Stochastic Loss Reserving with the Collective Risk Model

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ISO Innovative Analytics

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Outline of Presentation

- General Approach to Stochastic Modeling
  - Allows for better estimate of the mean
  - Quantify uncertainty in estimate
  - Predict distribution of “Statistics of Interest”
- The Paper - “Stochastic Loss Reserving with the Collective Risk Model”

Introduce Stochastic Modeling with an Example

- X ~ lognormal with $\mu = 5$ and $\sigma = 2$
- Two ways to estimate E[X] (= 1,097)
- Straight Average – $\hat{E}_N[X] = \frac{1}{n}\sum_{i=1}^{n} X_i$
- Lognormal Average – $\hat{E}_L[X] = e^{\hat{\mu}+\hat{\sigma}^2/2}$

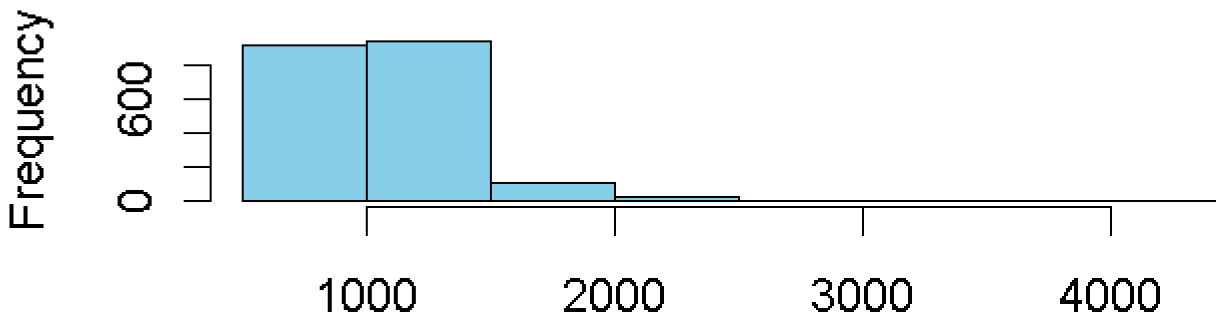
where $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\log(X_i)$, $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\log(X_i)-\hat{\mu}\right)^2}$

Which Estimator is Better? $E_N[X]$ or $E_L[X]$?

- Straight Average, $E_N[X]$, is simple.
- Lognormal Average, $E_L[X]$ is complicated.
  - But derived from the maximum likelihood estimator for the lognormal distribution
- Evaluate by a simulation
  - Sample size of 500
  - 2,000 samples
- Look at the variability of each estimator

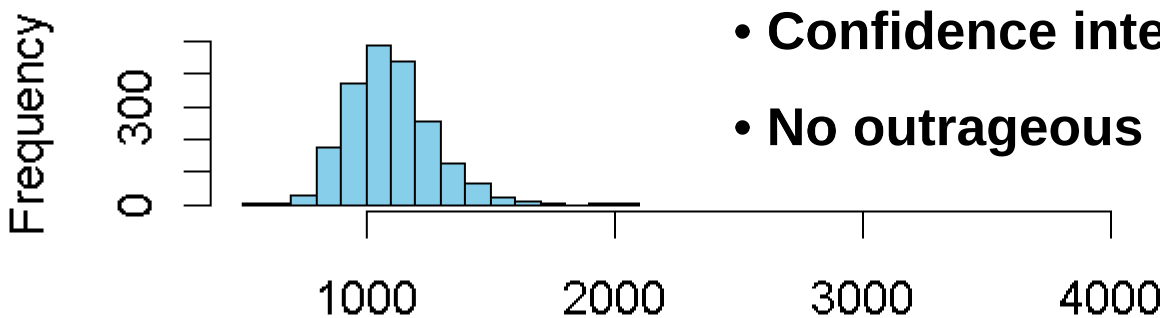
Results of Simulation

Straight Average



95% Confidence Interveal = (707.2 , 1902.9)
Maximum = 4267

Lognormal Average



- **Confidence interval is narrower**
- **No outrageous outliers**

95% Confidence Interveal = (817.1 , 1479.5)
Maximum = 2003.1

Lesson from Example

- ***Knowing the distribution of the observations can lead to a better estimate of the mean!***
- Actuaries have long recognized this.
  - Longtime users of robust statistics
    - Calculate basic limit average severity
    - Fit distributions to get excess severity
- More recently recognized in the growing use of the Generalized Linear Model

Parameter Uncertainty and Markov Chain Monte Carlo (MCMC) Methods

- MCMC methods used for Bayesian analyses.
  - Gibbs Sampler
  - Metropolis Hastings Algorithm
- It randomly generates parameters in proportion to posterior probabilities.

Gibbs Sampler on a Lognormal

Example from February 2008 Actuarial Review

- Simulate μ and σ from a prior distribution of parameters.
- Calculate the likelihood of each simulated μ and σ.
- Select a random uniform number U.
- Accept μ and σ into the posterior distribution if

$$\frac{\text{Likelihood}}{\text{Maximum Likelihood}} < U$$

Figure 1

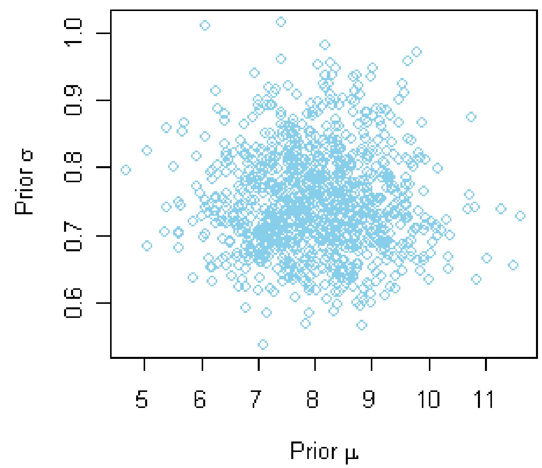
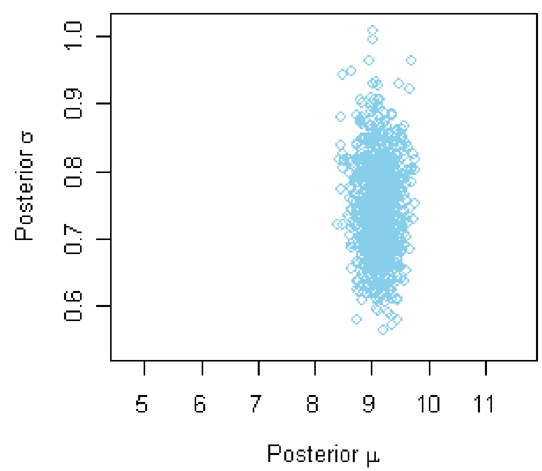


Figure 2



Posterior Distribution of μ and σ is Only of Temporary Interest!

- Most often we are interested in functions of μ and σ.
- For example:

Mean

$$e^{\mu+\sigma^2/2}$$

Limited Expected Value

$$e^{\mu+\sigma^2/2}\cdot\Phi\left(\frac{\log(L)-\mu-\sigma^2}{\sigma}\right)+L\cdot\left(1-\Phi\left(\frac{\log(L)-\mu}{\sigma}\right)\right)$$

Layer Expected Value
25,000 to 30,000

- Some posterior parameters generated by Gibbs sampler

□	□	LEV
9.194	0.723	392
9.206	0.708	383
8.817	0.707	119
8.944	0.644	120
9.461	0.785	836
9.150	0.651	252
9.043	0.739	280
9.240	0.773	514
9.392	0.863	845
9.018	0.781	311

Figure 3

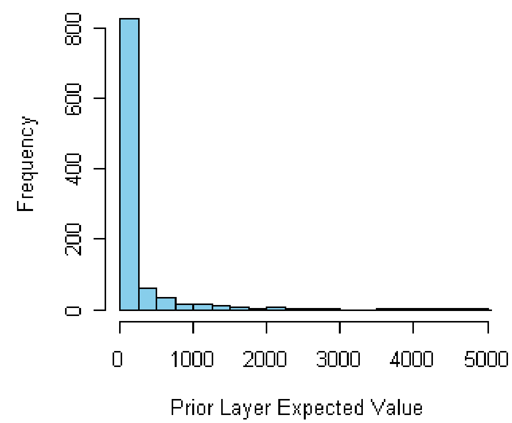
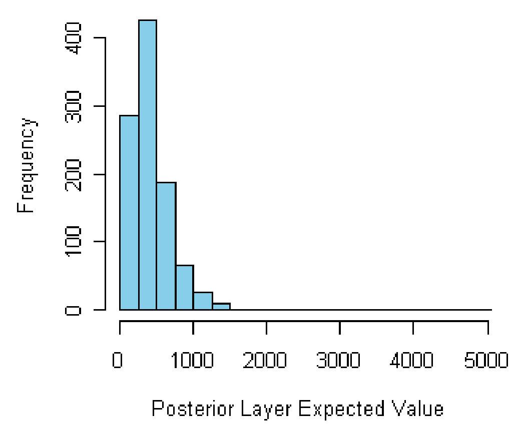


Figure 4



Evolving Strategy for Modeling Uncertainty

- Point Estimates
  - Based on MLE or (Bayesian) Predictive Mean
- Ranges - Bayesian
  - "Quantities of Interest" weighted by posterior probabilities of the parameters
    - Discrete prior or MCMC Samples
- Some Applications
  - Claim severity models – COTOR Challenge
  - Loss reserve models – Today's topic

S&P Report, November 2003

Insurance Actuaries – A Crisis in Credibility

"Actuaries are signing off on reserves that turn out to be wildly inaccurate."

Prior Work on Loss Reserve Models

- Estimating Predictive Distributions for Loss Reserve Models – 2006 CLRS and 2007 ***Variance***
  - Initial application of the strategy to loss reserves
  - Tested results on subsequent loss payments
    - **Set a standard for evaluating loss reserve formulas**
- Thinking Outside the Triangle – 2007 ASTIN Colloquium
  - Tested a formula based on simulated outcomes
  - Provided an example
    - **Model parameters from MLE understated range**
    - **Bayesian mixing (spreading out) provided accurate range**

Stochastic Loss Reserving with the Collective Risk Model

- Focuses mainly on “How to do it”
  - “Data” is simulated from collective risk model
  - Code for implementing algorithms included
- Secondary Objective
  - Illustrate my current favorite tools
  - Illustrate MCMC Methods
    - as does Verrall (2007) in ***Variance***

Method Illustrated on Data

Incremental Paid Losses

AY	Premium	*Lag* 1	*Lag* 2	*Lag* 3	*Lag* 4	*Lag* 5	*Lag* 6	*Lag* 7	*Lag* 8	*Lag* 9	*Lag* 10
1	50,000	7,168	11,190	12,432	7,856	3,502	1,286	334	216	190	0
2	50,000	4,770	8,726	9,150	5,728	2,459	2,864	715	219	0	$X_{2,10}$
3	50,000	5,821	9,467	7,741	3,736	1,402	972	720	50	$X_{3,9}$	$X_{3.10}$
4	50,000	5,228	7,050	6,577	2,890	1,600	2,156	592	$X_{4,8}$	$X_{4,9}$	$X_{4,10}$
5	50,000	4,185	6,573	5,196	2,869	3,609	1,283	$X_{5,7}$	$X_{5,8}$	$X_{5,9}$	$X_{5,10}$
6	50,000	4,930	8,034	5,315	5,549	1,891	$X_{6,6}$	$X_{6,7}$	$X_{6,8}$	$X_{6,9}$	$X_{6,10}$
7	50,000	4,936	7,357	5,817	5,278	$X_{7,5}$	$X_{7,6}$	$X_{7,7}$	$X_{7,8}$	$X_{7,9}$	$X_{7,10}$
8	50,000	4,762	8,383	6,568	$X_{8,4}$	$X_{8,5}$	$X_{8,6}$	$X_{8,7}$	$X_{8,8}$	$X_{8,9}$	$X_{8,10}$
9	50,000	5,025	8,898	$X_{9,3}$	$X_{9,4}$	$X_{9,5}$	$X_{9,6}$	$X_{9,7}$	$X_{9,8}$	$X_{9,9}$	$X_{9,10}$
10	50,000	4,824	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$	$X_{10,5}$	$X_{10,6}$	$X_{10,7}$	$X_{10,8}$	$X_{10,9}$	$X_{10,10}$

Plan of Attack

- Specify stochastic model needed to calculate likelihood of the data
- Calculate MLE and parameters from Metropolis-Hastings algorithm
- Quantity of Interest = Percentiles of OS Loss

$$R = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} X_{AY,Lag}$$

Model for Expected Losses

- Two models for expected loss
  - Independent Factor Model

$$\mathrm{E}\left[Loss_{AY,Lag}\right] = Premium_{AY} \cdot ELR_{AY} \cdot Dev_{Lag}$$

  - Beta Model

$$Dev_{Lag} = \beta\left(Lag/10 \mid a,b\right) - \beta\left((Lag-1)/10 \mid a,b\right)$$

- {ELR_{AY}} and {Dev_{Lag}} and/or {a,b} parameters estimated from data

Need a Stochastic Model to Calculate Likelihoods

Use the collective risk model.

- Select a random claim count, $N_{AY,Lag}$ from a Poisson distribution with mean λ.
- For i = 1, 2, …, $N_{AY,Lag}$, select a random claim amount, $Z_{Lag,i}$.
- Set, $X_{AY,Lag} = \sum_{i=1}^{N_{AY,Lag}} Z_{Lag,i}$

or if $N_{AY,Lag}$ = 0, then $X_{AY,Lag}$ = 0..

Details of Distributions

- Pareto severity distribution $F(z) = 1 - \left(\frac{\theta}{z + \theta} \right)^{\alpha}$
- for all lags – $\alpha = 2$
- Claims subject to limit of \$1 million
- Table of θ's

Lag	1	2	3	4	5	6	7-10
θ (000)	10	25	50	75	100	125	150

- Severity increases with lag
- Approximate likelihood calculated by matching moments with a Tweedie distribution

The Tweedie Approximate Likelihood

Tweedie distribution as collective risk model

1. Select claim count, N, at random from a Poisson distribution with mean λ.
2. For each claim $i = 1,\ldots,N$
   - Select claim amount, Z_i, from a Gamma distribution with scale parameter θ and shape parameter α.
3. Set loss $X = \sum_{i=1}^{N} Z_i$

The Tweedie Approximate Likelihood

- Choose θ and α to match moments of the Pareto distribution for each settlement lag
- Translate λ, θ and α into R's Tweedie package parameters.

$$p = \frac{\alpha + 2}{\alpha + 1}, \quad \mu = \lambda \cdot \alpha \cdot \theta, \quad \phi = \frac{\lambda^{1-p} \cdot (\alpha \cdot \theta)^{2-p}}{2-p}$$

- Use the function "dtweedie" in R to estimate log likelihood.
  - See Appendix B

Maximum Likelihood Parameter Estimates for the Two Models

Table 1

	Independent Factor		Beta	
	ELR	*Dev*	*ELR*	*Dev*
AY/Lag 1	0.88832	0.16760	0.88496	0.16995
2	0.67147	0.27635	0.65567	0.26388
3	0.64720	0.23451	0.65236	0.23094
4	0.56222	0.15660	0.55986	0.16322
5	0.49539	0.07751	0.48969	0.09764
6	0.57450	0.04825	0.57342	0.04885
7	0.58392	0.02267	0.57112	0.01936
8	0.56703	0.01101	0.59260	0.00536
9	0.60360	0.00108	0.63075	0.00077
10	0.54760	0.00443	0.56753	0.00002
			a =	1.75975
			b =	5.25776

Introduction to (MCMC) Markov Chain Monte Carlo Methods

- Consider a chain μ_t of random vectors, with $\mu_{t+1} \sim h(\mu_t)$ for some operator, *h*.
- For sufficiently large *t*, $\{\mu_t\}$ converges to a limiting distribution.
- With the right choice of *h*, that limiting distribution is a posterior distribution.
- Popular choices for *h*
  - Gibbs sampler
  - Metropolis Hastings algorithm

The Metropolis-Hastings Algorithm

1. Select a random candidate value, μ^* from a proposal density function $p\left(\mu^* \mid \mu_{t-1}\right) = \Gamma\left(\mu^* \mid \mu_{t-1} / \alpha_p, \alpha_p\right)$

2. Compute the ratio $R = \dfrac{f\left(x \mid \mu^*\right) \cdot g\left(\mu^*\right) \cdot p\left(\mu^* \mid \mu_{t-1}\right)}{f\left(x \mid \mu_{t-1}\right) \cdot g\left(\mu_{t-1}\right) \cdot p\left(\mu_{t-1} \mid \mu^*\right)}$

   - *f* comes from the Tweedie distibution
   - *g* is the prior distribution
   - *f*·*g* is the posterior distribution

3. Select a random *U* from a uniform distribution on (0,1).
4. If $U < R$ then set $\mu_t = \mu^*$. Otherwise set $\mu_t = \mu_{t-1}$.

Single Variable Example of Tuning the Metropolis-Hastings Algorithm

Figure 2 - $\alpha_p = 2{,}500$

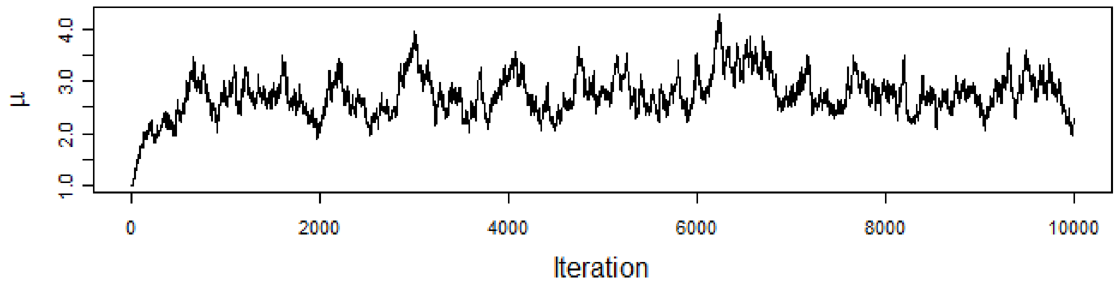


Figure 3 - $\alpha_p = 25$

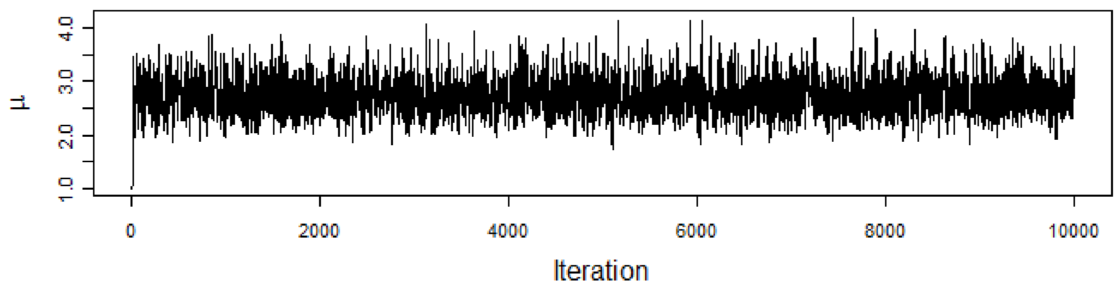
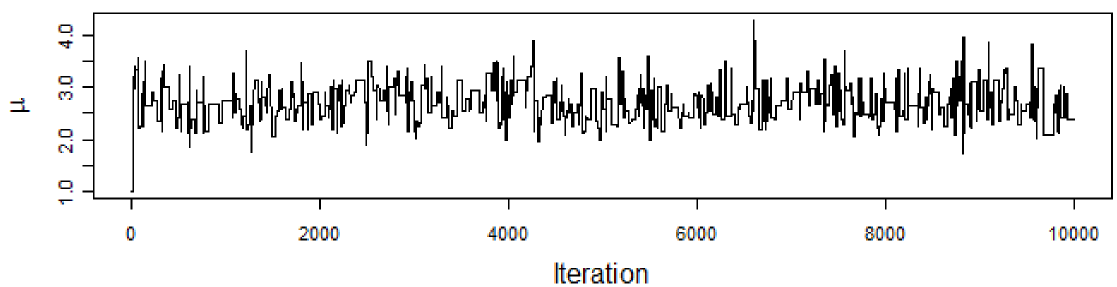


Figure 4 - $\alpha_p = 0.25$



Bayesian Analyses
Specify Prior Distributions

$$ELR_{AY} \sim \Gamma(\alpha, \theta) \text{ with } \alpha = 100 \text{ and } \theta = 0.07$$

Beta Model

$$a \sim \Gamma(\alpha, \theta) \text{ with } \alpha = 75 \text{ and } \theta = 0.02$$

$$b \sim \Gamma(\alpha, \theta) \text{ with } \alpha = 25 \text{ and } \theta = 0.20$$

Independent Factor Model

$\Gamma \backslash Lag$	1	2	3	4	5	6	7	8	9	10
α_d	11.0665	64.4748	189.6259	34.8246	10.6976	4.4824	2.1236	1.0269	0.4560	0.1551
θ_d	0.0206	0.0041	0.0011	0.0040	0.0079	0.0101	0.0097	0.0073	0.0039	0.0009

- Prior parameters were derived from looking at “Estimating Predictive Distributions … “ paper.

Sample from Metropolis-Hastings Algorithm Applied to $\{Dev_{AY,t}|a_{AY,t},b_{AY,t}\}$ and $\{ELR_{AY,t}\}$ parameters – Beta Model

ELR_1	ELR_2	ELR_3	ELR_4	ELR_5	ELR_6	ELR_7	ELR_8	ELR_9	ELR_{10}
0.91503	0.66796	0.62778	0.58480	0.56635	0.67332	0.56119	0.68528	0.69505	0.70776
0.91503	0.66796	0.62778	0.58480	0.56635	0.67332	0.56119	0.68528	0.69505	0.70776
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0.86193	0.63186	0.67501	0.57013	0.60554	0.64775	0.61769	0.74869	0.68954	0.68855
0.85805	0.62464	0.68672	0.55612	0.58922	0.63364	0.65857	0.70962	0.67289	0.64800

Dev_1	Dev_2	Dev_3	Dev_4	Dev_5	Dev_6	Dev_7	Dev_8	Dev_9	Dev_{10}
0.16546	0.25163	0.22465	0.16499	0.10414	0.05589	0.02427	0.00762	0.00131	0.00005
0.16546	0.25163	0.22465	0.16499	0.10414	0.05589	0.02427	0.00762	0.00131	0.00005
0.16321	0.24844	0.22338	0.16574	0.10598	0.05781	0.02564	0.00827	0.00148	0.00006
0.16613	0.24962	0.22293	0.16463	0.10487	0.05701	0.02520	0.00811	0.00144	0.00006
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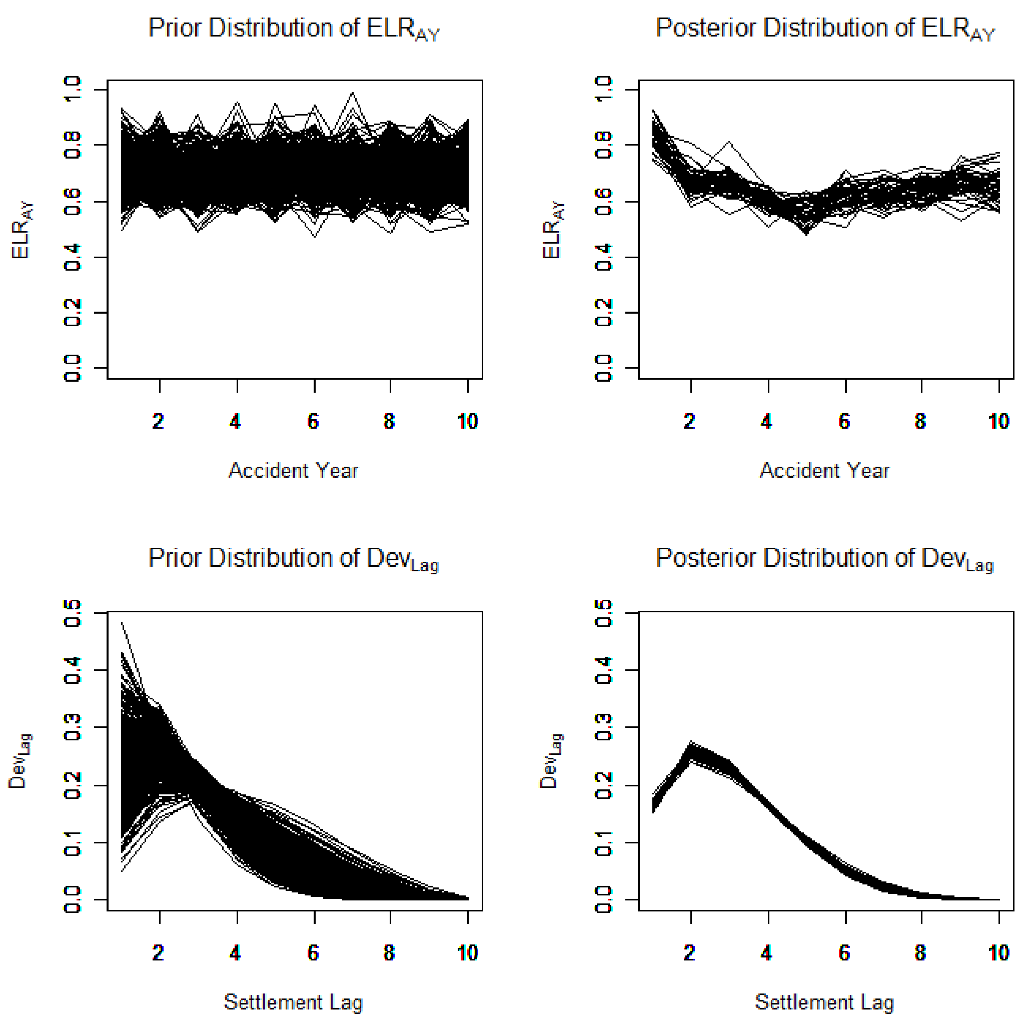
0.16613	0.24962	0.22293	0.16463	0.10487	0.05701	0.02520	0.00811	0.00144	0.00006
0.15732	0.24804	0.22578	0.16815	0.10736	0.05822	0.02555	0.00810	0.00141	0.00006
0.15732	0.24804	0.22578	0.16815	0.10736	0.05822	0.02555	0.00810	0.00141	0.00006

Graphical Representation of Metropolis-Hastings Sample



Note that the posteriors are tighter, showing how the data narrows the range of results.

Statistics of Intrest

Incremental Paid Losses

AY	Premium	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8	Lag 9	Lag 10
1	50,000	7,168	11,190	12,432	7,856	3,502	1,286	334	216	190	0
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8	50,000	4,762	8,383	6,568	$X_{8,4}$	$X_{8,5}$	$X_{8,6}$	$X_{8,7}$	$X_{8,8}$	$X_{8,9}$	$X_{8,10}$
9	50,000	5,025	8,898	$X_{9,3}$	$X_{9,4}$	$X_{9,5}$	$X_{9,6}$	$X_{9,7}$	$X_{9,8}$	$X_{9,9}$	$X_{9,10}$
10	50,000	4,824	$X_{10,2}$	$X_{10,3}$	$X_{10,4}$	$X_{10,5}$	$X_{10,6}$	$X_{10,7}$	$X_{10,8}$	$X_{10,9}$	$X_{10,10}$

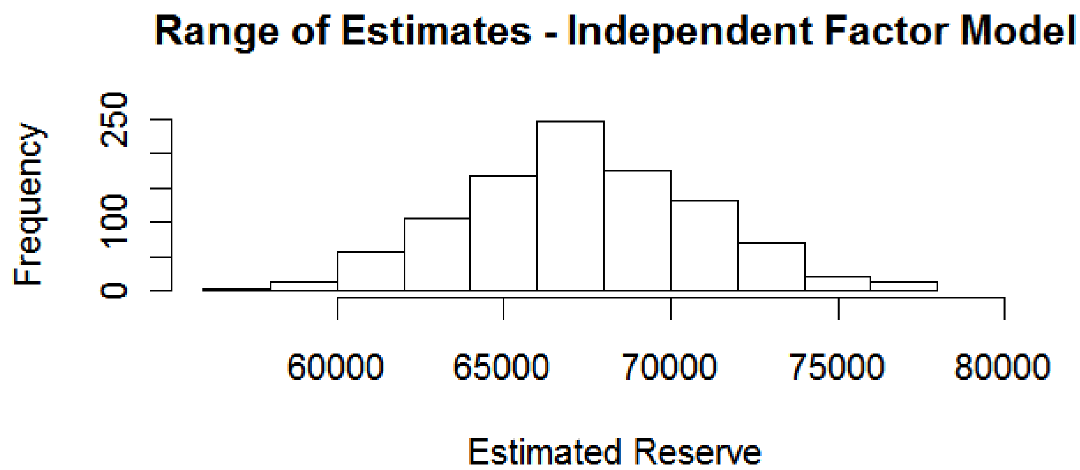
"Range of Reasonable Estimates"
Distribution of

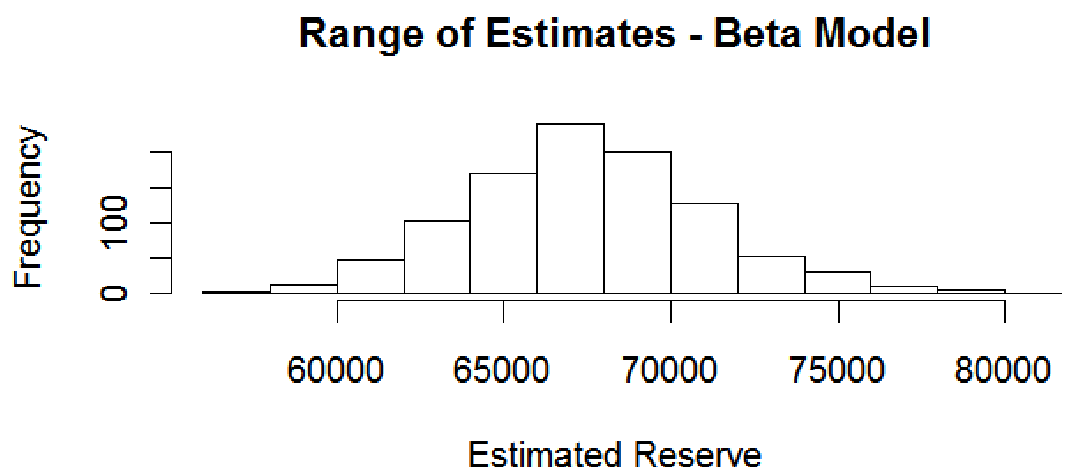
$$E = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} Premium_{AY} \cdot ELR_{AY} \cdot Dev_{Lag}$$

Predictive Distribution
of Reserve Outcomes

$$R = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} X_{AY,Lag}$$

“Range of Reasonable Estimates”

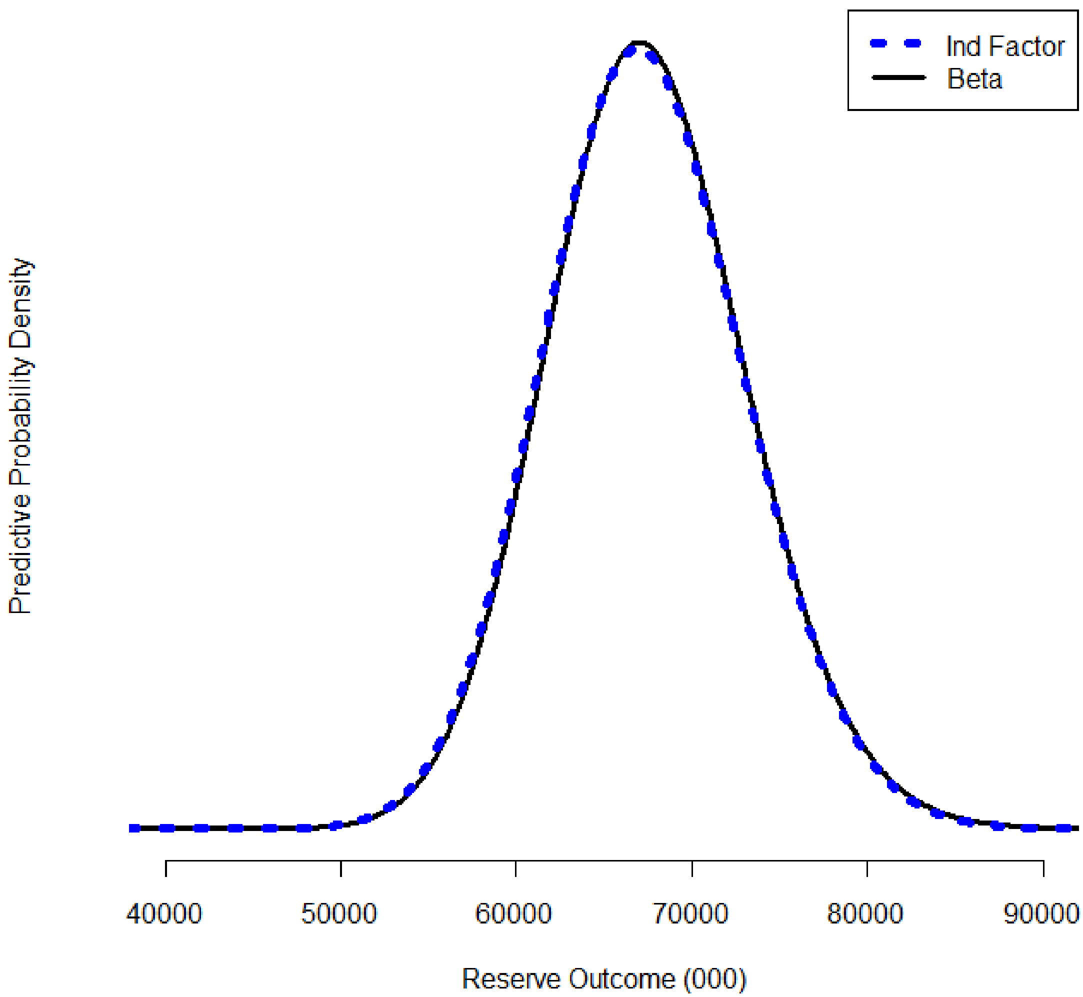




Statistics of Interest
Predictive Distributions of Reserve Outcomes

- Collective risk model – With Pareto distributions
- Simulation
  - Randomly select $\{ELR_i\}$ and $\{Dev_j\}$
  - Simulate $R = \sum_{AY=2}^{10} \sum_{Lag=12-AY}^{10} X_{AY,Lag}$ with collective risk model.
- Use the Fast Fourier Transform
  - Faster, more accurate, but uses some math
  - Used in this paper

Statistics of Interest
Predictive Distributions of Reserve Outcomes



Independent Factor Model
Mean = 67,343
StDev = 5,677

Beta Model
Mean = 67,511
StDev = 5,685

Further Development Needs

- Extend this approach to incorporate incurred data
  - Halliwell and Munich Chain Ladder
- Combine lines of insurance – “Correlation”
- Test predictive distribution on subsequent development – Meyers [2007]
- Apply results
  - Solvency II
  - Monitor results for changing environment