# Insurance Risk-Based Capital with a Multi-Period Time Horizon

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**Abstract:** There are several alternative views on how to determine capital for an insurer whose loss liabilities extend for several time periods until settlement. These views differ in their focus on the immediate time period (one-year) or runoff (until ultimate loss payment) time frame, and will generally produce different amounts of required capital.

For an insurer whose liabilities and corresponding assets extend over a single time period, Butsic [2013] showed how to determine the optimal capital level when the value of the insurance to the policyholder is maximized, while providing a fair return to the insurer's owners. This paper extends those results to determine optimal capital when liabilities last for several time periods until settlement. The multi-period analysis determines the optimal capital for one period and uses backward induction to find the optimal capital for successively longer time frames.

A key ingredient in this approach is the stochastic process for loss development; another is the choice of capital funding strategy, which must dynamically respond to the evolving loss estimate. In addition to the variables that affect the optimal one-period capital amount (such as the loss volatility, frictional cost of capital and the policyholder risk preferences), I show that the horizon length, the capitalization interval (time span between potential capital flows), and the policy term will influence the optimal capital for multiple time periods. Institutional and market factors, such as the conservatorship process for insolvent insurers and the cost of raising external capital, also play a major role and are incorporated into the model.

Results show that, for the same annual loss volatility, more capital is required as the time horizon increases; essentially, the optimal capital depends on *both* the annual volatility and the ultimate volatility. Also, less capital is needed if capital flows can occur frequently and/or if the policy term is shorter. Insurers that are able to more readily raise capital externally will need to carry less of it.

This paper applies the same loss-based techniques to develop asset risk capital; depending on the asset volatility, *the same or less* capital is generally needed as the time horizon increases.

The resulting optimal capital forms the basis for pricing, corporate governance and regulatory applications. The paper extends the model to incorporate features, such as present value and risk margins, that are necessary for practical applications. Although the primary focus is property-casualty insurance, the method can be extended to life and health insurance. In particular, the method used to determine capital required for multi-period asset risk will apply to these firms.

**Keywords**: Stochastic loss process, certainty equivalent, technical insolvency, capitalization interval, policy term, risk margin, backward induction, fair-value accounting

## 1. INTRODUCTION AND SUMMARY

There is considerable literature on how to determine the appropriate risk-based capital for an insurance enterprise. Generally, the analysis applies a particular risk measure (such as VaR or expected policyholder deficit), calibrated to a specific valuation level (e.g., VaR at 99.5%) to determine the proper amount of capital. However, most of the commonly-used risk measures apply most readily to short-duration risks, for example, property insurance, where the liabilities are settled within a single time period. Application of these methods is more problematic when addressing long-term insurance claims, such as liability, workers compensation and life insurance.

How to treat long-term, or multi-period, liabilities and assets is the subject of much debate in the actuarial and insurance finance literature. For a good, practically-oriented discussion of this topic, see Lowe *et al* [2011]. Essentially there are two camps: one side advocates using an annual[1] or one-period time horizon, wherein the current capital amount must be sufficient to offset default risk based on market values over the *upcoming year* (*period*) with the market value of liabilities reflecting the cost of capital over the remaining runoff horizon. The other side argues that the current capital must offset the default over the *entire duration* (the runoff horizon) required to settle the liability. Essentially, the issue is whether capital depends on the loss volatility only for the upcoming year, or the ultimate loss volatility. This controversy has gained momentum with the upcoming implementation of the Solvency II risk-based capital methodology, which uses an annual (single-period) time horizon.[2]

As shown in the subsequent analysis here, the problem may be solved by extending the one-period model to a longer time frame. I have used the concept of an *optimal capital strategy* to determine the appropriate capital amount for the current period, which is the first period of a multi-period liability. For a one-period liability, there is a theoretically optimal amount of capital that depends on the insurer's cost of holding capital and the nature of the policyholders' risk aversion. These results are derived in *An Economic Basis for Property-Casualty Insurance Risk-Based Capital Measurement* (Butsic [2013]), which develops the appropriate risk measure (adjusted ruin probability) and calibration method (using the frictional cost of capital) for *a one-period* insurer in an equilibrium insurance setting. In fact, the analysis here can be considered as an extension to this paper which, for reference, I shorten to EBRM.

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[1] More generally, the period could be shorter than one year, but most applications use the annual time frame. In this paper I use the more general concept of time *periods*.

[2] See the European Parliament Directive [2009]; Article 64.

With multi-period risks, we can use the same fundamental assumptions that drive optimal capital for a single period. The main point is that, as in a one-period model, the optimal capital over several periods depends on the balance between capital costs and the amount that the policyholders are willing to pay to reduce the value of default to the policyholders.

This paper shows in general how to address these issues using economic and financial principles, and points the way to practical applications.

Capital in this paper is defined in the general accounting sense as the difference between assets and liabilities. For practical applications, capital will need to be defined according to a standard accounting convention such as fair value[3] or statutory accounting.

Although the analysis primarily develops a model for optimal capital for insurance losses, the methodology also applies to long-term asset risk.

## 1.1 Summary

The main result of this paper is that the optimal capital for an insurer with multi-period losses depends on *both* the volatility of losses for the current year and the volatility of the ultimate loss value. The ultimate loss volatility is a factor because, when an insurer becomes insolvent, it generally enters conservatorship and the losses will develop further, as if the insurer had remained solvent. This further development depends on the *ultimate* loss volatility. For a constant annual volatility, the optimal capital generally increases as the time horizon lengthens, but at a decreasing rate.

For a multiple-period time horizon, the amount of optimal capital depends on the same variables as for an insurer with a single-period horizon: the frictional cost of holding capital (primarily the cost of double-taxation), the degree of policyholder risk aversion, loss/asset volatility and guaranty fund participation. However, with multiple periods, optimal capital also depends on

[3] The fair value of unpaid claim liabilities is often treated as the best estimate of the unpaid claims plus a risk margin. Sections 2-7 treat liabilities as the best estimate of unpaid claims. The effect of risk margins is discussed in Section 8.

1. The underlying stochastic process for loss development; the horizon length is also a random variable.
2. What happens to unpaid losses when an insolvency occurs. In particular, conservatorship for an insolvent insurer has a strong effect.
3. The capital strategy used by the insurer. The ability to add capital when needed is particularly important.
4. The cost of raising external capital. In the case of some mutual insurers or privately-held insurers, the limitation on the *ability* to raise capital is a key factor.
5. The length of time between capital flows. The shorter this time frame, the less capital is needed.
6. The policy term. More capital is needed for a longer term, since if default occurs early in the term, the remaining coverage must be repurchased.
7. The interest rate. As the interest rate increases, less capital is necessary for default that will occur in the future.
8. The risk margin (or market price of risk) embedded in the premium. This amount acts as policyholder-supplied capital and reduces the amount of ownership capital needed.

As identified in items 3 through 5, optimal capital depends on how the insurer raises capital, its ability to do so and the cost of doing so. For most insurers, the best feasible strategy is to add capital when it will improve policyholder welfare, and withdraw capital when it becomes too expensive to carry it. This strategy (called AC in the paper) is conditional on the insurer remaining solvent. An alternative strategy (full recapitalization, or FR), adds capital even when the insurer is insolvent. Under FR, only the current-period loss volatility is considered and thus is implied in the Solvency II risk-based capital methodology. However, the FR strategy is not feasible, so the Solvency II method will understate risk-based capital for long-horizon losses.

Interestingly, the optimal capital for asset risk generally does not depend on the ultimate volatility of asset values. When an insurer becomes technically insolvent, asset risk is virtually eliminated, as a consequence of entering conservatorship (where the insurer's investments are replaced with low-risk securities). If the cost of raising external capital is low, the optimal capital will actually decrease as the horizon length increases.

## 1.2 Outline

The remainder of the paper is summarized below:

Key Results from the One-Period Model

Section 2 summarizes the results for a one-period model, showing how the cost of holding capital and the policyholder risk preferences will provide an optimal capital amount. Coupled with the insurer's capital strategy, the one-period optimal capital amounts will generate optimal capital for longer-duration losses spanning multiple periods.

Multi-Period Model Issues

Section 3 briefly discusses issues presented in a multi-period model that are not applicable to the one-period case. These issues are further addressed in subsequent sections. A key concept is the stochastic loss development process, wherein the estimate of the ultimate loss fluctuates randomly from period to period, with the current estimate being the mean of the ultimate loss distribution; this process determines expected default values in future periods. Another important issue is the impact of technical insolvency, where a regulator forces an insurer to cease operations when its assets are less than its liabilities; in this case, losses continue to develop after the insurer has defaulted. I describe capital funding strategies, which are necessary to address the period-to-period loss evolution. This section also discusses the distinction between ownership capital and policyholder-supplied capital; this issue is not relevant in a one-period model.

Basic Multi-period Model

Section 4 establishes a basic model of an insurer with multiple-period losses for liability insurance. I describe characteristics of the loss development stochastic process (with an asset process as well), including a parallel certainty-equivalent process needed to value the default from the policyholders' perspective. This section establishes a premium model, which allows the calculation of the value of the insurance contract to both policyholders and the insurer, and thus the optimum capital amount for both parties. I examine the distinction between ownership capital and total capital, which also includes policyholder-supplied capital. Also, I discuss capital funding strategies, where insurers attempt to add or withdraw capital to maintain an optimal position over time; the strategies vary according to efficiency (value to policyholders) and feasibility. I show that the most efficient feasible strategy is where capital is added if the insurer remains solvent; this is denoted as AC.

Optimal Two-period Capital

Section 5 determines the optimal capital for a two-period model under the AC strategy. Here I evaluate the certainty-equivalent value of default under technical insolvency, which is a key component of the analysis. This section introduces a stochastic loss process with normally distributed incremental development, as well as a similar lognormal process. The AC model is enhanced to incorporate an additional cost of providing capital from external sources.

Optimal Capital for More Than Two Periods

Section 6 extends the two-period model to multiple periods using backward induction. This procedure provides optimal initial capital for the various capital strategies.

Capitalization Interval and Policy Term

Section 7 examines the effect on optimal capital of the time required to recapitalize, or capitalization interval. It also shows how optimal capital depends on the policy term.

Extensions to the Multi-period Model

Section 8 extends the basic multi-period model to include features necessary for a practical application. I apply a stochastic horizon, where the loss development continues for a random length of time. Also, the analysis shows the effect of using present value and risk margins. The section concludes with a brief discussion of applying the methodology to life insurance.

Multi-Period Asset Risk

Section 9 determines optimal capital for asset risk. An important concept used here is that when an insurer ceases new business and comes under regulatory control, as in a technical insolvency, its assets are converted to essentially risk-free investments and the asset risk is essentially eliminated. Another fundamental notion is that the certainty-equivalent return on risky assets must equal a risk-free rate.

## 2. KEY RESULTS FROM THE ONE-PERIOD MODEL

This discussion shows how optimum capital is determined in a one-period model. More details can be found in EBRM.

### 2.1 Certainty Equivalent Losses

Since a policyholder is presumed to be risk-averse, the perceived value of each possible loss, or claim, amount is different from the nominal value. For a policyholder facing a random loss, the *certainty equivalent* (CE) value of the loss is the amount the policyholder is willing to pay in exchange for removing the risk of the loss. Let $L$ denote the expected value of the nominal loss and $p(x)$ the probability of loss size $x$. We assume that $p(x)$ has a non-zero variance. The expected value of the loss is $L = \int_0^\infty xp(x)dx$. The translation from nominal loss amounts to the CE value of the amounts can done using an adjusted probability distribution $\hat{p}(x)$:

$$\hat{L} = \int_0^\infty x\hat{p}(x)\,dx\,. \tag{2.11}$$

Here, $\hat{L}$ is the CE expected loss, with $\hat{L} > L$. The value of the default to the policyholder is called the *certainty-equivalent* expected default (CED) value and is denoted by $\hat{D}$. Its expression is parallel to that of the nominal expected default $D$:

$$\hat{D} = \int_A^\infty (x - A)\hat{p}(x)\,dx\,. \tag{2.12}$$

Here $A$ is the insurer's asset amount. We have $\hat{D} > D$; for asset values significantly greater than the mean loss $L$, the CED can be an extremely high multiple of the nominal expected default amount.

### 2.2 Consumer Value, Capital Costs and Premium

In purchasing insurance, the policyholder pays a premium $\pi$ in exchange for covering the loss. However, the coverage is only partial, since if the insurer becomes insolvent, only a portion of a loss (claim) is paid. Thus, the value $V$ of the insurance to the policyholder, or *consumer value*, equals the CE loss minus the premium minus the CED, or

$$V = \hat{L} - \pi - \hat{D} \,. \tag{2.23}$$

If $V > 0$, then the policyholder will buy the insurance.

In the basic model described in EBRM (see the assumptions in Section 4) the only costs to the insurer are the loss and the frictional cost of capital (FCC), denoted by $z$. The FCC is primarily income taxes, but may include principal-agent, regulatory restriction or other costs. Assuming that the capital cost is strictly proportional to the capital amount $C$, the premium is

$$\pi = L + zC \,. \tag{2.24}$$

Since adding capital reduces the CED but increases premium (through a higher capital cost), there generally will be an optimal level of capital that maximizes $V$ and therefore provides the greatest policyholder welfare. By taking the derivative of $V$ with respect to the asset amount $A$, we get the requirement for optimal assets, and therefore optimal capital:

$$\hat{Q}(A) = z \,. \tag{2.21}$$

Here $\hat{Q}(A)$ is the default, or ruin, probability under the adjusted probability $\hat{p}(x)$ and equals the negative derivative of $\hat{D}$ with respect to $A$.

Meanwhile, the insurer's owners are fairly compensated for the capital cost through the $zC$ component of the premium, so their welfare is also optimized. Since policyholder and shareholder welfare are maximized, this theoretical optimal capital level can form the basis for pricing, regulation and internal insurer governance.

Notice that if there were no prospect of the insurer's default and the cost of capital were zero, the consumer value of insurance would be the CE loss minus the expected loss, or $\hat{L} - L$. Call this amount the *risk value*. It is the maximum possible value that the policyholder could obtain by purchasing insurance. In the basic model, the prospect of default introduces the frictional capital cost and the CE expected default as costs that are subtracted from the risk value to produce the net consumer value. A useful term for the sum of these two costs is the *solvency cost*. Since the risk value is not a function of the insurer's assets (the model assumes riskless assets), minimizing the solvency cost is equivalent to maximizing the consumer value.

## 3. MULTI-PERIOD MODEL ISSUES

Determining optimal capital for multiple periods presents several challenges not evident in the one-period situation. These issues are introduced in subsections 3.1 through 3.5 and are addressed in greater depth in the subsequent sections 4 through 9.

### 3.1 Stochastic Loss Development

In the one-period case, the loss is initially unknown, but its value is revealed at the end of the period. For multiple periods, the loss value may remain unknown for several periods. Consequently, in order to establish the necessary capital amount for each period (using the accounting identity that capital equals assets minus liabilities), we need to estimate the ultimate loss; this assessment is known as the *loss reserve*. The reserve estimate will vary randomly from period to period until the loss is finally settled. The stochastic reserve estimates will form the basis for a dynamic capital strategy.

### 3.2 Default Definition and Liquidation Management

In a multi-period model, the loss reserve values are *estimates* of the ultimate unpaid loss liability. If the estimated loss exceeds the value of assets at the end of a period, the insurer is deemed to be *insolvent*, even though it is possible that the reserve may subsequently develop downward and there is ultimately no default. This condition is called a *technical insolvency*. If the insurer adds sufficient capital to regain solvency, then there is the further possibility that the insurer may yet again become insolvent in future periods, either technically or on a cash basis. Thus, multiple insolvencies are possible for a recapitalized individual insurer that emerges from insolvency.

Generally, in the U.S. or Europe, when an insurer becomes technically insolvent, regulators transfer its assets and liabilities to a *conservator*, or receiver, who manages them in the interests of the policyholders. This usually means that the assets are invested conservatively in low-risk securities[4] and claims are paid to ensure that each policyholder gets the same pro-rata share of the assets according to their claim amounts.

There are several important consequences to receivership. First, the liabilities remain "alive" and are allowed to develop further. Second, there is no source of additional capital to mitigate the ultimate default amount (however, no capital can be withdrawn either, unless the assets become significantly larger than the liabilities). Third, the conservative asset portfolio will most likely have a significantly reduced asset risk compared to that of the insurer prior to conservatorship. These features profoundly affect the multi-period capital analysis, as shown in the subsequent sections.

## 3.3 Dynamic Capital Strategy

In a one-period model the capital is determined once, at the beginning of the period. In a multi-period model, capital is also determined initially, but it also must be determined at the beginning of each subsequent period. Thus, the capital-setting process is a predetermined strategy. This strategy is dynamic: the subsequent capital amounts will depend on the values of the assets and of the insurer liabilities as they evolve. Even though the capital strategy is dynamic, there will be an optimal starting capital amount. Also, for each strategy there will be a distinct *expected* amount of capital at the beginning of each subsequent period.

## 3.4 Capital Funding

Since there is a cost to the insurer for holding capital, the insurer must be compensated for this cost. This cost is included in the premium. In a one-period model, the premium is paid up front and the loss is paid at the end of the period; there is no need to consider subsequent capital contributions. In a multi-period model, the liability estimate may increase over time, leaving the insurer's assets insufficient to adequately protect against insolvency. In such an event, the policyholders will be better off if the insurer's shareholders contribute additional capital. However, the insurer will be worse

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[4] For example, the state of California uses an investment pool for its domiciled insurers in liquidation. The pool contains only investment grade fixed income securities with duration less than 3 years (see California Liquidation Office 2012 Annual Report). New York is more conservative: funds are held in short-term mutual funds containing only U.S. Treasury or agency securities with maturities under 5 years (see New York Liquidation Bureau 2012 Annual Report).

off due to the added capital cost. Nevertheless, if the premium includes the cost of additional capital funding, consistent with a particular funding strategy, it is economically practical for the insurer to make the capital contribution. Conversely, if the loss reserve decreases, it may be mutually beneficial for the insurer to remove some capital, consistent with the capital funding strategy.

For an ongoing insurer, there is an incentive to add capital as needed, since failure to do so may jeopardize the ability to acquire new business or renew existing policies. However, if technical insolvency occurs, it may not be feasible for the shareholders to add capital, since the prospect of a fair return on the capital may be dim. Thus, there are some limitations on capital additions. For a true runoff insurer, however, there is no incentive to add capital, so capital can only be withdrawn (which may occur if allowed by regulators).

### 3.5 Capital Definition

In a multi-period model, the premium will include the expected frictional cost of capital, as it does in the one-period case. However, at the end of the first period, only the first-period capital cost is expended for the multi-period model, and so the balance becomes an asset that is available to pay losses. This premium component thus can be considered as *policyholder-supplied* capital, since serves to mitigate default in exactly the same way as the owner-supplied capital in the one-period model. Similarly, if the premium contains a provision for the insurer's cost of bearing risk (a risk margin), that amount will also function as capital. Section 4.4 discusses the distinction between ownership capital and policyholder-supplied capital. Section 8.3 develops optimal capital with a risk margin.

## 4. BASIC MULTI-PERIOD MODEL

This section extends the one-period model to $N$ periods and discusses some important differences between the two cases. The basic model is designed to contain a minimal set of features that will allow the optimal capital calculation. Other features, which may be necessary for practical applications, are discussed in sections 5 through 8.

The basic multi-period model follows a specific cohort of policies insuring losses that occur at the start of the first period and which are settled at the end of the $N$th period.

The model assumes that the insurer is ongoing, so that other similar policies are added at the beginning of the other periods. The basic model does not track these other policies; however, they provide an incentive to add more capital to support the basic model cohort, if necessary.

## 4.1. Model Description and Assumptions

I start by adopting the basic assumptions of the one-period model, as developed in EBRM, and modifying some of them to fit the requirements of the multi-period model, as indicated below.

(1) Policyholders are risk averse and have homogeneous risk preferences. Thus, the certainty equivalent value of a particular loss amount is identical for each policyholder.

(2) There are no expenses (administrative costs, commissions, etc.). The only relevant costs are the frictional capital costs and the losses. These costs determine the premium.

(3) The cash flows for premium and the initial capital contribution occur at the beginning of the first period. The frictional capital cost is expended at the end of each period (before the loss is paid).[5] The entire loss is paid at the end of the *last* period. Other capital contributions or withdrawals may occur at the beginning of each subsequent period, depending on the chosen capital strategy.

(4) The interest rate is zero. This simplification makes the exposition less cluttered (since the nominal values equal present values) and does not affect the key results. Section 8.2 provides results with a positive interest rate.

(5) Losses represent pure gambles (having no correlation with economic factors) and have no risk margin. Thus, since the investment return is also zero, the expected return on owner-supplied capital is also zero.[6] Section 8.3 analyzes results with a risk margin.

(6) The frictional capital cost rate is $z \geq 0$. If capital becomes negative (a technical insolvency occurs), the FCC is zero.

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[5] I chose this assumption to be consistent with the one-period model in EBRM. For the one-period model, this assumption avoids the issue of policyholder-supplied capital vs. ownership capital. Also, if the loss is paid *before* the capital cost is expended, the optimal capital is determined from $\hat{Q}(A) = z \,/\, (1+z)$, instead of $\hat{Q}(A) = z$, which is a simpler result that gives approximately the same optimal capital.

[6] This is a standard financial economics assumption. Including a risk margin (discussed in section 8.3) will provide a positive expected return.

(7) There is no cost to raising external capital (section 5.4 develops results that include this cost).

(8) There is no guaranty fund or other secondary source of default protection for policyholders. The only insolvency protection for policyholders is the assets held by the insurer.

Additionally, we require some assumptions specific to the multi-period case that do not apply to a one-period model:

(1) The ultimate loss is not necessarily known when the policy is issued, but is definitely known at the end of the $N$th period (or sooner). This situation requires an intermediate estimate (the reserve amount) of the ultimate loss at each prior period. The reserve value is unbiased: it equals the expected value of the ultimate loss.

(2) The premium includes the *expected* FCC, since under a dynamic capital strategy, the capital amounts in future periods will depend on the random loss valuation and thus are also random. The expected capital cost excludes the possibility that the insurer becomes insolvent. Section 5.21 discusses this further.

(3) A capital strategy is used, wherein for each possible pair of loss and asset values at the end of each period, the insurer will add or withdraw a predetermined amount of capital.

(4) The policy term is one period. Section 7.4 discusses the case where the term is longer than a single period.

Since the certainty-equivalent value of losses and related expected default amounts are assessed from the perspective of each individual policyholder, we scale the insurer model to portray each policyholder's *share* of the results. Therefore, it is useful to consider the model as representing an insurer with only a *single* policyholder.

In the multi-period model with $N$ periods, variables that have a time element are generally indexed by a subscript denoting the particular period as time moves forward. The index begins at 1 for the first period and ends at $N$ for the last period. Balance sheet quantities such as assets and capital are valued at either the beginning or end of the period, depending on the context. For example, $C_1$ represents capital at the beginning of the first period and $A_1$ denotes the assets for the first period after the capital cost is expended. For simplicity, I drop the subscript for the first period where the situation permits.

When developing optimal capital with backward induction (section 6) the index represents the number of *remaining* periods: e.g., $C_3$ denotes the initial ownership capital for a three-period model.

Optimal values are represented by a asterisk (e.g., $C^*$), certainty-equivalent quantities by a carat (e.g., $\hat{D}$), market values (used in risk margins) by a bar (e.g., $\bar{L}$) and random values by a tilde (e.g., $\tilde{C}$).

Note that under this simplified model, it is not necessary to distinguish between *underwriting* risk (the risk arising from losses on premiums yet unearned) and *reserve* risk (the risk arising from development of losses already incurred from prior-written premiums).

## 4.2 Stochastic Process for Losses and Assets

To analyze capital requirements, it is useful to categorize property-casualty losses into two idealized types, which are approximate versions of real-world processes. The first loss type is *short-duration*, e.g., property, which has no lag between its estimated value when incurred and when ultimately settled. Further, the value of a loss incurred in one period does not affect the amount in another period. The second type is *long-duration*, e.g., liability coverage, where the lag is at least one period, and the value in a subsequent period (of a loss incurred in a particular period) will depend on the value in the earlier period.

For analyzing capital under the section 4.1 basic model, short-duration losses are one period, since the loss value cannot carry over to a subsequent period. Also, the expected value of losses in a subsequent period is independent of losses occurring in an earlier period. Since the per policy mean loss (adjusted for inflation) does not change much over time, property losses generally follow a *stationary* stochastic process. Since short-duration losses under the basic model can be considered one-period,[7] determining optimal capital is straightforward (see section 2), and so I turn to liability losses.

[7] An exception is where the policy term is more than one period. This case is discussed in section 7.4.

**4.21 Long-Duration Loss Stochastic Process**

Under a one-period model, the expected loss is $L$, which is a component of the premium. With a multi-period model, we use the same notation for the initial loss estimate. However, there will be intermediate reserve estimates $\{L_1, L_2, \cdots, L_{N-1}\}$ at the end of the periods 1 through $N-1$. The realized value of the ultimate loss is denoted by $L_N$. Because we have assumed that the reserve estimates are unbiased, each reserve value $L_t$ is the *mean* of the possible values for the next reserve estimate $L_{t+1}$. In other words, the difference $X_{t+1} = L_{t+1} - L_t$, or the *reserve increment*, has a zero mean. The sequence of reserve estimates is a *random walk*, which is a type of Markov process.[8] In a Markov process the future evolution of the value of a variable does not depend on the history of the prior values. In other words, conditional on the present reserve value, its future and past are independent. There cannot be a correlation between successive reserve amounts if the estimates are unbiased. The normal loss model in section 5.31 is an example of this stochastic process, which is an *additive* model since the increments are summed to determine successive values.

An alternative stochastic process that may characterize loss evolution is a *multiplicative* model. Here we define $Y_{t+1} = L_{t+1} / L_t$, which has a mean of 1 for all $t$. The product of the multiplicative random $Y_t$ factors and the initial loss estimate $L$ will give the ultimate loss value $L_N$. The lognormal loss model in section 5.32 is an example of this stochastic process. Notice that $\ln(Y_{t+1}) = \ln(L_{t+1}) - \ln(L_t)$, which is an additive random walk with a zero mean as described above.

We assume that the range of the possible $X_t$ values has the same probability distribution for all time values $t$. Also assume that the *variance* of $X_t$ (denoted by $\sigma^2$) is constant per period.[9] Also assume the same regularity for the lognormal distribution.

[8] See Bharucha-Reid[1960].

[9] This assumption can be modified to provide a specific variance for each period, as will be necessary for practical applications. The actual distribution may vary according to the elapsed claim duration. For example, the long discovery (or IBNR) phase for high-deductible claims will imply a low variance for the reserve estimates for the first few years. Scant information regarding the claims arrives over this time span, so there is little basis to revise the initial reserve.

Notice that the variance of the ultimate loss $L_N$ is the sum of the variances of the $X_t$ sequence, or $N\sigma^2$. There is no covariance between any of the reserve increments due to the memory-less property of the Markov process (a non-zero correlation would imply that the prior reserve history could help predict the future reserve values). The $X_t$ variance exists because the flow of information (positive and negative) regarding the ultimate loss value is random. The subsequent estimates of ultimate value are determined by information that becomes revealed over time, such as how many claims have occurred, the nature of the claims, the legal environment, inflation and so forth.

**4.22 Certainty-Equivalent Stochastic Process**

We also assume that the certainty equivalent loss values evolve according to a stochastic process parallel to that of the underlying losses. Generally, the risk value embedded in the CE losses is approximately proportional[10] to the loss variance. Thus, to model the CE loss process, I assume that the relationship is exact. For an additive process, the CE loss at the end of *N* periods is then

$$\hat{L}_N = L + aN\sigma^2 , \qquad (4.221)$$

where $a$ is a constant that indicates the degree of risk aversion. Therefore, the CE expected loss increases each period by the risk value $a\sigma^2$. Since the CE loss mean increases linearly with the time horizon, we can create a parallel CE stochastic process by keeping the variance for each period at $\sigma^2$ and adjusting the mean to satisfy equation 4.221.

Although the mean of the evolved loss for each successive period will change, the variance does not, so the CE value of the remaining loss will be the new conditional unadjusted mean plus $a\sigma^2$ times the number of remaining periods. Notice that equation 4.221 represents the CE expected loss value with *N* periods remaining; as the loss evolves there will be fewer periods left and the risk value will diminish (it will be zero when the loss is settled).

Appendix A provides a numerical example illustrating the stochastic process for long-duration losses. It includes both unadjusted and certainty-equivalent results.

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[10] For some distributions and risk aversion combinations, the relationship is exact (such as with exponential expected utility). See Panjer *et al* [1988], page 137.

**4.23 Asset Stochastic Process**

Asset values over time will also vary according to a stochastic process. Indeed, stochastic processes have seen wide use in finance to model both equity (stock) and bond values. The well-known Black-Scholes option pricing model uses a lognormal stochastic model wherein the log of stock price changes are independent increments. The option price is found using an adjusted (risk-neutral) probability distribution,[11] which is similar to the CE distribution used here. The mean of the adjusted distribution is obtained by shifting the mean of the underlying normal distribution.

As further discussed in section 9, the CE expected asset return for the policyholder must equal the *risk-free* interest rate if the policyholder has the same risk aversion as the average investor. Otherwise the CE value of the asset will differ from the value for which it can be purchased for certain,[12] which, by definition, is contradictory.

## 4.3 Premium and Balance Sheet Model

Following the one-period model, the premium for the multi-period case equals $L$ plus the expected capital cost. However, the capital for each period after the initial period will be determined by the evolving loss estimate, so it also will be a random variable. Consequently, the capital cost component of the premium will be the *expected value* of the sequence of capital costs. Let $C$ denote the ownership capital contributed initially (here I drop the subscript 1 for the first period). For a specific capital funding approach, under an $N$-period model, let $\tilde{C}_i$ be the capital amount at the beginning of the $i$th period.

I assume that the frictional capital cost is proportional to the *ownership* capital at the rate $z$. The expected capital cost for all periods is then $K = zC + E[\tilde{C}_2 + \tilde{C}_3 + \cdots + \tilde{C}_N]$. Then the fair premium equals $\pi = L + K$, which has the same form as in the one-period case.

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[11] See Hull [2008] (pages 307-309).

[12] To illustrate this, suppose that a risky investment of \$100 has an expected return of 10% in one year, while a riskless investment will yield 4%. Since the value of the riskless investment is \$104 for certain after one year, the certainty-equivalent value of the risky investment at one year must also equal \$104. The CE value now of the future CE value is the future value discounted back at the riskless rate, or \$100. This equals the amount originally invested.

The expected value of the future capital amount or of the capital cost should be calculated using *unadjusted* probabilities, since the capital costs to the insurer do not depend on its policyholders' risk preferences. Also, the insurer is already compensated for the risk it bears through the risk margin built into the premium (although the risk margin is zero here in the basic model, a more general model, such as in section 8.3, will include it).

This premium model forms the basis for pricing methods that use the present value of expected future costs and whose losses have embedded risk margins (see sections 8.2 and 8.3). The present value of the expected capital costs is determined by discounting them at a risk-free rate.

When the policies are written, the initial assets equal the owner-contributed capital plus the premium, or $C+\pi=C+L+K$. These assets are cash in the basic model. The liabilities are the expected losses (an obligation to the policyholders) the expected capital cost (another third-party obligation[13]), and the ownership capital, which is the residual of assets minus the obligations to other parties. At the end of the first period, before the loss is paid, the capital cost is expended, leaving the amount of assets available to pay losses, denoted by $A$, as

$$A=C+L+K-zC\,. \tag{4.31}$$

## 4.4 Ownership Capital and Total Capital

For the basic one-period model, the capital definition is straightforward. At the beginning of the period, the insurer's owners supply a capital amount $C$, and the policyholders supply the premium, equal to $L+zC$. Since the capital cost amount $zC$ is expended before the loss is paid, the amount of assets available to pay the loss is $A=L+C$.

For a multi-period model, however, the amount available to pay losses after the first period is *greater* than $L+C$ by the amount $K-zC>0$, which represents the expected

[13] As discussed in EBRM, essentially this amount will be an income tax liability.

capital cost for the remaining periods. Since this additional amount reduces default in exactly the same way as the owner-supplied capital, it may be considered as *policyholder-supplied* capital. Therefore it is useful to define the *total capital* as the available assets minus the expected loss, which for the basic model is

$$T = C + K - zC \,. \tag{4.41}$$

Notice that for a one-period model, we have $T = C$, and for two or more periods, $T > C$.

It is important to measure the ownership capital consistent with the premium determination. Here I use *fair-value* (also known as mark-to market) accounting, where the value of obligations is the amount they would be worth in a fair market exchange.[14] From equation 4.41 it is simple to determine the fair-value capital from the total capital and vice-versa. For brevity, I abbreviate ownership capital to OC.

With a risk margin, discussed in section 8.3, we have a similar situation: the risk margin compensates the insurer for bearing risk and is a premium component in addition to the expected loss. Like the unexpended expected capital cost, it provides additional default protection. However, in fair-value accounting, the risk margin is not considered as OC.

For the subsequent sections, I present most results using *the total capital* definition. Where appropriate, I show the OC for comparison.

## 4.5 Capital Funding Strategies

In order to determine the expected capital cost, we need to know how much capital will be used for each period. As discussed in section 3.4, the amount will depend on the loss amount at the end of the prior period: if the amount is large, it may be necessary to add capital; if the amount is small enough, capital might be withdrawn. Define a

[14] An important property of fair value accounting is that, if the product is fairly priced (so that its components are priced at market values), there is no profit generated when the product or service is sold. Instead, the profit is earned smoothly over time as the firm's costs of production or service provision are amortized. For an insurer, this means that the profit will emerge as the risk of loss is borne.

*capital funding strategy* as a set of rules that assigns a specific (but not necessarily unique) amount of capital to the beginning of each period, corresponding to each possible loss value at the end of the prior period.

There are several basic capital funding strategies that an insurer might use. I describe the most relevant ones below, starting from the least to the most dynamic method.

*Fixed Assets* (FA): under this approach, the insurer's owners supply an initial capital amount, with no subsequent capital flows until the losses are fully settled. Thus, the initial assets remain constant until the losses are paid. Note that the capital amount will vary over time, since loss estimates will fluctuate and the capital equals assets minus liabilities. This method is used in Lowe *et al* [2011] to determine capital for a runoff capital model. Although it is viable for a true runoff insurer, it will not be for an ongoing insurer, whose capital level must respond dynamically to the level of its loss liabilities. For example, if an insurer's losses develop favorably, causing its capital level to increase above a reasonable solvency level, then the insurer will usually reduce its capital amount.

*Capital Withdrawal Only* (CW): with this strategy, capital is withdrawn if the asset level becomes high relative to the losses (and therefore satisfying a particular target leverage ratio). A common method for withdrawing capital is through dividends to shareholders. [15] However, no capital is added if assets become lower than the target level. Except possibly for some mutual insurers, this method also does not represent actual practice, where, within limits, insurers will add capital if its leverage ratio drops below the norm.

*Add Capital if Solvent* (AC): here, capital is withdrawn if a particular solvency level is reached, and capital is added if assets are below the solvency level. However, if the insurer becomes technically insolvent, then no capital is added. In this event, the insurer usually is taken over by a conservator. The incentive for shareholders to fund capital additions comes from the prospect of adding new business, which is difficult to accomplish without adequate capital. Note that a threshold *below* insolvency might be

[15] For a mutual insurer, the dividends will go to policyholders, who are the insurer's owners and therefore serve as shareholders. A mutual insurer's dividends can also be used as part of its pricing strategy.

used in the event that shareholders consider the franchise value of the insurer to be valuable enough. However, the results of this assumption would be analytically similar to using a solvency threshold. The main point here is that there is an upper limit beyond which the capital is no longer added.

A variation of the AC strategy, discussed in section 5.4, is where there is a cost to raising capital externally. I have labeled this strategy as ACR.

*Full Recapitalization* (FR): this approach is similar to AC, but the insurer, even if technically insolvent, will add sufficient capital to regain the target solvency level. However, in order to provide an adequate incentive for the shareholders to provide capital if the insurer is insolvent, the policyholders accept a cash settlement of their claims equal to the assets. The insurer (or a different insurer) then agrees to insure the loss liability and the policyholders pay a new premium for the reinstated coverage. The insurer's owners then provide adequate capital for the insurance. This transaction, in effect, converts the technical insolvency into a cash or "hard" insolvency. Thus, it is possible for the insurer to default multiple times before the loss is settled. As discussed in section 5.2, the FR approach is theoretically superior to the other three methods in that it provides the highest consumer value for the insurance coverage. However, it is not practically feasible: normally, the policyholders will enter receivership rather than take back their liabilities and insure them again with a different insurer.[16]

Other strategies, such as only adding capital, are possible. However, I have included only the strategies that are used in practice or which illustrate important concepts.

Let $T^*_{t+1}(L_t)$ represent the required total capital amount at the beginning of period $t+1$ given that the value of the loss at the end of period $t$ is $L_t$. Thus the required assets at the beginning of period $t+1$ are $L_t + T^*_{t+1}(L_t)$, and the indicated capital flow (i.e., addition or withdrawal) is the required assets minus the prior-period assets:

[16] One huge impediment to practically applying the FR method is that the insurer and the policyholders may have different opinions on the value of the loss reserve estimate. Another impediment is that this capital funding method also requires either that policyholders *without claims* contribute enough to pay for their possible future incurred-but-not-reported (IBNR) claims or for the IBNR reserve to be divided among the existing claimants.

$$CF_t = L_t + T^*_{t+1}(L_t) - A_t \,. \tag{4.51}$$

The above four capital funding strategies, plus the ACR variant, can be characterized by the regions of $L_t$ for which the indicated capital flow $CF_{t+1}$ is permitted. The first region is $A_t < L_t$, where the insurer is technically insolvent. The second is $[A_t - T^*_{t+1}(L_t)] < L_t < A_t$, where the insurer is solvent and capital can either remain the same[17] or increase if permitted. The third is $L_t < [A_t - T^*_{t+1}(L_t)]$, where capital is withdrawn if permitted.

To illustrate, assume that the required total capital is 600 and is independent of the loss value (i.e., it depends only on the variance, as under the normal distribution). Prior assets are 1400, so there is potentially 800 in excess assets. Region 1 has losses exceeding 1400, region 2 has losses between 800 and 1400 and Region 3 contains losses less than 800. For region 1, capital is added only for FR. For region 2, capital is added for AC and FR. For region 3, capital is withdrawn for all funding strategies except FA.

Table 4.51 summarizes the capital flows permitted by the different capital strategies. A minus indicates a withdrawal, a plus represents an addition and a zero indicates that capital can remain the same.

*Table 4.51*
*Permissible Capital Flows by Capital Strategy*

| | FA | CW | AC | FR |
|---|---|---|---|---|
| Region 1 | 0 | 0 | 0 | $CF_t$ (+) |
| Region 2 | 0 | 0 | $CF_t$ (0, +) | $CF_t$ (0, +) |
| Region 3 | 0 | $CF_t$ (−) | $CF_t$ (−) | $CF_t$ (−) |

Each of these strategies may have a different expected capital cost and therefore the premium will depend on the strategy used. Notice that after the initial capital $C_1$ is

[17] Under the ACR strategy and the FR strategy with a capital-raising cost, there may be a sub-region of Region 2, bordering on Region 3, where capital remains the same. As shown in section 5.4, due to the cost of raising capital, it will be sub-optimal to add capital in this region, and also sub-optimal to withdraw it.

established, the chosen strategy will produce a unique sequence of subsequent capital amounts corresponding to the sequence of loss estimates.

Since the insurer is fairly compensated up front for its capital costs, the capital suppliers (shareholders) will provide whatever capital amount (both for initial and subsequent periods) is desired by the policyholders. This also means that the investors are indifferent to the capital *strategy* desired by the policyholders. Therefore, for each capital strategy, we can determine the initial capital amount that maximizes the policyholder's consumer value. Then the strategy with highest consumer value (or the lowest solvency cost) is the optimal strategy and can be used to determine capital for similar types of insurance.

## 4.6 Efficiency and Feasibility of Capital Funding Strategies

Assume a two-period model and that initial assets for each strategy are fixed at $A_1$. At the end of the first period, whatever the loss estimate $L_1$, there is a single period remaining. We already know how to find the optimal capital for one period. Defining the required total capital in section 4.5 as the optimal capital, the optimal capital for the beginning of period 2 is $T_2^*(L_1)$. Thus, if the actual capital $T_1$ exceeds $T_2^*(L_1)$, the additional capital cost (from carrying the capital into the second period) will be greater than the reduction in the CE expected default value for the second period (by definition of the optimal capital), so policyholders will gain by a capital withdrawal to attain optimal capital. Note that this situation occurs in region 3 of Table 4.51. Consequently, CW is a superior strategy to FA, which we can represent as CW > FA.

A similar argument shows that AC > CW. If the loss estimate is between initial assets minus $T_2^*(L_1)$ and initial assets (region 2), increasing capital will increase the capital cost less than it changes the CED value. In parallel fashion, we have FR > AC.

However, as discussed in section 4.5, FR is not feasible in practice. AC is feasible for most insurers and CW, although feasible, is less efficient than AC. So it is not a good choice unless it is not possible to raise capital externally. Therefore it is important to determine optimal capital under the AC strategy. Nevertheless, it is informative to compare results between the different strategies. In particular, the FR strategy provides an important baseline, since it produces the highest consumer value and thus theoretically is the most efficient strategy. It also has the important feature that it converts a multi-period model into a *series of one-period models*.

Because of the single-period conversion property of the FR strategy, the required adjusted probability distributions can be analytically tractable, and it is relatively easy to calculate the optimal capital for the start of each period. This is usually not the case for the AC and CW strategies.

## 5. OPTIMAL TWO-PERIOD CAPITAL

In order to determine multi-period optimal capital, it is necessary to start with one period and extend the result to two periods. From that point, we extend the result to additional periods. In the two-period exercise, we gain valuable insight regarding multi-period capital dynamics. The two-period results are readily extended to additional periods in section 6 using backward induction. The results here in section 5 use an example with a normal or lognormal stochastic loss process. However, I also describe the general method to derive optimal capital for other stochastic processes.

First, I address the simple case where there is no cost to raising capital from external sources. Then, in section 5.4, I introduce a cost of raising capital and show how this changes the AC and FR optimal capital.

### 5.1 Expected Default with the AC Strategy

An important constraint in modeling capital for multi-period losses is that a technical insolvency normally forces an insurer into conservatorship. From section 3.2, this event means that losses will continue to develop while assets remain fixed until the losses are settled. Here I assume that the insurer enters conservatorship *immediately* when the technical insolvency occurs at the end of a particular period.

Conservatorship adds another dimension to the CE expected default calculation that is absent for a one-period model. To illustrate this, I approximate a normal stochastic loss process using a discrete probability distribution for the independent loss increments. This numerical example is shown in Appendix A.

From section 2, the CED for a one-period loss is denoted by $\hat{D}$. Define $\hat{G}$ as the unconditional ultimate CED for an insurer entering technical insolvency at the end of the current, or first period. For a discrete loss process, such as used here, let $x_i$ for

$i = 1, \cdots, n$ denote each possible value of the first-period loss $L_1$ that exceeds initial assets. Let $\hat{p}(x_i)$ represent the certainty equivalent probability that $x_i$ occurs and $\hat{D}_2(x_i)$ the CE expected second-period default given $x_i$. The CE expected default due to a technical insolvency is therefore

$$\hat{G} = \hat{p}_1(x_1)\hat{D}_2(x_1) + \hat{p}_2(x_2)\hat{D}_2(x_2) + \cdots + \hat{p}_n(x_n)\hat{D}_2(x_n)\,. \qquad (5.11)$$

Using this framework, in the above numerical example (see Appendix A) we have $x_1$ = 1350, $x_2$ = 1400, $\hat{p}(x_1)$ = 0.00520, $\hat{p}(x_2)$ = 0.00054, $\hat{D}_2(x_1)$ = 152.59 and $\hat{D}_2(x_2)$ = 200.72. Thus, $\hat{G}$ = 0.9029 = 0.00054(200.72) + 0.00520(152.59).

Assuming a non-zero variance of loss development beyond the first period, the value of $\hat{G}$ (0.9029 in the example) will always be greater than that of $\hat{D}$ (0.3144 in the example), as discussed in Appendix A. Therefore, the optimal capital will also be greater under the AC strategy than for the FR strategy.

Observe also that $\hat{G}$ depends on the variance of loss development *beyond* the first period (i.e., the ultimate variance), while $\hat{D}$ only depends on volatility *during* the first period.

## 5.2 Optimal Two-period AC Capital

A particular value of initial assets will establish the assets $A$ available to pay the loss at the end of the first period (equation 4.31) this amount will thus uniquely determine the CE expected default for the first period as discussed in section 5.1. The amount $A$ will also uniquely determine the CED for the second period since the capital strategy is predetermined. The total CE expected default for the insurer is the sum of the CED values for the first and second periods.

From section 5.1, $\hat{G}$ is the CE expected default for the first period (under technical insolvency). For a continuous distribution of losses, with $x$ denoting the first period loss value, the equivalent of equation 5.11 is

$$\hat{G} = \int_A^{\infty} \hat{p}(x)\hat{D}_2(x)dx\,. \qquad (5.211)$$

If the insurer remains solvent at the end of the first period, there is one period remaining: it can become insolvent at the end of the second period. However, from section 2, for each loss value there is an optimal amount of capital and a corresponding optimal CED amount, represented by $\hat{D}^*(x)$. The insurer will add or withdraw capital to reach the optimal beginning second-period capital. The unconditional expected default in the second period if the insurer stays solvent for the first period is then

$$\hat{H} = \int_0^A \hat{p}(x)\hat{D}^*(x)\,dx\,. \tag{5.212}$$

In words, $\hat{H}$, the CED for the second period, is the sum of the optimal one-period CED for each first-period loss value less than the asset amount, weighted by the CE probability of the loss value. Observe that the limits of integration span loss amounts from 0 to $A$, while the $\hat{G}$ limits span amounts greater than $A$. Consequently, the insurer's total CE expected default for both periods is $\hat{G}+\hat{H}$.

From section 4.3, the premium for a multi-period loss coverage is $\pi = L + K$, where $K$ is the expected total capital cost for all periods. For two periods, the expected amount of ownership capital used is the initial first period OC (a fixed amount) plus the expected second-period initial OC (a random amount determined by the first-period loss). Let $C_2^*(x)$ be the optimal second-period initial OC given that $L_1 = x$. If the insurer is technically insolvent after one period (i.e., $L_1 > A$), there is no capital used at the start of second period. If the insurer remains solvent, under the AC strategy the second-period initial OC is the optimal OC for a one-period insurer with expected loss $L_1$. Therefore we have

$$K = zC + z\int_0^\infty p(x)C_2^*(x)dx\,. \tag{5.213}$$

Here $p(x)$ is the *unadjusted* probability of loss, since we assume that the insurer will incorporate the actual expected amount of capital into the premium.[18] The consumer

[18] The capital is not adjusted for risk in the basic model, since the expected return to the insurer is zero.

value of the insurance transaction is then $V = \hat{L} - \pi - \hat{G} - \hat{H}$. The optimal initial available asset value is found by maximizing $V$, or alternatively, minimizing the *solvency cost*

$$S = \hat{G} + \hat{H} + K \, . \tag{5.214}$$

Because $\hat{G}$ is not analytically tractable for important probability distributions such as the normal, we need to use numerical approximation methods to find the optimal capital in these cases. Once the optimal assets are found, we use equation 4.31 to determine the optimal capital. Section 5.3 outlines an approach for the normal and lognormal stochastic processes.

For the FR strategy, the insurer is recapitalized at the end of the first period to the optimal second-period amount. So, viewed from the beginning of the first period, the solvency cost for the second-period is the optimal amount for that period as if we had just begun that period. Therefore, the initial capital for the first period is independent of the second period loss distribution, and depends only on the potential loss values for the first period.

Section 5.1 showed that, for a given initial asset level, the CE expected default for the AC strategy is higher than that for the FR strategy. This implies that the optimal initial total capital for the AC strategy is *higher* than for the FR strategy, which is the theoretically most efficient strategy. This result is reflected in the section 5.3 numerical examples with the normal and lognormal stochastic loss processes.

To prove this result, assume that we use an AC strategy, but the initial total capital is the optimal total capital for an FR strategy. The AC certainty-equivalent default $\hat{G}$ is greater than the optimal CED under FR. Also, the derivative $\partial\hat{G} / \partial A$ is a weighted average of the $\hat{Q}$ values for losses greater than $A$. Each of the component $\hat{Q}$ values in the weighted average is higher than $z$, so adding capital at the margin will reduce $\hat{G}$ more than it will increase the capital cost. Consequently, the optimal AC total capital will be greater than the optimal FR total capital[19] and the optimal AC solvency cost

[19] Since the premium contains the expected capital cost for both periods, the optimal first-period FV capital equals the optimal capital for a one-period model, less the expected capital cost for the second

will be greater as well.

## 5.3 Optimal Two-period Capital for Normal Stochastic Processes

Because it is meant to provide a simple illustration of the effects of capital strategies, the example described in section 5.1 and evaluated numerically in Appendix A, is highly unrealistic. The present section primarily uses the normal stochastic process (from section 4.2) whose period-ending loss distribution is normal. This distribution is continuous, and more closely represents actual loss development. I also discuss the lognormal loss process, which although multiplicative, gives similar results. The results here are intended to illustrate the general method for determining optimal capital; a practical application may involve more complex modeling.

A useful technique for creating an adjusted probability distribution that satisfies the requirements for certainty-equivalent values is to shift the mean of the unadjusted distribution,[20] in proportion to a risk-aversion parameter, as outlined in section 4.2. The shifted mean becomes the CE expected loss for the evolving loss until ultimate settlement. As the variance of the developing loss increases with additional time periods (viewed from the beginning of the first period), the CE value of the loss estimate will grow, in proportion to the cumulative loss variance.

With a normal loss process, the optimal capital and CED for one period are constants *independent* of the expected loss (but are a function of the standard deviation). However, for a lognormal process, the one-period optimal capital and CED are *proportional* to the expected loss. These properties facilitate the calculation of optimal capital for two or more periods.

### 5.31 Optimal Two-period Capital for the Normal Process

Appendix A develops a numerical example to illustrate optimal capital under the normal stochastic loss process. This example is further used to illustrate results for

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period. Essentially, in this case, compared to the one-period model, the policyholder has prepaid the second-period capital cost, so the optimal initial capital is *less than* in the one-period model by the amount of the prepayment.

[20] The Black-Scholes option pricing model can be derived using a mean shift. Here the expected future value of the underlying asset under the risk-neutral distribution has a shifted mean equal to the initial asset value and not its true expectation. See Panjer [1998], page 168. Using a mean shift for the lognormal distribution implies a power utility function.

subsequent sections of the paper.

The example uses a two-period normal stochastic loss process with a mean of 1000 and variance of the loss increment equal to $100^2$ for each period. The CE of the expected loss after one period is 1100 and the risk value (the CE of the loss minus its expected value) at each development stage is strictly proportional to the cumulative variance as in section 4.22. Thus, the CE value of the ultimate loss at the end of the second period is 1200. At each stage of loss development the CE loss is normally distributed with the same variance as the unadjusted distribution, but with the adjusted mean equal to the CE value of the evolved loss.

The frictional capital cost is $z = 2\%$. The optimal one-period total capital is 305.37 and the optimal two-period initial total capital is 357.67.

Table 5.311 summarizes the optimal AC results. Here I compare the optimal two-period AC strategy with that of the optimal FR strategy. The table also shows results for the AC strategy using the optimal FR initial total capital as the initial capital for the AC strategy.

*Table 5.311*
*Optimal AC and FR Strategy Comparison*
*Normal Stochastic Process Example*

| Strategy | Initial Total Capital | 1st Period CE Default Probability | 1st Period CED | 2nd Period CED | 1st Period Capital Cost | 2nd Period Capital Cost | Solvency Cost |
|---|---|---|---|---|---|---|---|
| FR Optimal | 305.37 | 0.0200 | 0.7343 | 0.7343 | 5.9852 | 6.1075 | 13.5613 |
| AC Using FR | 305.37 | 0.0200 | 2.8274 | 0.7196 | 5.9852 | 6.1075 | 15.6398 |
| AC Optimal | 357.67 | 0.0050 | 0.6829 | 0.7307 | 7.0312 | 6.1075 | 14.5507 |

Notice that the optimal solvency cost for the AC strategy has a lower CED (1.4136) than does the FR strategy (1.4686). However, the AC capital cost is larger, giving a higher AC optimal solvency cost.

**5.32 Optimal Two-period Capital for the Lognormal Distribution**

To further illustrate the optimal AC capital, I turn to a *lognormal* stochastic process, which is more realistic than a normal model. Loss development tends to follow claims inflation, which is a multiplicative process, rather than an additive one. However, for the lognormal distribution, the *logarithms* of the loss values are normally distributed. The lognormal distribution has been used by several authors (see Wacek [2007] and Han and Goa [2008]) to analyze the variability of loss reserves. To apply the lognormal model, I have used a mean shift (as in the normal case) to determine certainty equivalent values. Also, the *coefficient of variation* for each period remains constant as the loss amount varies (rather than the standard deviation, as in the normal model). Notice that for the lognormal model, the optimal second period capital is *proportional* to the ending first period loss (under the normal model, the optimal second period capital is a constant independent of the first period loss amount). The CE expected default is also proportional to the ending first period loss.

We use the same parameters as in the above normal example: the mean loss is 1000, the standard deviation of loss is 100 per period and the capital cost rate is 2%. The calculation is parallel to that in section 5.31, and the results are similar. The CE expected loss is proportional to the variance, so the CE expected ultimate loss is also 1200. Table 5.321 compares the optimal AC and FR results for both the normal and lognormal examples.

*Table 5.321*
*Optimal AC Strategy Comparison*
*Two-Period Model*
*Normal and Lognormal Stochastic Process Examples*

| | Initial Total Capital | Capital Cost | CED | Solvency Cost |
|---|---|---|---|---|
| Normal | 357.67 | 13.14 | 1.42 | 14.56 |
| Lognormal | 396.44 | 14.79 | 2.00 | 16.79 |

As shown in section 6, the similarity between the normal and lognormal processes is maintained as the time horizon is extended beyond two periods.

## 5.4 Two-Period AC Model with Cost of Raising Capital

An ongoing insurer usually manages its capital flows by having positive earnings, so that most of the time it will withdraw capital (usually as distributions to owners) to maintain the desired capital level. The positive earnings represent internally generated capital. If earnings are negative, it may be necessary to raise ownership capital externally, through issuance of bonds or equity capital. The initial basic model assumed that the cost of raising capital externally is zero. This is not realistic, since it is generally considered that there is a positive cost of raising external capital for businesses (see Myers and Majluf [1984]), including insurers (see Harrington and Niehaus [2002]).

A portion of this cost is due to the administrative expense of the capital issuance, such as investment bank fees. The other part of the cost is due to *signaling*, where if an insurer needs additional capital due to negative earnings, investors may believe that the management is poor. Thus, the capital suppliers will require a high return on the capital provided and the value of the company to existing shareholders will be diluted. This is especially the case when most other insurers do not require additional capital.[21]

### 5.41 Linear Model for Cost of Raising Capital

To model the cost of raising capital (abbreviated by CRC), assume that the cost is a rate $w$ times the amount of capital raised.[22] We continue to assume that no capital is raised if the insurer is technically insolvent. Also assume that the insurer is already in business, so that its first-period capital is not raised externally.[23]

For a two-period model, at the end of the first period, there is one period remaining.

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[21] In the event of an *industry-wide* catastrophe or pricing cycle downturn, the signaling effect may not be significant. In fact, the prospect of near-term increased insurance prices can spur investment in the property-casualty industry.

[22] An alternative formulation is to assume that the cost of raising capital increases as the insurer nears insolvency, but this will be more difficult to model.

[23] The one-period model in EBRM implicitly assumed that there was no cost of raising capital. A solvent ongoing insurer with one-period losses will need to raise capital (to the optimal level for the next group of policyholders) if the ending loss amount is large enough. This effect will change the optimal initial capital slightly.

If it is not necessary to raise capital, the optimal capital for the beginning of the second period is determined by equation 2.2. However, if capital is raised at that point, there is an additional capital cost $w$ to the insurer beyond $z$, the cost of holding capital.

Let $C_R$ represent the initial second-period ownership capital after having raised capital and $C_E$ the ending first period OC. Thus, the amount of capital raised is $C_R - C_1$. The total capital cost in the second period is then $zC_R + w(C_R - C_E)$.

Because the marginal amount of capital raised carries a frictional cost of $z + w$, following the section 2.2 analysis, the optimal second period capital is determined by

$$\hat{Q}(A_2) = z + w\,, \tag{5.411}$$

where $A_2$ is the second-period assets. Since we assume that $w$ is positive, the optimal second-period capital under CRC is *less* than that if there were no CRC: since new capital raised is expensive under CRC, the insurer will use less of it; the policyholder is satisfied, having achieved the optimal balance of price and security. Denote the optimal second-period OC, given that capital is raised at the end of the first period, by $C_R^*$.

**5.42 Optimal Two-Period Capital Under CRC**

Under the dynamic AC strategy, the additional cost under CRC is not borne if the first-period loss estimate $L_1$ is low enough so that the ending first-period OC is higher than $C^*$, the optimal one-period OC amount for the second period given that no capital is raised. The capital flows depend on four distinct regions based on the first-period ending OC amount $C_E$. Thus, we expand Table 4.51 by splitting region 2 into two sub-regions. Table 5.421 shows the capital flows by region:

*Table 5.421*
*Capital Flows by Region*
*Two-Period AC Strategy with Cost of Raising Capital*

| Region | Capital Carried Forward | Capital Raised | Capital Withdrawn |
|---|---|---|---|
| 1: $C_E < 0$ | 0 | 0 | 0 |
| 2a: $0 < C_E < C_R^*$ | $C_E$ | $C_R^* - C_E$ | 0 |
| 2b: $C_R^* < C_E < C^*$ | $C_E$ | 0 | 0 |
| 3: $C^* < C_E$ | $C^*$ | 0 | $C_E - C^*$ |

In region 1, the insurer is technically insolvent, so there are no capital flows. In region 2a, the ending OC is lower than the optimal capital needed if raising capital, so capital is added to reach $C_R^*$. In region 2b, the ending capital is greater than $C_R^*$, so no capital is raised. The ending capital is also lower than $C^*$, so none can be withdrawn either. In region 3 the ending capital is more than the optimal second-period capital, so the excess is withdrawn.

Denote the region 2a expected amount of capital carried forward by $EF_a$. We have $EF_a = \int_{A-C_R^*}^{A} (x-A)p(x)dx$, where $x$ is the ending first-period loss value and $p(x)$ is the unadjusted probability of $x$ occurring. This integral equals

$$EF_a = D(A - C_R^*) - D(A) - C_R^* Q(A - C_R^*), \tag{5.421}$$

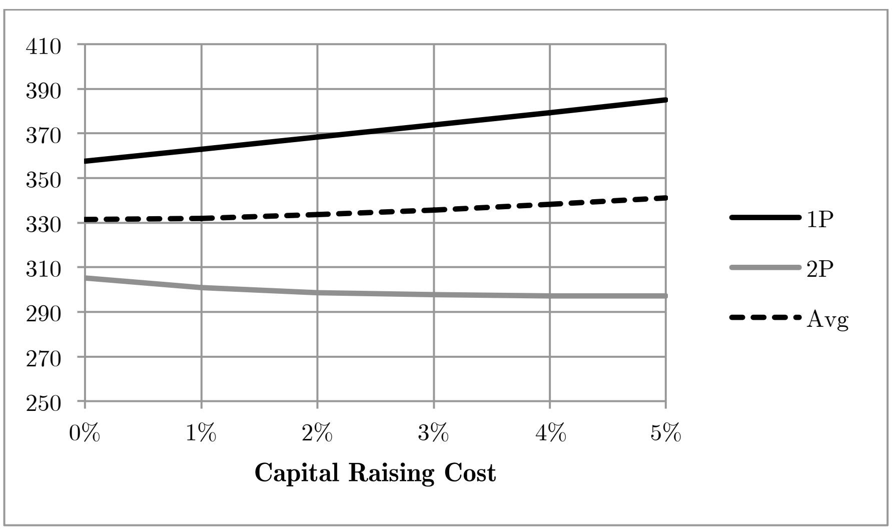
where the expected default and the default probability values are determined by unadjusted probabilities. The expected amount of capital carried forward for region 2b is developed in a similar fashion, and equals

$$EF_b = D(A - C^*) - D(A - C_R^*) - C^* Q(A - C^*) + C_R^* Q(A - C_R^*). \tag{5.422}$$

The calculation of the optimal initial capital under the AC with a CRC strategy can be illustrated by extending the section 5.31 normal example. Appendix A shows this analysis.

Figure 5.421 compares the optimal initial, expected second-period and average total capital obtained by varying $w$ in this example from 0 to 5%.

*Figure 5.421*
*Optimal Total Capital Amount by Cost of Raising Capital*
*Two-Period AC Strategy*
*Normal Example*




Notice that increasing the CRC raises the initial first-period optimal total capital and lowers the expected second-period optimal capital. The second-period capital is diminished because the second-period capital cost goes up due to the CRC and the insurer (on behalf of the policyholder) will use less of it. The initial capital increases because the insurer will avoid some of the high second-period cost by having a higher initial capital and carrying more of it into the second period. Also notice that increasing the CRC also raises the average amount of capital over both periods.

The capital raising cost will vary by insurer, and is likely to be lower for established insurers with better access to the capital markets. Thus, the CRC is another variable to be considered when assessing risk-based capital.[24]

To distinguish an AC capital strategy with a positive cost of raising capital from one with a zero cost, I abbreviate the CRC version to ACR.

## 5.5 Insurers with Limited Ability to Raise Capital

Besides depending on the cost of raising capital, the optimal capital amount also depends on the *ability* of insurers to raise capital. It is well-known that the organizational form of insurers dictates how they may raise capital (see Harrington and Niehaus [2002] and Cummins and Danzon [1997]). In particular, depending on the details of their structure, mutual insurers may have difficulty raising capital externally.[25] In the case where an insurer cannot raise external capital, the best capital strategy is capital withdrawal (CW). Note however, that this strategy will represent an upper limit to optimal capital for a mutual insurer, since the insurer can raise additional capital internally by charging its policyholders a higher premium.[26]

Under CW, all capital flows (except for the initial capitalization) are withdrawals; capital increases arise from positive earnings. Using the section 5.4 example, the optimal initial CW total capital amount is 481.29, with an expected second-period optimal capital of 305.37 and average over the two periods of 393.33. The solvency cost of this optimum position is 17.30. For comparison, the solvency cost of the section 5.42 AC strategy with a 3% CRC is 15.38.

Harrington and Niehaus show that mutual insurers carry more capital than stock insurers having the same risk. This result supports the analysis presented here.

Although the solvency cost (and hence the consumer value) for the mutual insurer is

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[24] With a CRC, even under a FR strategy the optimal initial capital will be larger than without the CRC, since capital must be stockpiled early to avoid the cost of subsequently raising it. Thus, the initial capital depends on the volatility of *future losses*, not just the behavior of current period losses.

[25] Some mutual insurers have issued *surplus notes*, which are similar to equity in terms of capital structure, but are a type of risky bond to investors. According to A.M. Best [2003], the major issuers of surplus notes were usually large insurers with more access to capital markets, while small or mid-size insurers could only issue surplus notes in relatively amounts with short maturity.

[26] However, this method is limited since the policyholders will tend to migrate to other insurers if the premium is too high.

worse than for the section 5.4 insurer, the policyholder is *not necessarily worse off.* A mutual policyholder is also an owner of the insurer and receives dividends if the mutual is profitable. These distributions are not taxable at the personal income level. However, a similar policyholder of a stock insurer with an equivalent stake in that insurer would be subject to income taxes on the capital distributions. This benefit increases the consumer value of the mutual insurance purchase. To illustrate, suppose that the personal income tax rate on the capital distributions is 15% and the expected return on capital is 8%. The average ownership capital for the section 5.4 stock insurer (with $w = 3\%$) is 332.60. Thus, the expected return to the policy/equity holder is 26.61 = 0.08(332.60). The tax on this amount is 3.99 = 0.15(26.61). The stock policyholder's consumer value after the personal income tax is the risk value minus the solvency cost minus the income tax: 180.63 = 200.00 – 15.38 – 3.99. The mutual policyholder's consumer value, with no personal income tax, is higher: 182.70 = 200.00 – 17.30.

To the extent that regulatory capital requirements are related to the optimal capital that insurers might carry, then the analysis here suggests that risk-based capital should be *higher for mutual insurers* than for stock insurers having the same risk.

## 6. OPTIMAL CAPITAL FOR MORE THAN TWO PERIODS

This section determines optimal capital for multiple periods by extending the two-period model for the capital strategies using the backward induction method. Here I outline the method generally and apply it to AC and ACR strategies.

### 6.1 General Backward Induction Method

The backward induction method determines a sequence of optimal actions or results by starting from the end of a problem with discrete stages and working backwards in time, to the beginning of the problem. It uses the output of each prior stage to form an optimal action or result based on the information available for the particular stage. This course proceeds backwards until one has determined the best action or result for every possible situation at every point in time. Backward induction is used extensively in dynamic programming and game theory.[27]

[27] For example, see Von Neumann and Morgenstern [1944].

To apply backward induction for a capital strategy where there is no CRC, define an index $i$ for each stage, where $i$ is the number of periods remaining until the ultimate loss is determined. At each stage $i$, there are three optimal quantities that have been determined from the prior stage, and may depend on the loss value $x$ from stage $i$: the optimal ownership capital $C^*_{i-1}(x)$, the optimal CED $\hat{D}^*_{i-1}(x)$ and the optimal capital cost $K^*_{i-1}(x)$.

For stage $i$, we start with the optimal asset amount from the prior stage $i - 1$ and calculate the solvency cost. We vary the asset amount until the optimal solvency cost is attained, and record the values of the above three optimal quantities. The process is repeated until the $N$th stage is complete. The result is the optimal initial capital, CED and capital cost for an $N$-period model. The intermediate stage results will give the optimal quantities for all models of lesser duration that have the same sequence of loss increment variances per period.[28] Accordingly, for a model with constant volatility per period, we will get the optimal results for all models with $N$ or fewer periods.

## 6.2 Backward Induction Method with AC Strategy

Under the AC capital strategy for stage $i$, the solvency cost has four components: (1) the CED for technical insolvency in the stage, (2) the expected CED for future insolvency, (3) the capital cost for the stage and (4) the expected future capital costs. The first two components represent the total CED for all periods through stage $i$, denoted by $\hat{D}_i$, and the last two represent $K_i$, the total capital cost for all periods. Therefore we can represent the solvency cost as $S_i = \hat{D}_i + K_i$, where

$$\hat{D}_i = \hat{G}_i + \hat{P}\int_0^A \hat{D}^*_{i-1}(x)\hat{p}(x)dx \tag{6.21}$$

and

[28] For example, suppose a three-period model has a standard deviation (SD) of 50 for the first period loss increment, 60 for the second period and 80 for the third period. This process will give the optimal results for a one-period model with an 80 SD, and a two-period model with a 60 SD for the first period and 80 for the second period.

$$K_i = zC_i + K^*_{i-1}\,. \tag{6.22}$$

Here $\hat{P}$ is the CE probability of remaining solvent with assets $A$, and $\hat{p}(x)$ is the CE probability density of loss size $x$.

We minimize the value of $S_i$ to get the optimal available asset value $A^*_i$ for this stage.[29] From equation 4.31, we get the optimal OC:

$$C^*_i = A^*_i - L - K^*_{i-1}\,. \tag{6.23}$$

The optimal total capital is $T^*_i = C^*_i + K^*_{i-1}$. We also have optimal values of the components $\hat{D}_i$, and $K_i$, which we label $\hat{D}^*_i$ and $K^*_i$. So now we have the three inputs needed to determine the optimal capital for the stage $i + 1$, and successive stages, until the optimal initial capital for the $N$th period is found.

To illustrate this process, consider the basic section 5 example from Appendix A. We have $\hat{D}^*_1 = 0.7343$ and $K^*_1 = 6.1075$. Applying equations 6.21 and 6.22, and optimizing gives $A^*_2 = 1357.67$, $\hat{D}^*_2 =1.4120$, $K^*_2 = 13.1387$ and $C^*_2 = 351.56$. The next iteration gives $\hat{D}^*_3 = 2.0560$, $K^*_3 = 20.4220$ and $C^*_3 = 364.16$. The corresponding optimal total capital amounts are $T^*_1 = 305.37$, $T^*_2 = 357.67$ and $T^*_3 = 377.30$.

For the lognormal version of the basic example, the process is similar. Figure 6.21 compares optimal initial total capital amounts for the normal and lognormal cases by number of periods from 1 to 15.

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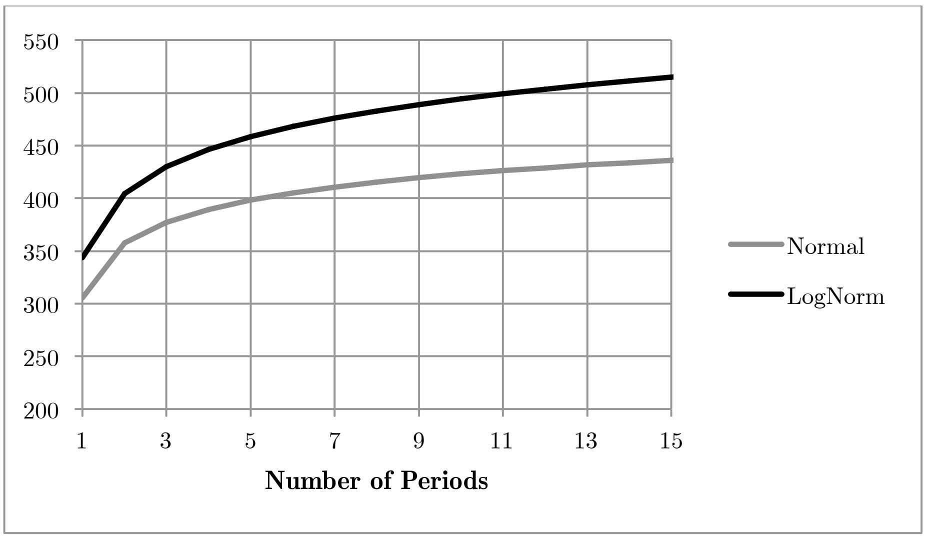
[29] Appendix B discusses the optimization technique, which uses two asset values whose difference is small.

*Figure 6.21*
*Optimal Total Capital Amount by Number of Periods*
*Normal vs. Lognormal Examples*
*AC Strategy*




Notice that the optimal initial total capital increases steadily, but at a declining rate, as the number of periods increases. Therefore, as the ultimate loss variance increases, the optimal initial capital also increases.

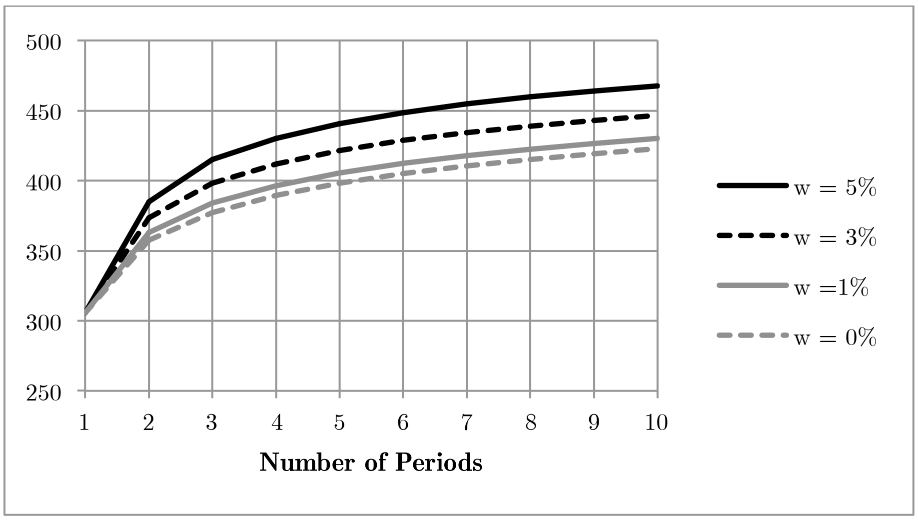
## 6.3 Multi-Period Capital with ACR Strategy

As shown in section 5.42, the two-period optimal ACR capital calculation requires *two* optimal capital relationships at each stage: one based on the cost of holding capital $z$ and a smaller amount based on $z$ plus $w$, the cost of raising capital. Appendix B develops the recursive relationships needed for the optimal ACR initial capital for $N$ periods. Here we need *six* optimal quantities at each stage: three similar to those in section 6.1 (based on no CRC) and three more based on the higher capital costs under the CRC.

Also, the CRC creates loss region 2b (where capital remains the same). This requires an additional calculation: at each stage, the expected CED and capital cost for this region must be found by numerical integration. Figure 6.31 extends the section 5.42

example to 10 periods and shows the optimal capital for $w$ ranging from 0% to 5%.

*Figure 6.31*
*Optimal Total Capital Amount by Number of Periods*
*ACR Strategy*
*Normal Example*




## 7. CAPITALIZATION INTERVAL

The preceding analysis has used an arbitrary period length, with capital flows occurring at beginning of each period. Since the period length governs the duration between capital flows, and to distinguish it from other insurance periods such as policy term, I specifically refer to the period length as the *capitalization interval* (abbreviated as CI).

The actual length of the CI will affect the optimal capital, since, for a given loss duration, a shorter capitalization interval will allow more opportunities to add or withdraw capital as the loss amount evolves. To analyze this effect, recall that the policy term is defined to be equal to the period length. Thus, the losses occur at the beginning of the policy term,[30] and capital flows also occur at the beginning and end of the policy term. Section 7.3 discusses the case where the period length is shorter than the policy term.

The frequency of potential capital additions and withdrawals will have a significant impact on the optimal capital and solvency cost, regardless of the capital strategy used.

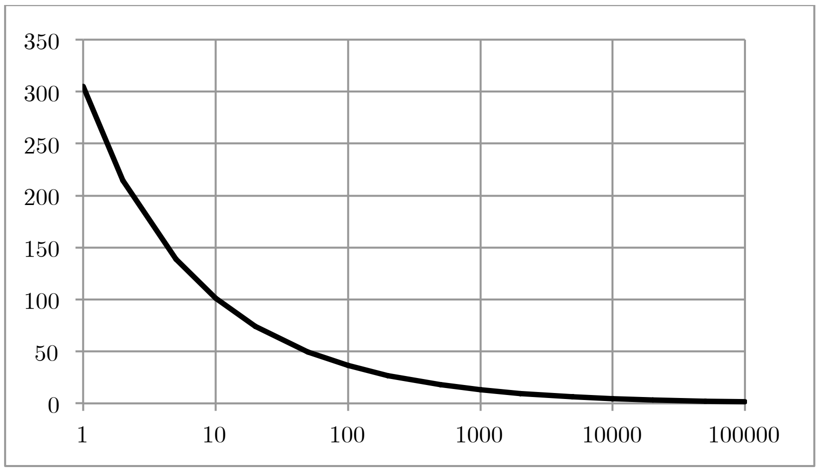
## 7.1 Capitalization Interval with the FR Strategy

To illustrate the effect of the CI, again assume the basic one-period normal example from section 5 with a standard deviation of 100. The optimal total capital is 305.37 with a solvency cost of 6.84. Suppose that the loss duration is *one year*: thus, capital is supplied at the beginning of the year and the amount of loss is known at the end of the year. Now suppose that we subdivide the one-year period into half-year periods, with capital flows allowed at the beginning of each. Each smaller period will now have a loss standard deviation of $70.71 = 100/\sqrt{2}$ and the capital cost rate is $0.01 = 0.02/2$. Under the full recapitalization (FR) strategy, the optimal beginning total capital for each half-year period is now 214.24 with a corresponding 2.38 solvency cost. The solvency cost for the entire year is twice this amount, or 4.77. Thus, by allowing more frequent capital movement, the consumer value has improved and less capital is required.

Figure 7.11 shows the effect of further subdividing the one-year period into more capitalization intervals:

[30] A more realistic assumption is that the loss may occur randomly throughout the policy term, with the average loss happening at the middle of the term. Here, I am merely attempting to show the effect of changing the capitalization interval length. A practical application would use the actual expected timing of the incurred losses.

*Figure 7.11*
*Optimal FR Total Capital by Number of Intervals*
*Basic Normal Example*




As the number of intervals becomes large, the optimal capital amount approaches zero! Although not shown in this graph, the annual solvency cost associated with the optimal capital also approaches zero (it is only 0.032 for 100,000 intervals). Since capital is added in response to infinitesimal changes in loss evaluation, there is only a tiny chance at any time that a default will occur, and if it does, the default amount will be infinitesimally small. Notice that this result depends on the assumption of a *continuous* stochastic process for losses: if the loss valuation can change in somewhat large increments, then the capital additions cannot "catch up." Consequently, in a theoretical world with a continuous stochastic loss process and the ability to add capital with no cost, there is no need for an insurer to carry capital. However, the loss process might not be continuous, and, as discussed next in section 7.2, important real-world imperfections, frictions and costs no not permit an infinitesimally small CI, so capital is indeed required.

## 7.2 Capital Strategies and Time Intervals

For other capital strategies, the optimal capital also declines as the CI becomes smaller. Figures 7.22a and 7.22b compare results of the FR, AC, CW and ACR (with 5% cost of raising capital) strategies, according to interval length. Here, I show the *average* amount of total capital over the year. Note that the initial capital for the first period will also decline with the number of intervals for the FR, AC and ACR with low capital-raising cost strategies. However, for the CW and high capital-raising cost ACR strategies, the first-period capital amount *increases* with the number of intervals. Nevertheless, since these strategies stockpile capital in the early periods and tend to withdraw more of it later than for the other strategies, the average amount of capital declines with the number of intervals.

*Figure 7.22a*
*Optimal Average Total Capital*
*by Number of Intervals and Capital Strategy*
*Basic Normal Example*

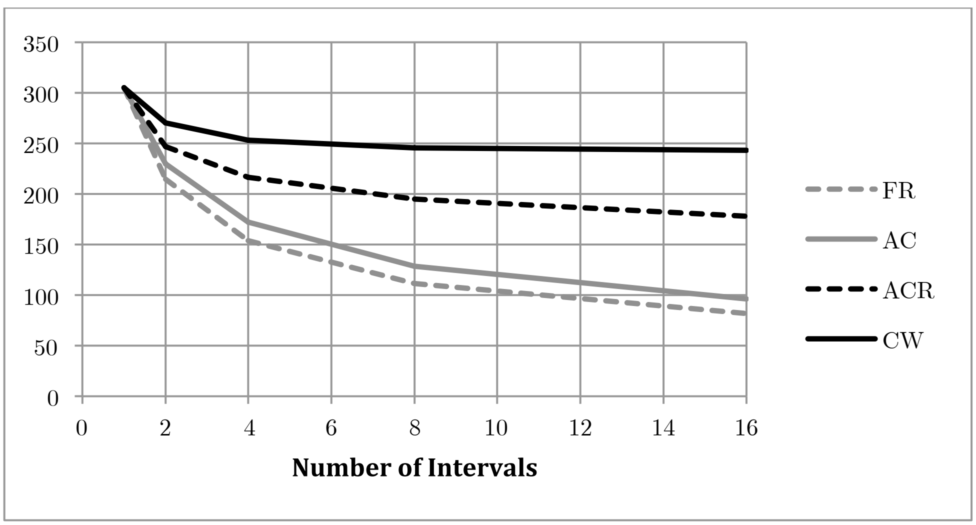
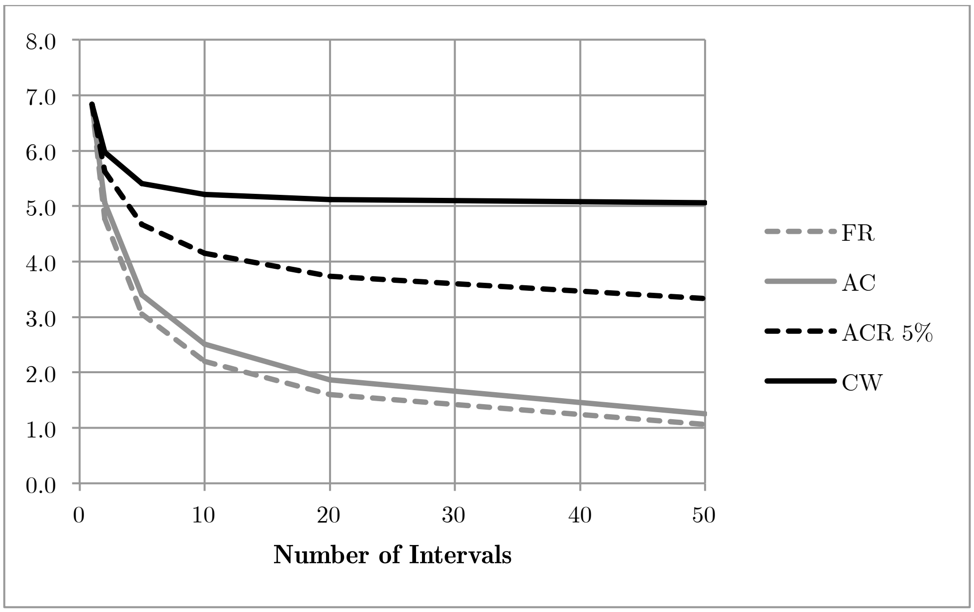


*Figure 7.22b*
*Optimal Solvency Cost by Number of Intervals by Capital Strategy*
*Basic Normal Example*




Here, all strategies provide smaller optimal capital amounts and solvency costs as the CI length decreases. The AC strategy follows the FR strategy in that the optimal capital approaches zero as the interval approaches zero. However, the optimal capital for the CW and ACR strategies decline much more slowly[31] because either capital cannot be added, or its addition is costly.

Although the optimal capital and solvency cost decline with shorter period length, there will be a practical limit to this effect. Even for a pure continuous stochastic loss process, the minimum interval length is governed by real-world considerations. The minimum length depends on a sequence of events, each of which requires some time. Among other factors, the loss reserve must be evaluated (for most insurers this occurs monthly or quarterly) and then management must decide to raise capital and then contact an investment bank. The bank then performs due diligence and offers the public

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[31] It appears that the average capital may reach a fixed limit, but I have not proved this.

an opportunity to supply capital. Even if the insurer has a prior commitment from an investment bank, this process may take several months.

Nevertheless, it is clear that policies with short capitalization intervals will require far less capital than longer ones, and will be more efficient (with lower solvency costs) as well. Because some insurers may be better-equipped to generate capital flows quickly, the minimum interval length will vary by insurer. Consequently, this factor should be considered in assessing specific insurer capital levels.

## 7.3 Capitalization Interval and Short-Duration Losses

The preceding sections have analyzed long-duration losses, where there is a lag between when the loss occurs and when its ultimate value is determined; the lag extends over several periods. Conversely, a one-period capital model may be appropriate for short-duration losses. Here, as with long-duration losses, the interval length will influence the optimal capital. To illustrate, we can modify the basic section 5 example to accommodate the short-duration stochastic process.

Assume that losses occur randomly and uniformly over time, with an annual amount whose value is normally distributed with a 1000 mean and 100 standard deviation. Loss amounts for claims occurring at any time are independent of prior losses. The other parameters (capital cost and risk aversion) are the same as for the long-duration example. Suppose that the policy term is one year with capital flows occurring at annual intervals. Since, at the end of the policy term, the loss distribution is identical to that of the long-duration case, the optimal capital and solvency cost are also the same, at 305.37 and 6.84.

Now suppose that we issue two policies, one covering the first half-year and the other the second half-year. Capital may supplied or withdrawn at the beginning of each policy term. Each period is independent of the other: if insolvency occurs in the first period, the policyholder absorbs the default at that point and buys an equivalent policy from another insurer to cover the second period. There is no technical insolvency to consider. Consequently, the situation is equivalent to the section 7.1 example under the FR strategy, giving the same results by interval length. So, for short-duration insurance, less capital is required as the capitalization interval decreases, and the optimal capital level also depends on the practical limit to the interval length. For any type of insurance, then, the interval length then is an important element when setting the amount of risk-based capital.

## 7.4 Effect of Policy Term and Capitalization Interval

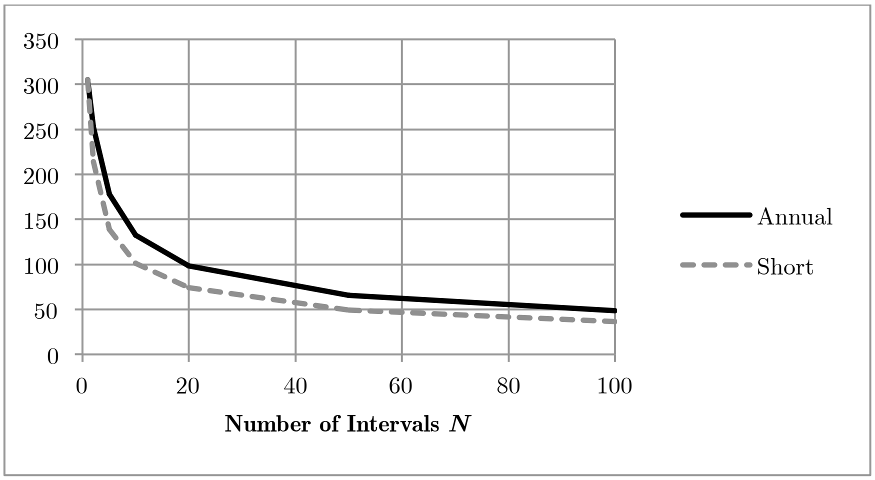
The preceding analysis has assumed that the capitalization interval equals the policy term. Generally, the policy term is one year,[32] but the capitalization interval will most likely be shorter than one year. Assume that the premium is paid at the beginning of the period. When the capitalization interval is shorter than the policy term, insolvency may occur early in the policy term. This event will effectively terminate coverage for losses that may occur in the remainder of the policy term, and will produce an additional solvency cost, since the full premium is paid up front.

To illustrate this effect, we extend the section 7.3 short-duration example. Appendix A shows the calculation details.

As the CI becomes shorter, the optimal capital and solvency cost values approach zero, as with the case where the policy term equals the CI. Figure 7.41 compares the optimal average total capital for an annual policy with $N$ capitalization intervals with the total capital for a shorter policy having the same term as each interval.

[32] Some automobile policies are six months and, less commonly, some commercial risks have multi-year coverage.

*Figure 7.41*
*Average Optimal Total Capital by Number of Intervals*
*Annual vs. Shorter Policy Term*
*Normal Short-Duration Example*




Here, more capital is needed if the policy term exceeds the CI, regardless of the number of intervals. Thus, besides the CI, which greatly affects the optimal capital, the *length of the policy term* is another variable that will influence the capital amount. This effect will be present with long-duration[33] losses as well as short-duration losses, since the cost of foregone coverage must be considered.

## 8. MULTI-PERIOD MODEL EXTENSIONS

This section extends the basic loss model to incorporate features that may be necessary for a practical application. Also, I briefly discuss how the results might apply to life insurance.

### 8.1 Stochastic Time Horizon

In a more realistic model of the development process for long-duration losses, the

[33] Modeling this effect is more complicated than for property, since one must assume a relationship between the losses of each interval. For liability losses, these will be correlated. A convenient approach is to assume that all losses move together.

ultimate duration of losses is not known. Here I relax the basic model assumption that the loss develops randomly for $N$ periods and is settled at the end of the $N$th period. Instead, assume that, although the value evolves according to the section 4.2 liability stochastic process, the process may terminate randomly at the end of each period, at which point the loss is settled. In this model, there are $N$ possible periods, extending to the longest possible claim duration. In effect, there are $N$ possible models, with fixed horizons from one to $N$ periods. Call this entity the *stochastic-horizon* (abbreviated as SH) loss model.

Let $q_i$ represent the probability of settlement at the end of period $i$. Then $q_1 + q_2 + \cdots + q_N = 1$. From section 4.21, the variance of the ultimate loss will be $\sigma^2[1 \cdot q_1 + 2 \cdot q_1 + \cdots + N \cdot q_N]$, since the variance of each period's loss increment is independent of the prior value. This is a simple weighted average of the loss variances of the component $N$ possible models.

Meanwhile, the CE expected value of the SH loss is proportional to the variance (as discussed in section 4.42). Consequently, the CE expected loss of the stochastic-horizon loss equals the weighted average of the CEL values of its $N$ component loss models, where the weights are the termination probabilities $q_i$.

Under the SH model, the optimal capital will be a weighted average of the optimal capital values for the component fixed-horizon models. Given the above analysis, to approximate the optimal SH capital, it is reasonable to use the *exact termination probabilities* (rather than a CE adjusted set of probabilities) to weight the optimal capital amounts.

To illustrate the stochastic-horizon model, we extend the basic normal model. Assume three periods with termination probabilities $q_1$ = 0.5, $q_2$ = 0.3 and $q_3$ = 0.2. The average loss duration is 1.7 periods and the respective amounts of loss paid at the end of each period are 500, 300 and 200. From section 6.2, the optimal initial capital amounts for the three component horizons are $C_1^*$ = 305.37, $C_2^*$ = 357.59 and $C_3^*$ = 377.22. Thus, the optimal initial capital for the basic SH model is 335.41 = 0.5(305.37) + 0.3(357.59) + 0.2(377.22).

Notice that, if the expected loss is independent of the loss duration (as in the basic model) the set of termination probabilities will represent the expected loss payment pattern.

## 8.2 Present Values

Because multi-period losses, especially for liability insurance, can be paid several years from when the loss occurs, it is necessary to use the *present value* of the solvency cost components in determining optimal capital. Since the present value of a certainty-equivalent amount must also be a CE value, its present value must be found using a *risk-free* interest rate, denoted by a rate $r$ per period. A similar logic applies to the capital cost component.

To illustrate the effect of using present values, I use the basic AC strategy. Let $S_i(r)$ denote the present value of the solvency cost with $r$. For a one-period loss, we have

$$S_1(r) = \hat{D}_1 / (1+r) + zC_1 / (1+r). \tag{8.21}$$

In setting the derivative of $S_i(r)$ to zero, the factor 1 + $r$ vanishes. Thus optimal capital with $r$ equals the optimal capital with a zero interest rate. For two or more periods under the AC strategy, however, the solvency cost does not scale in the same way. Using the section 6 backward induction indexing, define the present value of the expected capital cost as $K_i(r)$, which is the analog of $K_i$ in equation 6.22:

$$K_i(r) = [zC_i + K_{i-1}^*(r)] / (1+r). \tag{8.22}$$

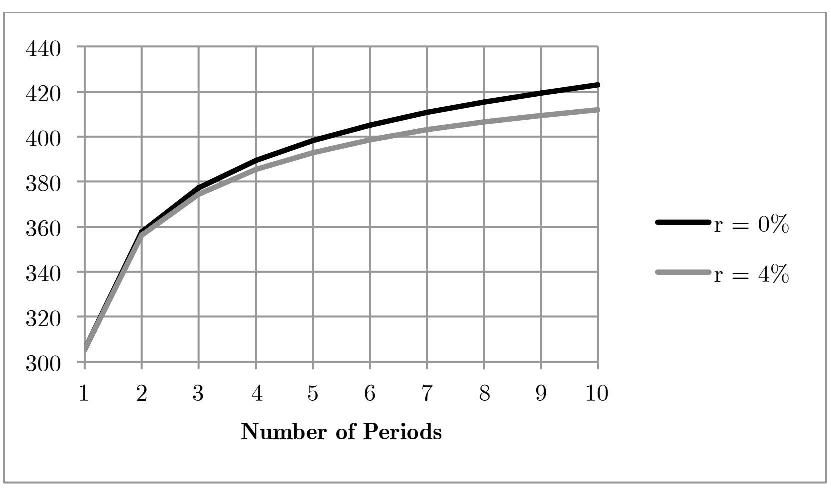
The solvency cost for stage $i$ is therefore

$$S_i(r) = \hat{D}_i(1+r)^{-i} + K_{i-1}^*(r). \tag{8.23}$$

Observe that $K_i(r)$ represent the accumulated present value of the capital costs for all but the first period. The default is realized at the end of $i$ future periods, so its CE expected value $\hat{D}_i$ is discounted for $i$ periods. However, we only discount the capital cost for the current ($i$th) period for *one* period. Thus, the present values and the CED amounts will not be proportional, and the optimal capital will be different from that with a zero interest rate. In fact, it will be *lower*, since the derivative of $\hat{D}_i(1+r)^{-i}$ (with respect to assets) in equation 8.23 is relatively lower than the derivative of $zC_i(1+r)^{-1}$, in comparison to the respective derivatives with $r = 0$.

The backward induction method easily incorporates present values. Starting with equation 8.21, we use equation 8.23 recursively to generate the successive optimal capital amounts. To demonstrate this calculation, let $r = 4\%$ in the basic AC example. We get the optimal two-period initial total capital of 356.18 and three-period total capital of 374.49. These values are *less* than the respective zero-interest amounts of 357.67 and 377.30. Figure 8.21 compares the optimal total capital for $r = 4\%$ with $r = 0\%$, for horizons of one to ten periods.

*Figure 8.21*
*Optimal Initial Total Capital*
*0% vs. 4% Riskless Interest Rate*
*Normal Example with AC Strategy*



Notice that the optimal capital here is *not the same* as what we would get by discounting the $N$-period optimal total capital with $r = 0$, by $N$ periods at the rate $r$. For example, the optimal initial total capital for 10 periods with $r = 0$ is 422.97. The present value of this amount at 4% is 285.75, but the true optimal capital amount (in Figure 8.21) is 411.84, which is only slightly less than the optimal capital with a zero interest rate.

## 8.3 Risk Margins

The preceding analysis assumed that the loss component of the premium included only the unadjusted expected value of the loss. The premium did not reflect a positive market price for bearing the risk. Here I assume that the market value of the loss is greater than the expected loss: i.e., it contains a *risk margin*, whose value is denoted by $M$. The expected loss, including the risk margin, can be determined from a *third* stochastic process with an adjusted probability distribution. For a one-period model, we have

$$\overline{L} = \int_0^\infty \overline{p}(x) x dx = L + M \,, \tag{8.31}$$

where $\overline{p}(x)$ denotes the adjusted probability underlying the risk margin. Here the relevant risk is systematic: it cannot be reduced through pooling, and therefore commands a price in financial markets. The value of the policyholder's underlying risk, before it is reduced through insurance pooling, will be larger per unit of expected loss than that of the insurer's risk (which is reduced through pooling). So we have $\hat{L} \geq \overline{L} \geq L$.

For a multi-period stochastic process with equal variance of loss increments for each period, I assume that the risk margin increases uniformly with the number of periods. Thus, if stochastic process is additive, the risk margin will also be additive. Let $m$ represent the risk margin as a ratio to the expected loss $L$ for one period. So, for an $N$-period loss, the risk margin will equal $mLN$, and the market value of the expected loss will be $L(1 + Nm)$.

For a multiplicative stochastic process, the market value of $L$ is $L(1+m)^N$. Observe that the present value of the market-value loss is $L(1+m)^N (1+r)^{-N}$, where $r$ is the risk-free interest rate. Therefore, the market value of the expected loss can be expressed as the expected value $L$, discounted at a risk-adjusted interest rate $r_a = (r - m) / (1 + m)$.[34]

The fair premium is $\pi = L + M + K$. The risk margin then provides additional total capital, beyond the amount from the expected capital cost $K$. The amount $M$ can also be considered as policyholder-supplied capital. Thus, for the same loss volatility, a higher risk margin will imply a lower optimal fair-value capital amount. The additional assets function as capital, so to give the same insolvency protection, the insurer will need less capital than without the risk margin. As discussed in section 4.4, the risk margin is equivalent to ownership capital in terms of solvency protection.[35] For multiple periods, the relationship holds as well, since the CED depends on the asset amount and not the accounting measure of the loss. To illustrate this effect, assume the basic AC normal model with a 0% interest rate, and let $m$ = 2%.[36] From section 5.31, for a one-period model without the risk margin, the optimal total capital is 305.37 and optimal assets are 1305.37. The expected default depends on the asset level, not the capital amount. With the risk margin, the same assets are also optimal: the CED is the same, and changing the asset amount through the initial owner-supplied capital will reduce the consumer value.

Thus, the premium and initial assets will be larger by 20 = 0.02(1000). Optimal one-period capital is now reduced by 20 to 285.37 to give the same CE default probability (equal to the capital cost rate). Since the risk margin in this example is proportional to the number of periods, the optimal initial capital is reduced by 20 units times the number of periods in the time horizon.

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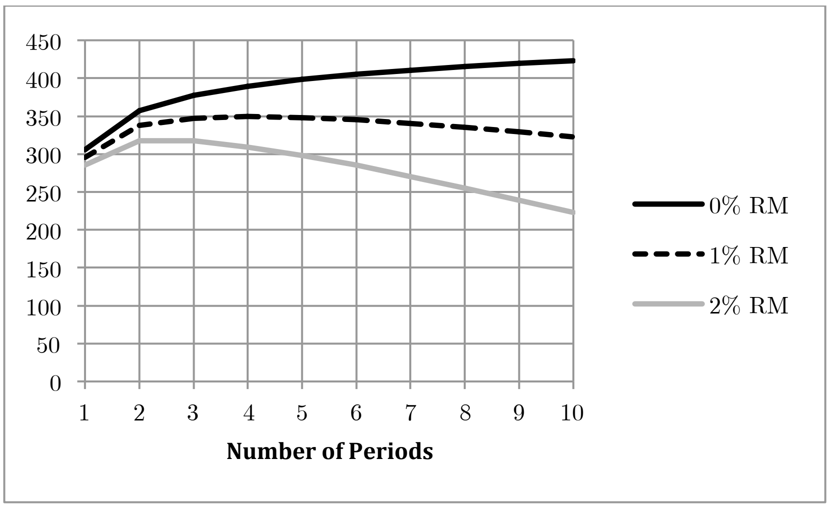
[34] Butsic [1988] develops the risk-adjusted interest rate for insurance reserving and pricing applications.

[35] In another sense, the risk margin may be considered as *ownership* capital in that it is not a third-party obligation: it "belongs" to the owners of the insurance firm and will be returned to the owners if the insurance proves to be profitable.

[36] In practice, the amount of risk margin is a small fraction of premium. It is straightforward to show that the risk margin is $m = [(R - r) / (1 - t)] / [C / L]$, where $R$ is the expected return on the capital $C$, $r$ is the risk-free return and $t$ is the income tax rate. For example, assume an insurer's current expected (after-tax) return on equity is about 4% above the risk-free investment return, the effective tax rate is 30% and the leverage ratio $C/L$ is 40%. A risk margin equal to 2.3% of expected loss will provide the required return on equity. Note also that the risk margin cannot exceed the difference between the CE value of the loss and its expected value; otherwise the policyholder is better off without insurance.

Figure 8.31 compares the optimal initial fair-value capital for the AC strategy by time horizon for no risk margin and for risk margins of 1% and 2%.

*Figure 8.31*
*Optimal Initial Fair Value Capital by Risk Margin and Time Horizon*
*Normal Example with AC Strategy*




Notice that if the time horizon is long enough and the risk margin is large enough, it is possible that the initial fair-value capital can be lower than with a short horizon. However, the *total* capital (represented by a 0% risk margin) continues to increase with the time horizon.

With a risk margin, the expected capital cost for future periods is not the same as without one, because the insurer values the loss according to the market value loss distribution $\overline{p}(x)$; the future capital amount depends on the future random loss value. Consequently, the expected capital should be determined from the market value loss distribution and not the unadjusted distribution (used in sections 5 through 7). However, this effect does not change the optimal initial *total* capital amount produced with the unadjusted loss distribution, since a change in assets will not produce a corresponding change in the expected future capital amount (it is fixed with respect to the asset value). But, since the expected capital costs increase, compared to the no-risk margin case, the optimal fair-value capital will be less. I have ignored the effect of the market value loss distribution on capital costs for this section.[37]

## 8.4 Life Insurance

This paper has focused on property-casualty insurance. As such, the scope of the study precludes a thorough development of optimal capital for of life insurance. However, below I briefly discuss some implications of the findings in this paper to an analysis of life insurance.

*Life Insurance Liability Risk*

Generally, for life insurance the risk of losses being higher than expected is low due to the lack of correlation between claims from separate policies. There is some chance of default from losses occurring earlier than expected (e.g., whole life insurance) or later than expected (e.g., annuities). The risk of default for the amount of claims and their timing can be addressed by the techniques presented in the earlier sections. Life claims risk has a different stochastic process than long-duration losses, since the periodic indemnity amounts are fixed but the horizon is stochastic. The process is not Markovian, since if more/fewer insureds die, then the probability of future deaths changes for the insured population.

*Embedded Policyholder Options*

A major source of risk for life insurers is the nature of the embedded options in policy contracts. These are not usually present for property-casualty insurance. For example, policyholders may stop paying premiums or they may add coverage after the policy has been in force; policyholders may be able to make loans at favorable terms; the policy may have other investment guarantees. The effect of any of these depends on policyholder behavior. Note that some policy features may not remain after the insurer becomes insolvent and is under conservatorship. Further, the policy features that create default risk have value to the policyholder, which should be incorporated into the consumer value component of the optimal capital calculation.

*Capital Funding Strategies*

Notwithstanding the above differences between life and PC insurance, the capital

[37] Using the section 5.3 basic example, this omission overstates optimal FV capital for two periods by 0.12 and for five periods by 2.11.

funding strategies available to life insurers are the same as for property-casualty insurers. The availability and cost of external capital will also be similar. These factors will have parallel impacts on the amount of capital needed for life insurers. Also, modeling the asset risk will be similar, since both types of insurers have the same categories of investments in their portfolios.

## 9. MULTI-PERIOD ASSET RISK

The preceding results for risky losses can be extended to the case where *assets* are risky. To isolate the capital for risky assets from that of risky losses, I assume that the insurer has a *riskless* $N$-period loss liability, but its assets are risky.[38] The intuition here is that there would be no insurance without the presence of the underlying loss, so the time horizon for asset risk should match that of the loss. Also assume (temporarily) that the CRC is zero.

### 9.1 One-Period Asset Model

For a one-period loss, where the loss is fixed and the ending asset amount is a random variable, the CE expected default value is

$$\hat{D}(A) = \int_0^L (L - x)\hat{p}(x)\,dx\,. \tag{9.11}$$

Here $x$ represents the asset value and $\hat{p}(x)$ its CE probability. With an assumed zero risk-free interest rate, and presuming that policyholders have the same risk preferences as investors,[39] the CE expected value of the ending assets must equal the initial asset value $A$. Otherwise the policyholder will have a net gain or loss in consumer value

[38] This technique is used to determine asset risk-based capital in Butsic [1994].

[39] This assumption is not essential, but it simplifies the analysis. If policyholders do not have the same risk preferences as investors, then there may be a tendency for less risk-avoiding policyholders to migrate toward insurers that have higher-than-average investment risk. Conversely, more risk-avoiding policyholders might gravitate to insurers with lower than-average investment risk.

merely by the choice of the insurer's investments.[40] Consequently, the expected return on the assets is not relevant to the CE calculations. However, it can be important for determining the unadjusted default values and the expected capital amounts for periods after the initial period.

The consumer value of the insurance contract with the riskless loss and risky assets equals the CE value of the loss, minus the premium, which is the loss value plus the solvency cost. The solvency cost has the some form as for losses: $S = zC + \hat{D}(A)$. Consequently, the consumer value of the insurance is the negative of the solvency cost, since the CE value of the loss equals the nominal (fixed) value of the loss. No prospective policyholder would knowingly buy such a contract. However, here we want to find the capital amount that minimizes the negative CV. In a more realistic context, the asset capital is combined with the loss capital and the consumer value is generally positive.[41]

By minimizing the solvency cost, the optimal capital is found by solving for $\hat{Q}(A) = z$, a result parallel to that of the loss capital case.

## 9.2 General Two-Period AC Asset Model

For two periods, however, the situation is not parallel to the case of risky losses. Here, the loss value will continue to evolve if technical solvency occurs at the end of the first period. With risky assets, in contrast, if the insurer becomes technically insolvent after the first period, the asset risk will drop to virtually zero since the insurer will enter conservatorship shortly after becoming technically insolvent. As discussed in section 3.2, the asset portfolio will be converted to an essentially riskless one by the conservator. For simplicity, I assume that the investment portfolio is immediately converted to riskless assets upon technical insolvency. Also assume that if the insurer remains solvent, the asset portfolio retains the same risk as the size of the portfolio changes. In other words, the variance of the asset portfolio value remains constant over time, even though the asset amount may change.

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[40] In finance, this concept is called risk-neutral valuation and is a cornerstone of analyzing financial instruments.

[41] This negative consumer value for risky assets is examined further in EBRM. It is not clear why insurers have risky investments, since there is no apparent benefit to policyholders that can overcome the cost of double taxation.

Therefore, if the insurer becomes technically insolvent after one period, the amount of the default is determined with certainty at that time: both the asset and the liability values are fixed and are no longer subject to random variation. If the insurer remains solvent, under the AC strategy it adjusts it capital to the optimum amount for the remaining period. The asset amount changes accordingly.

Denote the CE expected default at the end of the first period by $\hat{D}_1(A)$. If solvent at the end of the first period, the assets are adjusted to give the optimal capital amount for the remaining period.[42] Let $C_2^*$ represent the optimal second-period capital and $\hat{D}_2^*$ the corresponding optimal CED for the second period.

The unconditional CED for the second period is the optimal CED for the second period times the CE probability of remaining solvent, or $\hat{D}_2^*\hat{P}_1(A)$. Denote the first period capital amount by $C_1$.

The solvency cost for the asset risk is the sum of the CE default amounts for both periods and the expected capital cost:

$$S = \hat{D}_1(A) + \hat{D}_2^*\hat{P}(A) + zC_1 + zC_2^* . \tag{9.21}$$

Setting the derivative of $S$ with respect to assets equal to zero, we get the condition for the optimal beginning first-period capital:

$$\hat{Q}_1(A) = z + \hat{p}(A)\hat{D}_2^* , \tag{9.22}$$

where $\hat{p}(A)$ is the CE probability density at $A$. Since this density is positive, the initial capital for the two-period asset risk for an AC strategy is actually *less* than that for the FR strategy, where each period's capital is determined without regard to the risk of the other periods. Since the density will generally be quite small relative to $z$, the optimal beginning first-period capital is approximated by the FR strategy where $\hat{Q}_1(A) = z$.

[42] For losses, the optimal capital for the remaining period may depend on the ending first-period loss amount, such as with the lognormal distribution. However, for assets, the optimal capital will be constant, since the loss amount is fixed and the assets are adjusted to reach the optimal capital amount.

The reason why the optimal asset capital under the AC strategy is less than that with the FR strategy is that the risky asset portfolio is eliminated under technical insolvency, while it is not eliminated under full recapitalization. From equation 9.21, if we are at the beginning one-period optimal capital $C_1 = C_1^*$, and we lower the capital amount a tiny amount, the positive change in the first period CED virtually equals the negative change in the capital cost (because we are at the optimum). However, the second-period CED and capital costs are reduced because the probability of remaining solvent is reduced. Under FR, the policyholders may experience insolvency after the first period, but the full capital (and its cost) is present at the start of the second period and they may sustain insolvency a second time.

Appendix A provides a numerical example illustrating the optimal AC capital for two-period risk.

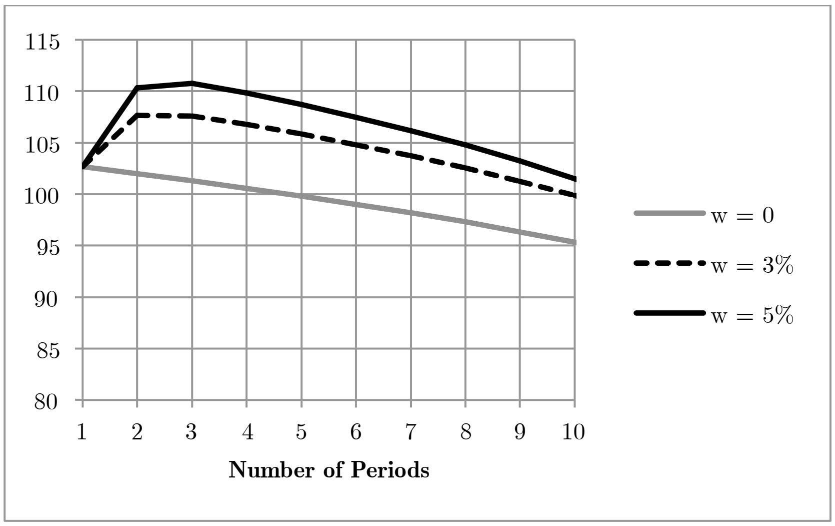
## 9.3 Asset Risk with Multiple Periods and Cost of Raising Capital

The methodology for determining initial capital for multi-period asset risk is parallel to that for losses in section 5.4. In fact, it is somewhat simpler, in that there is no need to calculate $\hat{G}$; a default in the first period is based on a one-period model, since investment risk is eliminated in conservatorship. Also, the investment portfolio has the same risk for each period, which simplifies the computation.[43] For more than two periods, we use backward induction to determine the optimal first-period capital.

Figure 9.31 shows the optimal first-period asset-risk total capital for one to ten periods, with a CRC of 0%, 3% and 5%, given the section 9.2 example.

[43] Under the alternative assumption that investment risk (standard deviation) is proportional to the asset value, there is not much difference in results. For example, in the Figure 9.31 results (with $w = 0$), the maximum difference in optimal capital is for $N = 10$: we have 95.36 for constant risk and 95.41 for proportional risk, a difference of only 0.05%.

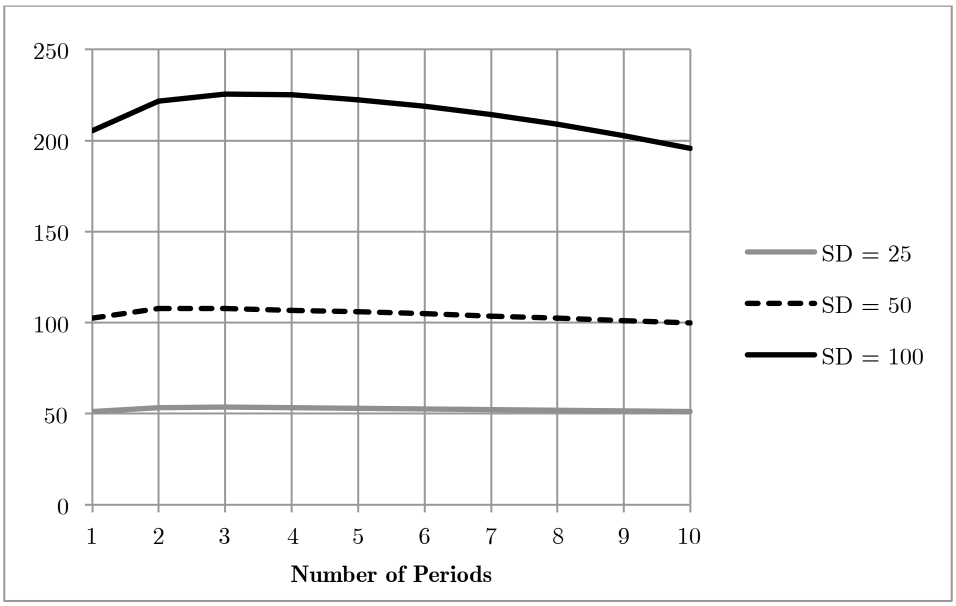
*Figure 9.31*
*Optimal AC Initial Total Capital Comparison*
*by Time Horizon and Cost of Raising Capital*
*Section 9.2 Example*




These results indicate that with a moderate CRC, the optimal initial capital rises somewhat for a few periods, and then declines as the number of periods grows. With no CRC, it strictly declines with the number of periods.

In Figure 9.32, I use a 3% CRC and vary the standard deviation of the per-period asset risk (a standard deviation of 25, 50 or 100).

*Figure 9.32*
*Optimal AC Initial Total Capital Comparison*
*by Time Horizon and Asset Standard Deviation*
*Section 9.2 Example*




These results suggest that if the asset risk is moderate (i.e., with an standard deviation of less than about 5% of assets) and the CRC is moderate, then the optimal capital for asset risk is *approximately the same* for each time horizon.

## 10. CONCLUSION

The main purpose of this study is to further the understanding of how to establish the risk-based capital for multi-period insurance losses and assets. I have attempted to accomplish this without making arbitrary assumptions about the choice of risk measure (e.g., VaR, TVaR and others) and which time horizon model (one-year vs. runoff) should be used. The volatility of each period is given its proper weight. It is important to understand the economic principles involved and which variables influence this process. Much of this undertaking is new territory. In particular, the notions of policyholder risk preferences and dynamic capital strategies may be unfamiliar to an actuarial audience. While falling short of a full practical application, I have provided numerical examples to illustrate how the concepts might be applied in a realistic setting.

The major qualitative results of this paper are summarized in section 1.1. Perhaps the chief among them are: (1) the optimal capital for long-horizon losses depends on *both* the annual loss volatility and the ultimate loss volatility, and will be greater than optimal capital based on the annual volatility, and (2) optimal capital for any horizon depends on the insurer's ability to raise capital, and its cost of raising capital. Analyzing the first relationship is largely a technical actuarial exercise, while analyzing the second involves understanding an insurer's connections to capital markets, ownership structure and internal information processes.

Knowing the optimal capital provides the basis for applications in product pricing, corporate governance and regulation. Due to the many variables involved, optimizing capital for multi-period insurance can be rather complicated and perhaps daunting, even when contemplating a simple model. However, as shown here, the simple one-period model provides the basic core of the multi-period model. The other parts can be assembled step-by-step to produce useful results. More complex models with multiple variables can be built using simulation techniques. In particular, asset risk could be analyzed jointly with loss risk.

The analysis in this paper has identified some important variables and factors that are not conventionally considered in setting capital standards for insurance. These include capital funding strategies, the cost of raising external capital, the capitalization interval, policy term, ownership structure and the effect of conservatorship. These areas provide a fertile source for future research.

## APPENDIX A: NUMERICAL EXAMPLES

### Section 4.2 Example

The loss stochastic process can be illustrated with a simple two-period binary example. The initial expected loss is 1000 and the reserve increments $X_1$ and $X_2$ each can be either 200, or –200 with probability 0.5, giving a per-period variance of $(200)^2$. Let $a = .0025$; from equation 4.221 we have $\hat{L}_1 = 1100$ and $\hat{L}_2 = 1200$. Thus the risk value per period is 100. The first period CE expected loss of 1100 is obtained by assigning a CE probability of 0.75 to the +200 reserve increment and 0.25 to the –200 increment.[44]

The evolution of the ultimate loss and its certainty-equivalent counterpart is shown in Figure 4.23 below. The first-period reserve increment probabilities and CE probabilities are denoted by $p_1$ and $\hat{p}_1$, with $p_2$ and $\hat{p}_2$ representing the second-period values.

[44] In this simple discrete example, the variance of the CE distribution for any period is less than that of the unadjusted distribution (30,000 vs. 40,000). However, for a continuous stochastic process, such as with normally distributed increments, it is possible to have the same variance for both distributions.

*Figure 4.231*
*Loss Reserve Evolution, Numerical Example*

| $L$ | $p_1$ | $\hat{p}_1$ | $L_1$ | $p_2$ | $\hat{p}_2$ | $L_2$ | $p_1 \times p_2$ | $\hat{p}_1 \times \hat{p}_2$ |
|---|---|---|---|---|---|---|---|---|
| | | | | 0.5 | 0.75 | 1400 | 0.25 | 0.5625 |
| | 0.5 | 0.75 | 1200 | | | | | |
| | | | | 0.5 | 0.25 | 1000 | 0.25 | 0.1875 |
| **1000** | | | | | | | | |
| | | | | 0.5 | 0.75 | 1000 | 0.25 | 0.1875 |
| | 0.5 | 0.25 | 800 | | | | | |
| | | | | 0.5 | 0.25 | 600 | 0.25 | 0.0625 |
| Expected Value | | | **1000** | | | **1000** | | |
| CE Expected Value | | | **1100** | | | **1200** | | |

Notice that for each period the variance of the loss increment is the same and that the variance of the evolved loss increases over time. Meanwhile, the mean for each subsequent period equals the value of the loss from the prior period: for instance, if $L_1$ becomes 1200 at the end of period 1, then 1200 is the mean for period 2. The CE value of the second-period loss conditional on the emerged 1200 amount is 1200 plus the 100 risk value for the second period, or 1300.

## Section 5.1 Example

Assume that the expected loss is 1000 and increments for each period range from –400 to 400 in steps of 50; the corresponding probabilities are generated by a binomial distribution having a base probability 0.5 with 16 trials. Thus the probability of a 400 increment is $(0.5)^{16}$, the probability of a 350 increment is $16(0.5)^{16}$, and so forth. The expected value of the increments is zero and the variance is $(100)^2$. For the parallel CE stochastic loss process, assume that the base probability is 0.625, giving a higher subjective likelihood of larger increments: the probability of a 400 increment is $(0.625)^{16} = 0.00054$ and the probability of a 350 increment is $16(0.625)^{15}(0.375) = 0.00520$. The CE expected value of the increment is 100, so the CE expected loss increases by 100 each period.

Now suppose that initial assets are 1300, so a technical insolvency occurs if the first-period loss is either 1350 or 1400 (the maximum possible loss). When the technical insolvency occurs, the assets remain fixed at 1300, but the loss can still develop for one more period. Consequently, if the first-period loss is 1350, its value at the end of the second period is one of $\{1350 - 400, 1350 - 350, \cdots, 1350 + 400\}$, or $\{950, 1000, \cdots, 1750\}$. However, only the amounts $\{1350, 1400, \cdots, 1750\}$ will produce a default when the loss is settled at the end of the second period. The respective CE probabilities for these amounts are $\{0.11718, 0.17361, \cdots, 0.00054\}$. Weighting the possible default amounts by their occurrence probabilities gives 152.59, the conditional CED given that the 1350 loss amount occurs.

For the 1400 first-period loss, the range of its possible second-period values that produce an ultimate default is larger: from 1350 to 1800. Thus, its conditional CED is larger, at 200.72, than that for the 1350 loss amount. Table 5.1 outlines these calculations.

*Table 5.1*
*Conditional Certainty-Equivalent Expected Default*
*Two-Period Numerical Example*
*Discrete Stochastic Process*

| One-period Loss | CE Probability | 0.00054 | 0.00520 | . . . | 0.06250 | 0.02625 | Total |
|---|---|---|---|---|---|---|---|
| **1400** | 2P Loss | 1800 | 1750 | . . . | 1400 | 1350 | |
| | Default | 500 | 450 | . . . | 100 | 50 | |
| | CE Expected Default | 0.27 | 2.34 | | 11.72 | 3.12 | **200.72** |
| **1350** | 2P Loss | 1750 | 1700 | . . . | 1350 | | |
| | Default | 450 | 400 | . . . | 50 | 0 | |
| | CE Expected Default | 0.2439 | 2.0817 | . . . | 5.8592 | 0 | **152.59** |

The unconditional CED is determined by weighting the above conditional amounts by the CE probabilities of the 1350 and 1400 loss values occurring. We get 0.9029 = 0.00054(200.72) + 0.00520(152.59). Notice that under the FR strategy, with the same 1300 in initial assets, the technical insolvency at the end of the first period is converted to a hard insolvency. So the CED equals the possible default amounts (50 = 1350 – 1300 and 100 = 1400 – 1300) multiplied by the respective CE probabilities: 0.3144 = 0.00520(50) + 0.00054(100). For comparison with the FR strategy, notice that for each loss value producing a default (e.g., 1350) the default amount (50 here) is fixed under FR, but will develop under AC (the CE expected value is 152.59). For a positive second-period variance, the mathematical properties of the default calculation ensure that the conditional expected ultimate default is *greater* than that of the original first-period default: the default value cannot be negative; it equals zero if the loss develops favorably. This asymmetry increases the expected default amount from its initial value.

## Section 5.31 Example

Consider a two-period normal stochastic loss process with a mean of 1000 and variance of the loss increment equal to $100^2$ for each period. Assume that the CE of the expected loss after one period is 1100 and that the risk value (the CE of the loss minus its expected value) at each development stage is strictly proportional to the cumulative variance as in section 4.22. Thus, the CE value of the ultimate loss at the end of the second period is 1200. At each stage of loss development the CE loss is normally distributed with the same variance as the unadjusted distribution, but with the adjusted mean equal to the CE value of the evolved loss.

Assume that the frictional capital cost $z = 2\%$. Thus for one period, the optimal assets are such that the CE default probability is 0.02. This occurs with 1305.37 of assets available to pay losses and ownership capital of 305.37. The premium of 1006.1075 is the 1000 of expected loss plus 6.1075 = 0.02(305.37) of capital cost, so initial assets are the premium plus the 305.37 of ownership capital. However, the capital cost is expended prior to the loss payment, so the assets available to pay the loss include the capital plus the expected loss portion of the premium. The CED is 0.7343.

For two periods, under the FR strategy, the optimal total capital for the first period is also 305.37. However, the ownership capital is less than 305.37 by the amount of the expected second-period capital cost (which is policyholder-supplied capital contained in the premium) of 6.1075, so the first-period OC equals 299.26. The optimal solvency cost is 13.5613 = 2(.07343) + 6.1075 + 0.2(299.26).

For two periods, under the AC strategy, suppose we begin with the optimal one-period optimal available assets of 1305.37. The expected second-period capital cost is the optimal one-period capital cost of 6.1075, since under the normal distribution, the optimal one-period capital depends only on the variance for that period.

However, the CE expected default is higher. Using equation 5.211,[45] with available assets of 1305.37, the CE expected default for technical insolvency after the first period is $\hat{G} = 2.8274$. Since the CE probability of default is 0.02, the CE probability of remaining solvent after one period is 0.98. Because the insurer begins the second period

[45] For this calculation, I used 1,000 discrete ending first-period loss values to determine an approximate value.

with optimal capital, the second-period CED is 0.7343. Thus, the CE expected default for remaining solvent after one period is $\hat{H} = 0.7196 = 0.98(0.7343)$. The total CED is $3.5471 = 2.8274 + 0.7196$. For the first period, the ownership capital is 299.26, which is less than the total capital by the 6.1075 expected second-period capital cost. Accordingly, the first-period capital cost equals 5.9852. Thus the total capital cost is $12.0927 = 5.98522 + 6.1075$ and the solvency cost is $15.6398 = 3.5471 + 12.0927$.

The AC strategy consumer value can be improved by increasing the initial capital: the optimal available asset amount for the AC strategy is 1357.67, which produces an optimal ownership capital of 351.56 and an optimal total capital of 357.67 (the difference is the expected unexpended second-period capital cost of 6.1075). The first period capital cost is 7.0312, giving a total expected capital cost of 13.1387. With higher available assets, the CE default probability for the first period drops to 0.0050 and $\hat{G}$ reduces to 0.6813. The CE probability of remaining solvent is 0.9950, so $\hat{H}$ is higher at $0.7307 = 0.9950(0.7343)$, giving a total CED of 1.4120.

The optimal solvency cost is $14.5507 = 1.4120 + 13.1387$, which is an improvement of 1.2114 over than the above case using the optimal one-period available assets. However, it is greater than the 13.5613 optimal FR solvency cost.

## Section 5.42 Example

Suppose that the cost of raising capital is $w = 3\%$ and initial assets are 1400. Thus we get $C_R^* = 264.49$ (from equation 5.411) and $C^* = 305.37$. We need to determine the expected cost of the capital and the CE value of the expected default.

Suppose that the initial first-period capital is 400. Table 5.422 shows the beginning second-period expected ownership capital amounts by region.

*Table 5.422*
*Expected OC by Region*
*Two-Period AC Strategy with Cost of Raising Capital*
*Normal Example*

| Region | Capital Carried (a) | Capital Raised (b) | Probability (c) | Expected Value [c(a + b)] |
|---|---|---|---|---|
| 1 | 0 | 0 | 0.00003 | 0.00 |
| 2a | 218.45 | 46.04 | 0.08765 | 23.18 |
| 2b | 286.52 | 0 | 0.08432 | 24.16 |
| 3 | 305.37 | 0 | 0.82799 | 252.85 |
| Total | | | 1.00000 | 300.19 |

The capital carried (a) and raised amounts (b) are the conditional expected values given that ending first-period capital is in the particular region. The region 2a unconditional expected capital carried amount is determined from equation 5.421, and equals 19.9382. The probability of the loss being in this region is 0.08765, so the conditional expected capital amount is 218.45 = 19.9382/0.08765. The sum of the expected capital carried and capital raised amounts for region 2a equals the optimal capital-if-raised amount 264.49. Thus, the conditional expected amount of capital raised is 46.04 = 264.49 – 218.45. The region 2b capital carried amounts are determined from equation 5.422 in a similar manner.

Since the expected amount of OC in the second period is 300.19, the expected cost of holding the capital is 6.0038 = 0.02(300.19). The first-period holding cost is 8.0000 = 0.02(400), so the total capital cost is 14.0038. The expected amount of capital raised is 4.0353 = 46.04(0.08765), giving an expected CRC of 0.1211 = 0.03(4.0353). Therefore, with initial capital of 400, the total expected capital cost is 14.1249.

The CE expected default amount can also be determined by region, as shown in Table 5.423.

*Table 5.423*
*Expected CE Default by Region*
*Two-Period AC Strategy with Cost of Raising Capital*
*Normal Example*

| Region | CE Default | CE Probability | Expected Value |
|---|---|---|---|
| 1 | 133.55 | 0.0014 | 0.1803 |
| 2 | 2.0893 | 0.3599 | 0.7519 |
| 3 | 1.4118 | 0.1602 | 0.2233 |
| 4 | 0.7343 | 0.4786 | 0.3514 |
| Total | | 1.000 | 1.3677 |

The solvency cost is the CED plus the expected capital cost: 15.6318 = 1.3677 + 14.1249. Minimizing the solvency cost by varying the initial OC gives an optimal initial OC of 367.57 and a total capital of 377.73.

## Section 7.4 Example

Assume that the capitalization interval is one-half year. If the policy period is one half year, the optimal capital is 214.50, the CED is 0.2396 and capital cost is 2.1450 (giving a 2.3846 solvency cost). However, if the policy term is one year, more capital is required at the beginning of the policy term. If we start with the 214.50 in capital the CE probability of default in the first half-year is 1% (equal to the capital cost rate for the period). If default occurs, the consumer value of the foregone coverage equals the CE value of the remaining loss, minus the CED that would accompany the coverage. The CE loss value is 550 (one-half of 1100) and the CED is 0.2396, so the value of the lost coverage is 5.4976 = 0.01(550 – 0.2396). The CE value of the second period default is the CED given that the insurer is solvent after the first period, times the CE default probability, or 0.2372 = 0.2396(0.99). The expected second-period capital cost is the capital cost given that the insurer is solvent after the first period, times the unadjusted default probability, or 2.1424 = 2.1450(0.9998); this gives a second-period solvency cost (excluding the foregone coverage cost) of 2.3796 = 0.2372 + 2.1424.

Thus the total solvency cost is 10.2618 = 2.3846 + 5.4976 + 2.3796. But, by increasing the initial capital, we can reduce the solvency cost: it is minimized at 5.4800 when the capital is 290.22. Notice that both of these values are lower than the case of an annual policy with an annual capitalization interval. Keeping the annual policy term and reducing the capitalization interval to ¼ year, we get (using the backward induction method from section 6) the optimal capital of 214.69 with a corresponding 3.3919 solvency cost.

### Section 9.2 Example

Assume that the fixed loss amount is 1000 and that the asset value follows a normal stochastic process (random walk) with an standard deviation of 50 per period. Following the section 5 example, assume that $z = 2\%$. Therefore the optimal CE default probability is 0.02. The optimal capital for a one-period loss is 102.69 with a CED of 0.3672. Thus the solvency cost is 2.4209 = 0.02(102.69) + 0.3672.

For a two-period fixed loss of 1000 and the same stochastic process for asset values, suppose that we use the same 102.69 of beginning capital under the AC strategy. The solvency cost for the first period is the same as for the one-period model. Since capital is added or withdrawn to maintain the optimal level for the second period, the CE expected default for the second period is the optimal first period CED times the CD probability of remaining solvent: 0.3596 = 0.3762(0.98).

Assume that the expected return on the assets is 4%. This gives an expected first-period ending asset amount of 1146.79 and thus the unadjusted probability of remaining solvent is 0.9983. The expected capital cost for the second period is then 2.0503 = 0.9983(0.02)(102.69), giving a second-period solvency cost of 2.4101 and a total solvency cost for both periods of 4.8311.

Using equation 8.12, the optimal first-period capital under the AC strategy is 102.07, which is slightly *less* than that of the optimal one-period and the two-period FR amount. Here the solvency cost is minimized at 4.8309. Under the FR strategy the solvency cost is twice the one-period amount, or 4.8418.

## APPENDIX B: BACKWARD INDUCTION WITH ACR STRATEGY

Under the ACR strategy, there are two optimal ownership capital amounts to consider at each stage $i$ of the iteration. The first is the optimal OC given the current loss value is *small enough* to withdraw capital. This is the amount $C^*_{i-1}(x)$ defined under the AC strategy. The second is the optimal capital $CR^*_{i-1}(x)$ given the current loss value is *large enough* to add capital (by raising it externally).

At each stage $i$, there are now six optimal quantities that we need to calculate: the three from the AC strategy (capital, CED and capital cost), and their counterparts given that capital is raised: the optimal capital is defined above, the optimal CED is $\hat{D}R^*_{i-1}(x)$ and the optimal capital cost $KR^*_{i-1}(x)$.

At each stage, the three capital-raising components are found by using a capital cost for the current period of $z + w$ instead of only $z$. We then have a parallel calculation of the solvency cost $SR_i = \hat{D}R_i + KR_i$, which is minimized by changing the asset amount.

Also, at each stage it is necessary to calculate the CED and capital cost components for region 2a (where capital is neither raised nor withdrawn) by numerical integration: we vary the capital amount in this region and weight the results by the corresponding loss probabilities.

To illustrate this process, we use the basic normal example with a 3% CRC. For one period we have the key variables $C^*_1 = 305.37$, $\hat{D}^*_1 = 0.7343$, $K^*_1 = 6.1075$, $CR^*_1 =$ 264.49, $\hat{D}R^*_1 = 2.0893$ and $KR^*_1 = 5.2897$. To obtain the optimal two-period value $C^*_2$, we start with the optimal one-period assets of 1305.37 and calculate the solvency cost based on the capital cost rate for the first period of $z = 0.02$: $S_2 = 17.1103$. This is done by adding the CED and capital cost components for the four regions of first-period loss outcomes (see section 5.42). This calculation uses the above six key variables. We perform a parallel calculation with the assets increased by a small amount $\Delta A$. With $\Delta A = 0.01$, we get $S_2 = 17.1097$. This gives a change in the solvency cost of $\Delta S_2 = -0.0006$. We raise the asset amount (and its $A + \Delta A$ counterpart) until $\Delta S_2 = 0$. This occurs when $A = 1377.73$ and $S_2 = 15.5035$, giving $T^*_2 = 377.73$, $C^*_2 =$ 367.57, $\hat{D}^*_2 = 1.8682$ and $K^*_2 = 13.6353$.

We next do a parallel calculation where the first period capital cost is $z + w = 0.05$. This provides the optimal values of the key variables for the case where capital is raised after the first period of a three-period horizon (we are preparing for the next stage of the induction procedure). Here we get $CR_2^* = 328.98$, $\hat{D}R_2^* = 3.1924$ and $KR_2^* = 12.8794$.

We continue the induction process to get the optimal key variables for longer horizons.

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## GLOSSARY OF ABBREVIATIONS AND NOTATION

| **Abbreviation** | Meaning | Section Where Defined |
|---|---|---|
| AC | Add capital (strategy) | 4.5 |
| ACR | Add capital (strategy) with cost of raising capital | 5.4 |
| CE | Certainty equivalent | 2.1 |
| CED | Certainty equivalent expected default | 2.1 |
| CI | Capitalization interval | 7.3 |
| CW | Capital withdrawal (strategy) | 4.5 |
| EBRM | Economic Basis … Risk Based Capital Measurement | 1 |
| FA | Fixed assets (strategy) | 4.5 |
| FCC | Frictional capital cost | 2.2 |
| FR | Full recapitalization (strategy) | 4.5 |
| OC | Ownership Capital | 4.4 |
| SH | Stochastic Horizon | 8.1 |
| VaR | Value-at-risk | 1 |
| TVar | Tail Value-at-risk | 1 |

| **Variable** | Meaning | Where Defined |
|---|---|---|
| $a$ | Risk aversion parameter | 4.2 |
| $A$ | Assets | 2.1 |
| $C$ | Capital (ownership) | 4.1 |
| $CF$ | Capital flow | 4.5 |
| $D$ | Expected default | 2.1 |
| $DR$ | Expected default if capital is raised | App. B |
| $E(\ )$ | Expectation operator | 4.3 |
| $EF$ | Expected capital carried forward | 5.4 |
| $G$ | Expected default under technical insolvency | 5.1 |
| $H$ | Expected default for remaining periods | 5.2 |
| $i$ | Period index | 4.3 |
| $K$ | Expected capital cost | 4.3 |
| $KR$ | Expected capital cost if capital is raised | App. B |

| | | |
|---|---|---|
| $L$ | Expected loss | 2.1 |
| $M$ | Risk margin value | 8.3 |
| $N$ | Number of periods | 4.1 |
| $p(\ )$ | Probability density | 2.1 |
| $P(\ )$ | Cumulative probability | 6.2 |
| $q$ | Probability of period length | 8.1 |
| $Q$ | Default probability | 2.2 |
| $r$ | Risk-free interest rate | 8.2 |
| $R$ | Expected return on capital | 8.3 |
| $S$ | Solvency cost | 5.2 |
| $t$ | Income tax rate | 8.3 |
| $T$ | Total capital | 4.4 |
| $V$ | Consumer value | 2.2 |
| $w$ | Cost of raising capital | 5.4 |
| $x$ | Loss or asset size | 2.1 |
| $X$ | Reserve increment | 4.2 |
| $Y$ | Ratio of successive reserve amounts | 4.2 |
| $z$ | Frictional cost of capital | 2.2 |
| $\partial$ | Partial derivative operator | 5.2 |
| $\Delta$ | Asset increment | App. B |
| $\pi$ | Premium | 2.2 |
| $\sigma$ | Loss standard deviation | 4.2 |
| **Subscript** | | |
| $a$ | Region 2a | 5.4 |
| $b$ | Region 2a | 5.4 |
| $E$ | Ending capital | 5.4 |
| $R$ | Raising capital | 5.4 |
| $t$ | Elapsed time | 4.2 |

## BIOGRAPHY OF THE AUTHOR

**Robert P. Butsic** is a retired actuary currently residing in San Francisco. He served as a member of the American Academy of Actuaries Property-Casualty Risk Based Capital Committee and is a member of the Casualty Actuarial Society's Risk-Based Capital Dependency and Calibration Working Group. He previously worked for Fireman's Fund Insurance and CNA Insurance. He is an Associate in the Society of Actuaries, has a B.A. in mathematics and an MBA in finance, both from the University of Chicago. He has won the Casualty Actuarial Society's Michelbacher Award (for best Discussion Paper) five times. Since the 2008 financial crisis he has enjoyed reading economics blogs, which have contributed to the development of this paper.