The Capital Asset Pricing Model: An Insurance Variant

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Overview

Part 1: Fundamentals (Ricardas Zitikis)

- Classics
- From CAPM to WIPM
- Weighted premiums
- Estimating β_w (analogue of β)
- Weighted Gini allocations
- Estimating $\beta_{w,Gini}$
- Tail WIPM

Part 2: Applications (Edward Furman)

Classics

CAPM

$$Expect[R_i] = r_f + \frac{\mathrm{Cov}[R_i, R_m]}{\mathrm{Cov}[R_m, R_m]}(Expect[R_m] - r_f)$$

where

- the expected return on the asset $Expect[R_i]$
- the expected market rate of return $Expect[R_m]$
- the risk free rate of return r_f

Statistically speaking, CAPM is

$$E[R_i|R_m] = E[R_i] + \frac{\text{Cov}[R_i, R_m]}{\text{Cov}[R_m, R_m]}(R_m - E[R_m])$$

when (R_i, R_m) is bivariate normal

We want to

- depart from normality
- work with all (light and heavy) tails
- instead of expectations, use better risk measures (e.g. CTE)

From CAPM to WIPM

- CAPM re-written

$$E[X|S] = E[X] + \frac{\text{Cov}[X, S\,]}{\text{Cov}[S, S\,]}(S - E[S])$$

when $(X, S\,)$ is bivariate normal

- WIPM idea

$$\Pi[X|S] = E[X] + \ \beta\,(\pi[S] - E[S])$$

- Compare with the classical CAPM

$$Expect[R_i] = r_f + \beta_i\,(Expect[R_m] - r_f)$$

How does WIPM work?

Hint: $\Pi[X|S] = E[X] + \beta\,(\pi[S] - E[S])$

Example. Modified variance premium and allocation

$$\pi[S] = \frac{E[S^2]}{E[S]} \qquad \text{and} \qquad \Pi[X|S] = \frac{E[XS]}{E[S]}$$

$$\Pi[X|S] - E[X] = \frac{E[XS]}{E[S]} - E[X] = \frac{\mathrm{Cov}[X,S]}{E[S]}$$

$$= \frac{\mathrm{Cov}[X,S]}{\mathrm{Cov}[S,S]} \frac{\mathrm{Cov}[S,S]}{E[S]} = \beta\, \frac{E[S^2] - (E[S])^2}{E[S]}$$

$$= \beta\,(\pi[S] - E[S])$$

Weighted premiums and allocations

$$\pi_w[S] = \frac{E[Sw(S)]}{E[w(S)]} \quad \text{and} \quad \Pi_w[X|S] = \frac{E[Xw(S)]}{E[w(S)]}$$

- Size-biased

$$w(s) = s^{\lambda}$$

- Esscher

$$w(s) = e^{\lambda s}$$

- Kamps

$$w(s) = 1 - e^{-\lambda s}$$

- Excess-of-loss

$$w(s) = 1\{s > \lambda\} \text{ that is } \Pi_w[X|S] = E[X|S > \lambda] = \frac{E[X1\{S > \lambda\}]}{E[1\{S > \lambda\}]}$$

How does weighted premium work?

$$\pi_w[S] = \frac{E[Sw(S)]}{E[w(S)]} = \int s \frac{w(s)f(s)}{E[w(S)]} ds = \int s f_w(s) ds$$

where

$$f_w(s) = \frac{w(s)f(s)}{E[w(S)]}$$

is the weighted density

$$\int f_w(s)ds = \frac{1}{E[w(S)]} \int w(s)f(s)ds = \frac{E[w(S)]}{E[w(S)]} = 1$$

How does WIPM work?

Hint: $\Pi[X|S] = E[X] + \beta(\pi[S] - E[S])$

$$\Pi_w[X|S] - E[X] = \frac{E[Xw(S)]}{E[w(S)]} - E[X] = \frac{\mathrm{Cov}[X, w(S)]}{E[w(S)]}$$

$$= \frac{\mathrm{Cov}[X, w(S)]}{\mathrm{Cov}[S, w(S)]} \frac{\mathrm{Cov}[S, w(S)]}{E[w(S)]}$$

$$= \beta_w[X, S] \frac{E[Sw(S)] - E[S]E[w(S)]}{E[w(S)]}$$

$$= \beta_w[X, S]\,(\pi_w[S] - E[S])$$

Calculating $\beta[X,S]$

$$\beta_w[X,S] = \frac{\mathrm{Cov}[X, w(S)]}{\mathrm{Cov}[S, w(S)]} = \frac{\mathrm{Cov}[E[X|S], w(S)]}{\mathrm{Cov}[S, w(S)]}$$

- When

$$E[X|S=s] = a + bs$$

  we have

$$\beta_w[X,S] = \frac{\mathrm{Cov}[a + bS, w(S)]}{\mathrm{Cov}[S, w(S)]} = b\,\frac{\mathrm{Cov}[S, w(S)]}{\mathrm{Cov}[S, w(S)]} = b$$

- In the bivariate normal case

$$b = \beta = \frac{\mathrm{Cov}[X,S]}{\mathrm{Cov}[S,S]} \in R$$

- *Note*: Edward will argue that in WIPM, we usually have $a = 0$ and $b \geq 0$

Estimating $\beta_w[X, S]$

- Individual "tricks" (such as linear regression, and slope β estimators)
- Brute force parametric stats (write $\beta_w[X, S]$ in terms of parameters)
- Non-parametric stats

$$\beta_w[X, S] = \frac{\mathrm{Cov}[X, w(S)]}{\mathrm{Cov}[S, w(S)]} \approx \frac{\sum(X_i - \bar{X})\left(w(S_i) - \overline{w(S)}\right)}{\sum(S_i - \bar{S})\left(w(S_i) - \overline{w(S)}\right)}$$

where

$$\bar{X} = \frac{1}{n}\sum X_i \quad \text{and} \quad \overline{w(S)} = \frac{1}{n}\sum w(S_i)$$

Weighted Gini premiums and allocations

$$\pi_{w,Gini}[S] = \frac{E[Sw(F(S))]}{E[w(F(S))]} \quad \text{and} \quad \Pi_{w,Gini}[X|S] = \frac{E[Xw(F(S))]}{E[w(F(S))]} \quad (F = cdf\ S)$$

- Proportional hazards

$$w(t) = p(1-t)^{p-1}$$

- Distortion

$$w(t) = g'(1-t)$$

- Aumann-Shapley

$$w(t) = e^{pt}$$

- Conditional tail expectation

$$w(t) = 1\{t > p\} \quad \text{that is} \quad \Pi_w[X|S] = E[X|S > s_p] = \frac{E[X1\{S > s_p\}]}{E[1\{S > s_p\}]}$$

How does Gini WIPM work?

$$\Pi_{w,Gini}[X|S] - E[X] = \frac{E[Xw(F(S))]}{E[w(F(S))]} - E[X] = \frac{\mathrm{Cov}[X, w(F(S))]}{E[w(F(S))]}$$

$$= \frac{\mathrm{Cov}[X, w(F(S))]}{\mathrm{Cov}[S, w(F(S))]} \frac{\mathrm{Cov}[S, w(F(S))]}{E[w(F(S))]}$$

$$= \beta_{w,Gini}[X, S] \frac{E[Sw(F(S))] - E[S]E[w(F(S))]}{E[w(F(S))]}$$

$$= \beta_{w,Gini}[X, S] \left(\pi_{w,Gini}[S] - E[S]\right)$$

What is $\beta_{w,Gini}[X,S]$?

- Gini correlation (Gini, a hundred years ago)

$$\frac{\mathrm{Cov}[X, F(S)]}{\mathrm{Cov}[S, F(S)]} \qquad \ldots\ w(u) = u$$

- Extended Gini correlation (Yitzhaki & Schechtman, few decades ago)

$$\frac{\mathrm{Cov}[X, (1-F(S))^{\nu}]}{\mathrm{Cov}[S, (1-F(S))^{\nu}]} \qquad \ldots\ w(u) = (1-u)^{\nu}$$

- Weighted Gini correlation (Edward and I, our CAS report)

$$\beta_{w,Gini}[X,S] = \frac{\mathrm{Cov}[X, w(F(S))]}{\mathrm{Cov}[S, w(F(S))]}$$

Calculating $\beta_{w,Gini}[X,S]$

$$\beta_{w,Gini}[X,S] = \frac{\text{Cov}[X, w(F(S))]}{\text{Cov}[S, w(F(S))]} = \frac{\text{Cov}[E[X|S], w(F(S))]}{\text{Cov}[S, w(F(S))]}$$

- When

$$E[X|S=s] = a + bs$$

we have

$$\beta_{w,Gini}[X,S] = \frac{\text{Cov}[a + bS, w(F(S))]}{\text{Cov}[S, w(F(S))]} = b\frac{\text{Cov}[S, w(F(S))]}{\text{Cov}[S, w(F(S))]} = b$$

- In the bivariate normal case

$$b = \beta = \frac{\text{Cov}[X,S]}{\text{Cov}[S,S]} \in R$$

- *Note*: Edward will argue that in WIPM, we usually have $a = 0$ and $b \geq 0$

Estimating $\beta_{w,Gini}[X,S]$

- Individual "tricks" (such as linear regression, and slope β estimators)
- Brute force parametric stats ($\beta_{w,Gini}[X,S]$ in terms of parameters)
- Non-parametric stats (more complex than for $\beta_w[X,S]$)

$$\beta_{w,Gini}[X,S] = \frac{\mathrm{Cov}[X, w(F(S))]}{\mathrm{Cov}[S, w(F(S))]} = \frac{\mathrm{Cov}[r(S), w(F(S))]}{\mathrm{Cov}[S, w(F(S))]}$$

$$= \frac{\mathrm{Cov}[r(F^{-1}(U)), w(U)]}{\mathrm{Cov}[F^{-1}(U), w(U)]}$$ where $r(s) = E[X|S=s]$

which connects with L-stats $L_F = \int_0^1 g(F^{-1}(u))w(u)du$

Estimating $\beta_{w,Gini}[X,S]$

$$L_F = \int_0^1 g(F^{-1}(u))w(u)du \approx \int_0^1 g\left(F_n{}^{-1}(u)\right)w(u)du$$

$$= \sum_{i=1}^{n} g(S_{i:n}) \int_{(i-1)/n}^{i/n} w(u)du$$

where $S_{1:n} < \cdots < S_{n:n}$ are the ordered observations $S_1, \ldots, S_n$ of S

$$\beta_{Gini}[X,S] = \frac{\mathrm{Cov}[r(F^{-1}(U)), w(U)]}{\mathrm{Cov}[F^{-1}(U), w(U)]} \approx \frac{\sum_{i=1}^{n} \hat{r}(S_{i:n}) \int_{(i-1)/n}^{i/n} w(u)du}{\sum_{i=1}^{n} S_{i:n} \int_{(i-1)/n}^{i/n} w(u)du}$$

One more thing: tail-WIPM (needed for Edward's talk)

$$\frac{E[Xw(F(S))|S > s_p]}{E[w(F(S))|S > s_p]} - E[X|S > s_p]$$

$$= \beta_{w,Gini,p}[X,S]\left(\frac{E[Sw(F(S))|S > s_p]}{E[w(F(S))|S > s_p]} - E[S|S > s_p]\right)$$

where

$$\beta_{w,Gini,p}[X,S] = \frac{\mathrm{Cov}[X,w(F(S))|S > s_p]}{\mathrm{Cov}[S,w(F(S))|S > s_p]} \;[\text{ when } w(u) = u\,] = \frac{\mathrm{TGini}_p[X,S]}{\mathrm{TGini}_p[S,S]}$$

where

$$\mathrm{TGini}_p[X,S] = \frac{4}{1-p}\mathrm{Cov}[X,F(S)|S > s_p]$$

is the tail-Gini measure of variability