Ruin Probability-Based Initial Capital of the Discrete-Time Surplus Process

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1. Model descriptions
1.1 Classical surplus process

$$U_0 = u, U_n = u + cT_n - \sum_{i=1}^{n} X_i, \tag{1}$$

Assumptions:

- Claims happen at the times T_i, satisfying

$$0 = T_0 \leq T_1 \leq T_2 \leq \cdots,$$

- The nth claim arriving at time T_n causes the claim size X_n,
- c represent the constant premium rate for one unit time,
- $U_0 = u \geq 0$ is the initial capital.

Remark:

- The quantity cT_n describes the inflow of capital by time T_n,
- $\sum_{i=1}^{n} X_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_n]$.

Therefore, the quantity U_n is the insurer's balance (or surplus) at time T_n.

1.2 Research Model

In this research, We consider the discrete-time surplus process (1) in the situation that the possible insolvency (ruin) can occur only at claim arrival times $T_n = n, n = 1, 2, 3, \cdots$. Thus, the model (1) becomes

$$U_0 = u,\ U_n = u + cn - \sum_{i=1}^{n} X_i \tag{2}$$

for all $n = 1, 2, 3, \cdots$.

Assumptions and notations

- The claim process $X = \{X_n, n \geq 1\}$ is assumed to be independent and identically distributed (i.i.d.).
- Let $F_{X_1}(x)$ be the distribution function of X_1, i.e.,

$$F_{X_1}(x) = \Pr\{X_1 \leq x\}. \tag{3}$$

- The *premium rate* c is calculated by the *expected value principle*, i.e.,

$$c = (1+\theta)E[X_1] \tag{4}$$

where $\theta > 0$ which is the *safety loading of insurer*.

1.3 Survival and ruin probability

Let $u \geq 0$ be an initial capital. For each $n = 1, 2, 3, \cdots$, we let

$$\varphi_n(u) := \Pr\{U_1 \geq 0, U_2 \geq 0, U_3 \geq 0, \cdots, U_n \geq 0 | U_0 = u\} \quad (5)$$

denote the *survival probability* at the times n.
Thus, the *ruin probability* at one of the time $1, 2, 3, \cdots, n$ is denoted by

$$\Phi_n(u) = 1 - \varphi_n(u). \quad (6)$$

Remark: The equivalent definition of the ruin probability given by

$$\varphi_n(u) := \Pr\{U_n < 0 \text{ for some } n = 1, 2, 3, \ldots, n | U_0 = u\}$$

1.4 Research Objective

There are many papers studied the ruin probability as a function of the initial capital.

In this research,

"we want to work in the opposite direction, i.e., we want to study the initial capital for the discrete time surplus process as a function of ruin probabilities."

2. Main results

Assume that all the processes are defined in a probability space $(\Omega, \mathcal{F}, \mathsf{Pr})$. Let $\{U_n, n \geq 0\}$ be a surplus process which is driven by the i.i.d. claim process $X = \{X_n, n \geq 1\}$ and $c > 0$ be a premium rate.

Definition 1.

Given $\alpha \in (0, 1)$ and $N \in \{1, 2, 3, \cdots\}$. Let $u \geq 0$ be an initial capital,

- if $\Phi_N(u) \leq \alpha$ then u is called an *acceptable initial capital* corresponding to (α, N, c, X).
- if $u^* = \min_{u \geq 0}\{u : \Phi_N(u) \leq \alpha\}$ exists, u^* is called the *minimum initial capital* corresponding to (α, N, c, X) and is written as

$$u^* := \mathrm{MIC}(\alpha, N, c, X). \tag{7}$$

2.1 Survival and ruin probability properties

Lemma 1.
Let $N \in \{1, 2, 3, \cdots\}$ and $c > 0$ be given. Then $\varphi_N(u)$ is increasing and right continuous and $\Phi_N(u)$ is decreasing and right continuous in u.

Theorem 2.
Let $N \in \{1, 2, 3, \cdots\}$ and $c > 0$ be given. Then

$$\lim_{u\to\infty} \varphi_N(u) = 1 \text{ and } \lim_{u\to\infty} \Phi_N(u) = 0. \tag{8}$$

Existence of acceptable initial capital

Corollary 3.

Let $\alpha \in (0,1)$, $N \in \{1,2,3,\cdots\}$ and $c > 0$ be given.
Then there exists $\tilde{u} \geq 0$ such that, for all $u \geq \tilde{u}$, u is an acceptable initial capital corresponding to (α, N, c, X).

2.2 Recursive formula 1st recursive formula

Theorem 4.

Let $N \in \{1, 2, 3, \cdots\}$, $c > 0$ and $u \geq 0$ be given. Then the ruin probability at one of the times $1, 2, 3, \cdots, N$ satisfies the following equation

$$\Phi_N(u) = \Phi_1(u) + \int_{-\infty}^{u+c} \Phi_{N-1}(u + c - x) dF_{X_1}(x) \qquad (9)$$

where $\Phi_0(u) = 0$.

2nd recursive formula

Corollary 5

Let $N \in \{1, 2, 3, \cdots\}$, $c > 0$ and $u \geq 0$ be given. Then the ruin probability at one of the times $1, 2, 3, \cdots, N$ satisfies the following equation

$$\begin{aligned} \Phi_1(u) &= 1 - \Pr(X \leq u + c), \\ \Phi_N(u) &= \Phi_{N-1}(u) + \Theta_N(u) \end{aligned}$$

where

$$\Theta_N(u) = \int_{-\infty}^{u+c} \left(\int_{-\infty}^{u+c-x} \Phi_{N-2}(u + 2c - x - v) dF_{X_1}(v) \right) dF_{X_1}(x)$$

for all $n = 2, 3, 4, \cdots$.

2.3 Existence of MIC

Lemma 6.

Let a, b and α be real numbers such that $a \leq b$. If f is decreasing and right continuous on $[a, b]$ and $\alpha \in \left[f(b), f(a)\right]$, then there exists $d \in [a, b]$ such that

$$d = \min\left\{x \in [a, b] : f(x) \leq \alpha\right\}. \tag{10}$$

Theorem 7.

Let $\alpha \in (0, 1)$, $N \in \{1, 2, 3, \cdots\}$, and $c > 0$. Then there exist $u^* \geq 0$ such that

$$u^* = \mathrm{MIC}(\alpha, N, c, \{X_n, n \geq 1\}).$$

2.4 Approximate the MIC

Theorem 8.

Let $\alpha \in (0,1)$, $N \in \{1,2,3,\cdots\}$, and $v_0, u_0 \geq 0$ such that $v_0 < u_0$. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be a real sequence defined by

$$\begin{cases} v_k = v_{k-1} & \text{and} \quad u_k = \frac{u_{k-1}+v_{k-1}}{2}, \quad \text{if } \Phi_N\left(\frac{u_{k-1}+v_{k-1}}{2}\right) \leq \alpha \\ v_k = \frac{v_{k-1}+u_{k-1}}{2} & \text{and} \quad u_k = u_{k-1}, \qquad \text{if } \Phi_N\left(\frac{u_{k-1}+v_{k-1}}{2}\right) > \alpha \end{cases}$$

for all $k = 1,2,3,\cdots$. If $\Phi_N(u_0) \leq \alpha < \Phi_N(v_0)$, then

$$\lim_{k\to\infty} u_k = \mathrm{MIC}(\alpha, N, c, \{X_n, n \geq 1\}) \tag{11}$$

and

$$0 \leq u_k - \mathrm{MIC}(\alpha, N, c, \{X_n, n \geq 1\}) \leq \frac{u_0 - v_0}{2^k} \tag{12}$$

for all $k = 1,2,3,\cdots$.

3 Numerical examples Example with exponential claim process

Theorem 9.

Let $N \in \{1, 2, 3, \cdots\}$ and $u \geq 0$. Assume that $\{X_n, n \geq 1\}$ is a sequence of exponential distribution with intensity $\lambda > 0$, i.e., X_1 has the probability density function $f(x) = \lambda e^{-\lambda x}$. The obtained ruin probability is in the following recursive form

$$\Phi_0(u) = 0, \ \Phi_n(u) = \Phi_{n-1}(u) + \frac{(u+c)\lambda^{n-1}(u+nc)^{n-2}}{(n-1)!} e^{-\lambda(u+nc)} \tag{13}$$

for all $n = 1, 2, 3, \cdots$, where the initial capital $u \geq 0$ and premium rate
$c > E[X_1] = 1/\lambda$.

We approximate the minimum initial capital of the discrete-time surplus process (2) by using Theorem (8) in the case of $X = \{X_n, n \geq 1\}$ a sequence of i.i.d exponential distribution with intensity $\lambda = 1$, by choosing model parameter combinations $\theta = 0.10$ and 0.25, i.e., $c = 1.10$ and $c = 1.25$, respectively; and $\alpha = 0.1$, 0.2, and 0.3.

Table 1:

	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
N	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$
10	4.31979	3.39733	2.89299	2.09364	1.99866	1.29821
20	5.80757	4.13270	3.98629	2.58739	2.84099	1.65474
30	6.79110	4.47565	4.69130	2.80479	3.37378	1.80597
40	7.52286	4.66050	5.20540	2.91736	3.75643	1.88242
50	8.09889	4.76749	5.60309	2.98061	4.04866	1.92467
100	9.81693	4.92644	6.74520	3.07093	4.86621	1.98377
200	11.13546	4.94953	7.56253	3.08341	5.42576	1.99174
300	11.60284	4.95021	7.83409	3.08377	5.60493	1.99197
400	11.79769	4.95024	7.94308	3.08378	5.67545	1.99197
500	11.88611	4.95024	7.99136	3.08378	5.70634	1.99197
1,000	11.96919	4.95024	8.03565	3.08378	5.73435	1.99197
5,000	11.97291	4.95024	8.03757	3.08378	5.73554	1.99197
10,000	11.97291	4.95024	8.03757	3.08378	5.73554	1.99197

Table 1 shows the approximation of $\mathrm{MIC}(\alpha, N, c, X)$ with u_{25}, choosing $v_0 = 0$ and $u_0 = 20$, and $\Phi_N(u)$ is computed from the recursive form (13).

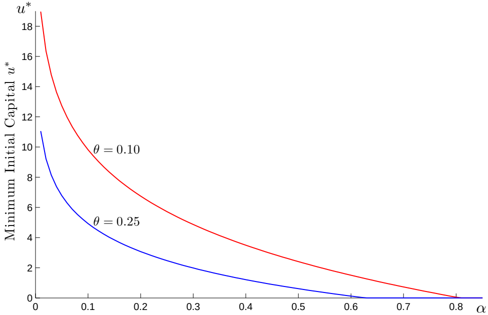

Figure 1:

Figure 1 shows the approximation of $\mathrm{MIC}(\alpha, N, c, X)$ for the various values of α with u_{25}. Here we choose $v_0 = 0$, $u_0 = 20$, and parameter combinations $\theta = 0.10$, $\theta = 0.25$, i.e., $c = 1.10$ and $c = 1.25$, respectively.

Thank you for Your Attention