e^2 \sqrt{I - H}$$

where:

$\hat{\sigma}_e^2$ is an estimate of the of the true variance of the error term of the observation

As a result the "Unscaled" Pearson residuals are "standardized" by:

$$r_{P_{ij}}^{**} = \frac{r_{P_{ij}}}{\sqrt{1 - h_{ii}}}$$

where:

h_{ij} is the diagonal of the Hat matrix

the denominator represents the standard error of the residuals

Standardized "unscaled" Pearson residuals - Practice

"Unscaled" Pearson Residuals: $r_{P_{ij}}$

Accident Year	Development Age 12	24	36	48	60
1	-1.233	0.295	-0.229	2.916	0.000
2	0.124	5.564	-5.327	-2.719	
3	0.052	-3.726	4.650		
4	0.752	-1.091			
5	0.000				

Diagonal of the Hat Matrix - h_{ij}

Accident Year	Development Age 12	24	36	48	60
1	0.628	0.390	0.379	0.509	
2	0.644	0.412	0.413	0.573	
3	0.707	0.487	0.526		
4	0.785	0.548			
5					

"Standardized" Pearson Residuals: $r_{P_{ij}}^{**}$

Accident Year	Development Age 12	24	36	48	60
1	-2.021	0.378	-0.291	4.160	0.000
2	0.207	7.256	-6.952	-4.160	
3	0.097	-5.200	6.752		
4	1.623	-1.623			
5	0.000				

- The adjustment in the residuals is equal to:

$$r_{P_{ij}}^{**} = \frac{r_{P_{ij}}}{\sqrt{1 - h_{ii}}}$$

- It makes sure that the residuals have the same variance as the underlying random variable
- When the residuals are "standardized", they do not need to be "corrected" by the degrees of freedom global adjustment described above

Other Considerations

- The bootstrapping stochastic mean might be different from the deterministic mean
- Outlier have a leveraged effect on the results of the bootstrap simulations
  - Outlier residuals could be “tempered” based on inter-quartile distances
- Tail variability can be introduced through Sherman inverse power curves
- The bootstrapping model is “flexible” enough to incorporate B-F adjustments
- The simulations from the Gamma distribution can be problematic in excel

Bootstrapping: Pros and Cons

Pros	Cons
■ Actual data "guides" the simulation ■ No assumption needed for simulation of parameters ■ It is a "modern" simulation technique	■ Data outliers can have a leveraged effect on the results ■ Needs additional complexity to measure process risk ■ Residuals might need to be divided into similar resampling groups

Aggregation of Liabilities

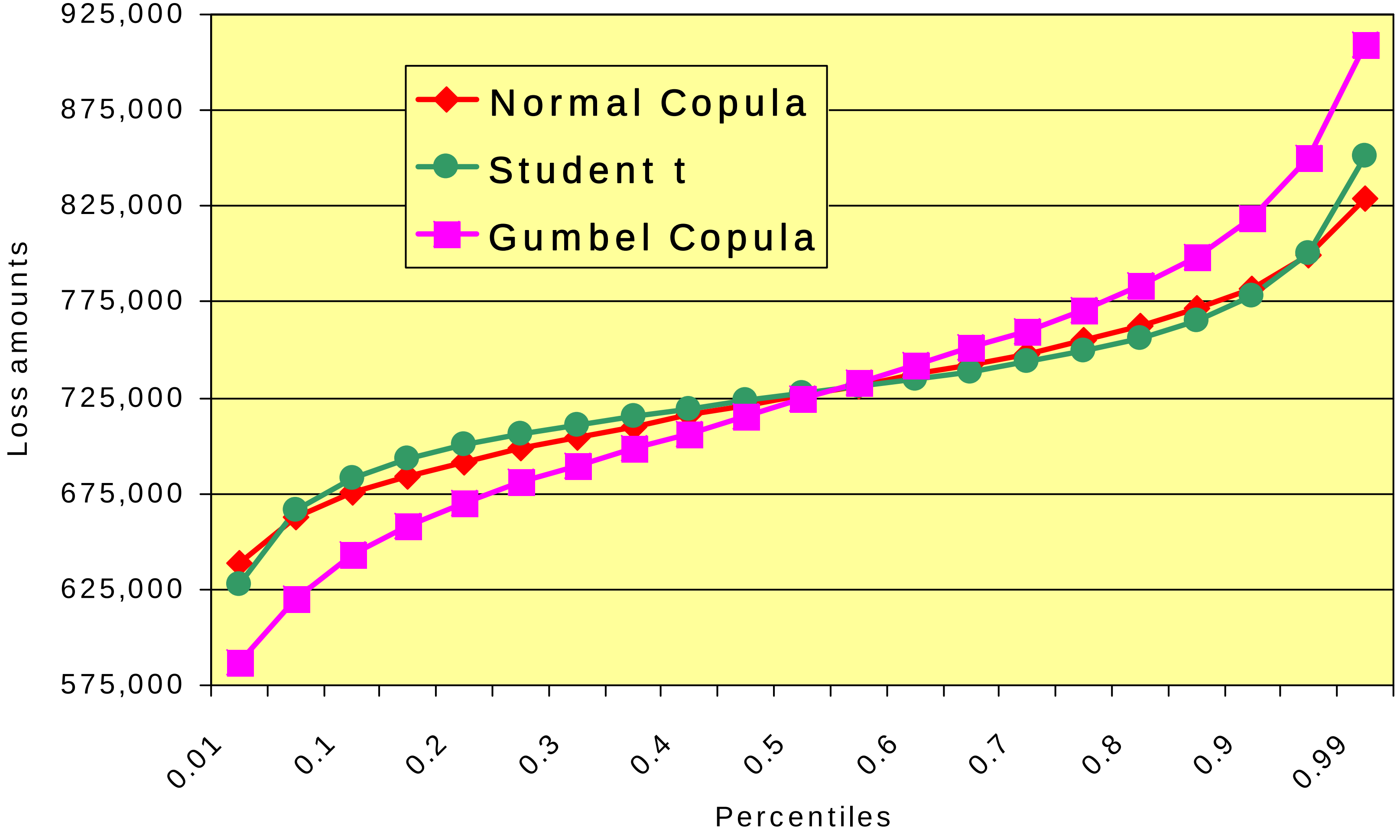
Aggregation: Correlation between Lines of Business

- Strength of the correlation is irrelevant if we only care about the mean reserve indication for two lines A and B:
  - mean(A + B) = mean(A) + mean(B)
- Strength of correlation matters when we look towards the ends of the aggregate distribution of (A+B)
- Generally, the aggregate distribution is less risky than the distribution of the individual lines:
  - 75^{th}percentile(A + B) < 75^{th}percentile(A) + 75^{th}percentile(B)
  - Equality only occurs in the case of perfect correlation across lines (this is very unlikely!)
- The volatility of the aggregate distribution increases:
  - By the volatility of the individual lines
  - By the correlation between the lines

Theory of Copulas

- Copulas provide a convenient way to express the aggregate distributions of several random variables
- Copula components:
  - The distributions of individual random variables
  - Correlations of these variables
- Correlation coefficients measure the overall strength of association across various distributions
- Copulas can vary that degree of association over the various parts of the aggregate distribution
  - Example: for workers comp and property losses the correlation is higher in the tail of the distribution

Comparison of Copulas



Questions?