Pricing in a Competitive Insurance Market Driven by Fractional Noise

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Pricing Process – No market concerns

Traditional premium calculation

constant safety loading

Expected Claims	Safety loading

Dynamic premium calculation

Lundberg(1903) - Cramer(1930) - Borch (1967)

Variable (& controllable) safety loading

$$F(t) = F(0) + \int_0^t (1 + \theta(s))\, p\, ds - S(t)$$

Pricing process – Competitive Market

Market premium calculation

Taylor (1986) and Emms, Haberman & Savoulli (2007)

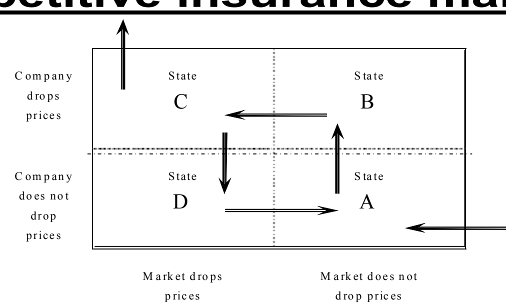
Variable (& controllable) safety loading

plus Market concerns

$$dF(t) = \delta(t) F(t) dt + [p(t) - c(t)] dV(t)$$

The level of the safety loading and consequently
the level of premium determines the volume of business

The payoff matrix of a competitive insurance market



Model Equations for the Insurance Company

$$dF(t) = \delta(t)F(t)dt + [p(t) - c(t)]dV(t)$$

V(t): is the volume of business at time t (input variable)

p(t): is the premium rate of the company at time t (controlled variable)

F(t): is the reserve value at time t (state variable) , $F(0)=F_0$

$\delta(t)$: is the force of interest earned by the fund at time t

c(t): is the total cost per policy (claims plus expenses and profit margin) at time t.

$$\min_{p(t)} \mathbf{E}\left[g\left(p(t), F(T)\right)\right] = \min_{p(t)} \mathbf{E}[F(T) - F_0]^2$$

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Modelling the Volume of Business

$$V(t) = f(p(t), \overline{p}(t), \tilde{\psi}(t))$$

$p(t)$ is the premium rate of the company,

$\overline{p}(t)$ is the market's average premium rate and

$\tilde{\psi}(t)$ is a vector of different parameters
(e.g., money spent in the advertisement campaign of the company)

$$\lim_{p(t)\to\infty} f(p(t), \overline{p}(t), \tilde{\psi}(t)) = 0$$

$$\lim_{p(t)\to 0} f(p(t), \overline{p}(t), \tilde{\psi}(t)) = E$$

E corresponds to the total market

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Modelling the market's average premium rate

$$d\overline{p}(t) = \mu dt + \sigma dW^H(t)$$

μ is the average claim rate (drift),

σ is the volatility of the claim process

$W^H(t)$ is a fractional Brownian motion

WHY Fractional Brownian Motion ?

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Fractional Brownian Motion
Basic Theory - (1)

Founders

Kolmogorov (1940)
Theoretical Derivation

Hurst (1951)
Nile River yearly water flows

Mandelbrot & Van Ness (1968)
Physical Derivation

Formal Description

$$W^{H} = \left\{ W^{H}(t) ; t \geq 0 \right\} \; , \; H \in (0.5, 1)$$

$$\Pr\left[W^{H}(0) = 0 \right] = 1$$

$$\mathbf{E}\left[W^{H}(t) \right] = 0$$

$$\mathbf{E}\left[W^{H}(s) \cdot W^{H}(t) \right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |s-t|^{2H} \right]$$

for any t,s

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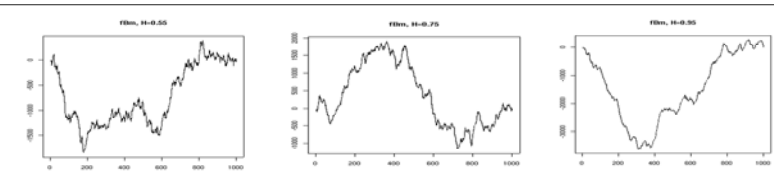
Fractional Brownian Motion
Basic Theory - (2)

Gaussian process - self-similar - stationary increments
exhibiting long-range dependency

Pictures



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Fractional Brownian Motion
Basic Theory - (3)

Estimation of the Hurst exponent

Given the data of a time series, the Hurst exponent may be regarded as a measure of smoothness and is defined as

$$H = \frac{\ln(R/S)}{\ln(T)}$$

R is the respective range,
S is the standard deviation and
T is the duration of the sample data.

Regarding the procedure for the estimation of the Hurst exponent, see the algorithm described in Zimbidis (2011)

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Fractional Brownian Motion Evidence

Greek motor insurance market

- A quite strange example within the European area
- Exhibits a low premium rate for MTPL (Motor Third-Party Liability), while the road deaths rates are among the highest within Europe.
- The tariff system for MTPL has been controlled by the government up to the end of 1996.
- Investigation period 1998-2006. The Hurst index is similar for each line of business, and always greater than the critical value 0.5 and near 0.6, supporting the evidence for long-range dependency

(see Zimbidis (2011))

Stochastic Optimal Control

TARGET

$$\min_{p(t)} \mathbf{E}\left[F(T) - F_0\right]^2$$

RESTRICTION

$$dF(t) = \left[\delta(t)F(t) + \mu E\left(1 - \frac{c(t)}{p(t)}\right)\right]dt + \sigma E\left(1 - \frac{c(t)}{p(t)}\right)dW^H(t)$$

SOLUTION

$$p(t) = \frac{c(t)}{1 + \frac{\mu}{\sigma^2 E} \cdot \frac{t^{0.5-H} \cdot (T-t)^{0.5-H}}{2H\,\Gamma(0.5+H)\cdot\Gamma(1.5-H)} \cdot \left(F(t) - F_0\right)}$$

Limiting Behavior of the Solution

	IF	THEN
a)	$F(t) \to \infty$	$p(t) \to 0$
b)	$F(t) \to F_0$	$p(t) \to c(t)$
c)	$F(t) \to 0$ and $\frac{\mu}{\sigma^2 E} \cdot \frac{t^{0.5-H} \cdot (T-t)^{0.5-H}}{2H\,\Gamma(0.5+H)\cdot\Gamma(1.5-H)} < \frac{1}{F_0}$	$p(t) \to \infty$
d)	$t \to 0 \; or \; T$	$p(t) \to c(t)$
e)	$\frac{t^{0.5-H} \cdot (T-t)^{0.5-H}}{2H\,\Gamma(0.5+H)\cdot\Gamma(1.5-H)} \cong 1$ and $\frac{\mu}{\sigma^2 E} \cong \frac{1}{F_0} \; or \; F_0 \cong E \cdot \frac{\sigma^2}{\mu}$	$p(t) \cong \frac{F_0}{F(t)} c(t)$

Calculation of the Safety Loading

H=0.5	Solvency Ratio = F(t) / F(0)				
	60%	80%	100%	120%	140%
t					
1 -19	67%	25%	0%	-17%	-29%

H=0.6	Solvency Ratio = F(t) / F(0)				
	60%	80%	100%	120%	140%
t					
1	32%	14%	0%	-11%	-20%
2	30%	13%	0%	-10%	-19%
3	28%	12%	0%	-10%	-18%
4	28%	12%	0%	-10%	-18%
5	27%	12%	0%	-10%	-18%
6	27%	12%	0%	-10%	-17%
7	26%	12%	0%	-9%	-17%
8	26%	12%	0%	-9%	-17%
9	26%	12%	0%	-9%	-17%
10	26%	12%	0%	-9%	-17%
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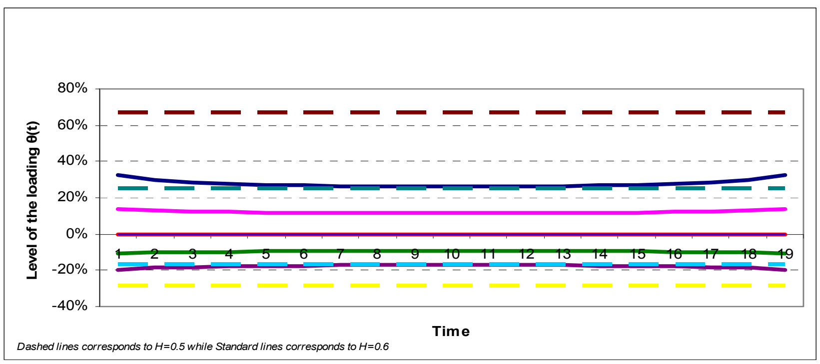
13	26%	12%	0%	-9%	-17%
14	27%	12%	0%	-10%	-17%
15	27%	12%	0%	-10%	-18%
16	28%	12%	0%	-10%	-18%
17	28%	12%	0%	-10%	-18%
18	30%	13%	0%	-10%	-19%
19	32%	14%	0%	-11%	-20%

Time Development of the Safety Loading



Final Remarks

- empirical evidence found within the data of the Greek motor insurance market
- fractional Brownian motion
  as the modeling tool for the driving force of the market’s behavior
- the optimal premium control strategy differs considerably from the straight line
  when there is some kind of long-range dependency,
  while it remains time invariant when the dependency does not exist .

References

■ Borch, K., “The Theory of Risk,” *Journal of the Royal Statistical Society* B29, 1967, pp. 432–452.

■ Cramér, H., “On the Mathematical Theory of Risk”, Skandia Jubilee Volume, Stockholm,1930

■ Emms, P., S. Haberman, and I. Savoulli, “Optimal Strategies for Pricing General Insurance,” *Insurance: Mathematics and Economics* 40, 2007, pp. 15–34.

■ Hurst, H. E., “Long-Term Capacity in Reservoirs,” *Transactions of the American Society of Civil Engineers* 116, 1951, pp. 400–410.

■ Kolmogorov, A. N., “Wienersche Spiralen und Einige Andere Interessante Kurven in Hilbertachen Raum,” C. R. (Doklady) Academy of Sciences USSR (N.S.) 26, 1940, pp. 115–118.

■ Lundberg, F., “Approximerand Framställning au Sannolikehetfunktionen. Ǻterförsäkering au Kollektivrisker,” Almqvist & Wiksell, Uppsala, 1903.

■ Mandelbrot, B. B., and J. W. Van Ness, “Fractional Brownian Motions, Fractional Brownian Noises and Applications,” *SIAM Review* 10, 1968, pp. 422–437.

■ Taylor, G. C., “Underwriting Strategy in a Competitive Insurance Environment,” *Insurance: Mathematics and Economics* 5, 1986, pp. 59–77.

■ Zimbidis, A., “Pricing in a Competitive Insurance Market Driven by Fractional Noise ” Variance 5:1, 2011, pp. 55-67.