The Capital Asset Pricing Model: An Insurance Variant

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Outline

WIPM versus CAPM

Let's confine to a pair of loss random variables (r.v.'s) (X, Y) with the aggregate loss r.v. $S = X + Y$. Similarly to the CAPM, the WIPM provides a price of the loss X, when it's considered a part of the risk portfolio (r.p.) (X, Y). However, there are differences:

	CAPM	WIPM
Risk measure (r.m.)	modified variance	any weighted r.m.
Heavy-tailedness	finite 2nd moment	finite 1st moment
Risks' distribution	real domains	?
	symmetric	?
	same (tail) dependence	?

Recall

We want to have a CAPM-like equation for r.p.'s (X, Y) of positive losses, possibly with heavy tails and positive skewness. Is it feasible at all?

WIPM versus CAPM

Positivity of the losses implies that $\mathbf{E}[X|\ S=0]=\mathbf{E}[Y|\ S=0]=0$, and so, we have immediately that:

	CAPM	WIPM
Assumption	$\mathbf{E}[X\|\ S=s]=a+bs$	$\mathbf{E}[X\|\ S=s]=bs$

Moreover, in the CAPM the 'beta' can in principle be negative, but not so in the WIPM. Indeed, $\mathbf{E}[\mathbf{E}[X|\ S]]=b\mathbf{E}[S]$ yields $b=\mathbf{E}[X]/\mathbf{E}[S]$.

Problem

Describe a class of r.p.'s (X,Y) that have decumulative distribution functions (d.d.f.'s) such that for $S=X+Y$, the regression of the loss r.v. X on the aggregate loss r.v. S is linear.

Recall

Let $(X, Y) \sim N_2(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} := (\mu_X, \mu_Y)'$ is the vector of means, and Σ is the variance-covariance matrix with diagonal entries σ_X^2 and σ_Y^2, and the off-diagonal entries equal to $\sigma_{X,Y}$. Then $(X, S) \sim N_2(\boldsymbol{\mu}^*, \Sigma^*)$ with

$$\boldsymbol{\mu}^* = (\mu_X, \mu_S)'$$

and

$$\Sigma^* = \begin{pmatrix} \sigma_X^2 & \sigma_{X,S} \\ \sigma_{X,,S} & \sigma_Y^2 \end{pmatrix},$$

where $\mu_S = \mu_X + \mu_Y,\ \sigma_{X,S} = \sigma_X^2 + \sigma_{X,Y}$. Then, as it's well-known that in the normal case, the regression is linear, we readily have that

$$\mathbf{E}[X|\ S = s] = \mu_X + \frac{\sigma_{X,S}}{\sigma_S^2}(s - \mu_S),\ s \in (-\infty, \infty).$$

Good for the CAPM; can be extended to jointly elliptical losses to augment heavy-tailedness. No good for the WIPM; so what about positive loss r.v.'s?

I.I.D. loss r.v.'s

Let X and Y be independent and identically distributed (i.i.d.) Then as

$$\mathbf{E}[X|\ S = s] = \mathbf{E}[Y|\ S = s] \text{ for all } s \geq 0$$

and, also,

$$\mathbf{E}[X|\ S = s] + \mathbf{E}[Y|\ S = s] = \mathbf{E}[S|\ S = s] = s \text{ for all } s \geq 0,$$

we must have that

$$\mathbf{E}[X|\ S = s] = \frac{s}{2} \text{ for all } s \geq 0.$$

Then, for every w weight function, such that the quantities below are well-defined and finite, the price of X is simply:

$$\Pi_w[X, S] = \frac{1}{2}\pi_w[S].$$

Note

We have that $b = 1/2$ doesn't depend on the w weight function, and that

$$\pi_w[S] := \frac{\mathbf{E}[Sw(S)]}{\mathbf{E}[w(S)]}$$

doesn't carry any information about the joint c.d.f. of the pair (X, S).

Note

If the losses X and Y are not i.i.d., then $\mathbf{E}[X|\ S = s] = bs,\ s \geq 0$ comes as a reasonable initial guess, which is equivalent to assuming that

$$\frac{\mathbf{E}[X|\ S]}{\mathbf{E}[Y|\ S]} = \frac{\mathbf{E}[X]}{\mathbf{E}[Y]}.$$

Question

In short, a simple (a là the CAPM) WIPM equation is feasible for positive i.i.d. r.v.'s. Can we relax the i.i.d. assumption and yet have the WIPM?

Identically distributed but dependent loss r.v.'s

Simple multiplicative background risk model

Let X_1, Y_1 and Z be independent and positive r.v.'s. The former two can be interpreted as idiosyncratic risk factors (r.f.'s) and the latter one - as a systemic r.f. Assume that X_1 and Y_1 are identically distributed, and the c.d.f. of Z is arbitrary. Then set $X = ZX_1$ and $Y = ZY_1$. In this case, we again have that the 'weighted' price of X is

$$\Pi_w[X, S] = \frac{1}{2}\pi_w[S],$$

for every w weight function such that the quantities above are well-defined and finite.

Example

Put $X_1 \sim \mathcal{E}_1$, $Y_1 \sim \mathcal{E}_1$ and assume that Z is distributed inverse gamma with some parameters. Then the r.p. $(X, Y) \stackrel{d}{=} (ZX_1, ZY_1)$ was considered by, e.g., Albrecher et al. (2011), IME in the context of Ruin Theory.

Non-identically distributed but independent loss r.v.'s

Simple additive background risk model

Let X and Y be two independent loss r.v.'s with infinitely divisible d.d.f.'s, such that the following equation holds $M_X(t) = (M_Y(t))^\gamma$, where M_X and M_Y are moment generating functions of X and Y, respectively, and $\gamma = \mathbf{E}[X]/\mathbf{E}[Y]$. Then, for every w weight function, such that the expressions below make sense, the price of X is

$$\Pi_w[X, S] = \frac{\mathbf{E}[X]}{\mathbf{E}[S]}\pi_w[S].$$

Note

Even though X and Y are independent, the r.v.'s X and S are not so.

Note

Gamma, inverse Gaussian (cont.), binomial (disc.), and compound Poisson with gamma secondary c.d.f. (mixed) are all infinitely divisible distributions. In fact, the log-normal and Pareto distributions are so too.

Non-identically distributed and dependent loss r.v.'s 1

Additive background risk model: continuation

Let X_1, Y_1 and Z be independent but not necessarily identically distributed r.v.'s. Set $X = Z + X_1$ and $Y = Z + Y_1$, which reduces to our previous example for $Z = 0$ almost surely. Then if there exist constants

$$b_1 = \frac{\mathbf{E}[X_1]}{\mathbf{E}[X_1] + 2\mathbf{E}[Z]} \quad \text{and} \quad b_2 = \frac{\mathbf{E}[Y_1]}{\mathbf{E}[Y_1] + 2\mathbf{E}[Z]},$$

such that

$$\mathbf{E}[X_1 \mid X_1 + 2Z = s] = b_1 s \quad \text{for all} \quad s \geq 0$$

and

$$\mathbf{E}[Y_1 \mid Y_1 + 2Z = s] = b_2 s \quad \text{for all} \quad s \geq 0,$$

then the 'weighted' price of the loss r.v. X is

$$\Pi_w[X,\ S] = \frac{\mathbf{E}[X_1] + \mathbf{E}[Z]}{2\mathbf{E}[Z] + \mathbf{E}[X_1] + \mathbf{E}[Y_1]} \pi_w[S].$$

Note

Even in this quite general case, we have that b doesn't depend on the w weight function.

Example

Let the idiosyncratic r.f.'s be distributed $\Gamma_{\gamma_1,\alpha}$ and $\Gamma_{\gamma_2,\alpha}$, and the systemic r.f. be distributed $\Gamma_{\gamma_0,2\alpha}$; all parameters are positive. Then we arrive at the set-up of, e.g., F& Landsman (2005), Alai et al. (2013) and Xu & Mao (2013), all published in the IME. For $\gamma_+ := \gamma_0 + \gamma_1 + \gamma_2$, we have that

$$b = \frac{\gamma_0 + 2\gamma_1}{\gamma_+},$$

and the price of the loss r.v. X is easy to calculate within the general weighted class of r.m.'s.
Same can be done for such other than gamma r.v.'s, as inverse Gaussian (Tweedie EDMs in general), Binomial, compound Poisson with gamma secondary c.d.f. Log-normal, Pareto?

Non-identically distributed and dependent loss r.v.'s 2

Multiplicative background risk model

Let the loss r.v. Z have any distribution on $[0,\infty)$, and let X_1 and Y_1 be jointly distributed Dirichlet with some parameters. Set $X = ZX_1$ and $Y = ZY_1$, then, for any w weight function, such that the quantities below are well-defined and finite, we have that the price of the loss r.v. X is

$$\Pi_w[X.S] = \frac{\mathbf{E}[X_1]}{\mathbf{E}[X_1] + \mathbf{E}[Y_1]} = \mathbf{E}[X_1]\pi_w[S].$$

Example

Assume that Z is distributed generalized Pareto with parameters γ, ξ, θ, all positive. Also, denote the Dirichlet parameters of X_1 and Y_1 by ξ_1 and ξ_2, which are positive and such that $\xi_1 + \xi_2 = \xi$. Then we arrive at the set-up in, e.g., Yang et al. (2011), IME.

Half-way through summary

In our discussion hitherto we have shown that the WIPM equation is feasible to have for loss r.v.'s that are:

- positive (versus real), and
- positively skewed (versus symmetric).

We have also shown that in the case when the WIPM holds, the underlying r.p.'s are such that:

- copulas of sub-portfolios can vary from one sub-portfolio to another (versus same copula for the entire r.p.), and consequently
- the tail dependence can be distinct for different sub-portfolios.

Last but not least, we have mentioned that

- the WIPM (Gini version) requires finiteness of the 1st moments only of the loss r.v.'s (versus 2nd moments, needed for the CAPM),

Question

Are infinite variances really supported by empirical evidences?

Infinite variances but finite expectations in real world

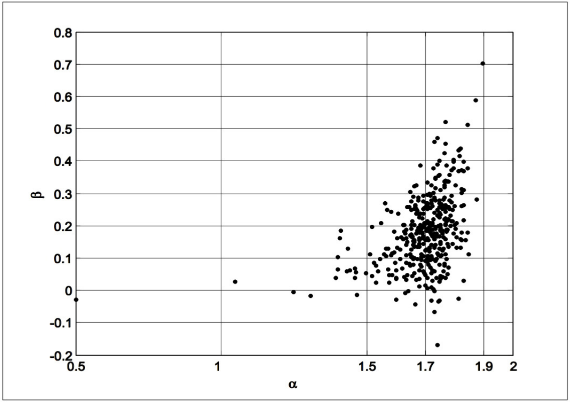

Figure 1: Scatter plot of the indices of stability and skewness for the daily returns of 382 stocks. Rachev et al. (2005)

An application to real data

Gini Shortfall

Let's look at the following r.m., for $p \in [0,\ 1)$:

$$GS_p^{\lambda}[X,S] := ES_p[X,S] + \lambda_p TGini_p[X,S],$$

where

$$\begin{aligned} ES_p[X,S] &= \mathbf{E}[X|\ S > s_p], \\ TGini_p[X,S] &= \frac{4}{1-p}\mathbf{Cov}[X, F_S(S)|\ S > s_p], \text{ and} \\ \lambda_p &= \frac{1-p}{2(1+p)}. \end{aligned}$$

We know that the GS r.m. is sub-additive, positively homogeneous, translation invariant and monotone - hence it's coherent. Moreover, the GS r.m. satisfies the SSD order, and it is also additive for co-monotonic loss r.v.'s. Last but not least, the GS r.m. is a weighted r.m. $(w(u) = u,\ u \in (0,\ 1))$.

Note

The GS r.m. is akin to the famous Standard Deviation (SD) r.m. Unlike the latter, however, the GS r.m. is coherent and exists for all loss r.v.'s with finite expectations, whereas the SD r.m. is not monotone (and so not coherent), and requires finite variances.

Note

We have that

$$\sum_{k=1}^{n} GS_p^{\lambda}[X_k, S] = GS_p^{\lambda}[S] := ES_p[S] + \lambda TGini_p[S].$$

Thus, the GS r.m. can be used as an additive allocation rule.

Example - Panjer and Jing (2001)

Consider a portfolio of ten $Student - t$ risks. Assume that

$$\boldsymbol{\mu} = (25.69,\ 37.84,\ 0.85,\ 12.70,\ 0.15,\ 24.05,\ 14.41,\ 4.49,\ 4.39,\ 9.56)'$$

and B is

$$\begin{pmatrix} 7.24 & 0 & 0.07 & -0.07 & 0.28 & -2.71 & -0.51 & 0.28 & 0.23 & -0.21 \\ 0 & 20.16 & 0.05 & 1.60 & 0.05 & 1.39 & 1.14 & -0.91 & -0.81 & -1.74 \\ 0.07 & 0.05 & 0.04 & 0.00 & -0.01 & 0.08 & 0.01 & -0.02 & -0.02 & -0.07 \\ -0.07 & 1.60 & 0.00 & 1.74 & 0.17 & 0.26 & 0.19 & -0.14 & 0.18 & -0.79 \\ 0.28 & 0.05 & -0.01 & 0.17 & 0.32 & -0.24 & 0.01 & -0.02 & 0.08 & -0.01 \\ -2.71 & 1.39 & 0.08 & 0.26 & -0.24 & 14.98 & 0.43 & -0.33 & -1.89 & -1.60 \\ -0.51 & 1.14 & 0.01 & 0.19 & 0.01 & 0.43 & 2.53 & -0.38 & 0.13 & 0.58 \\ 0.28 & -0.91 & -0.02 & -0.14 & -0.02 & -0.33 & -0.38 & 0.92 & -0.16 & -0.40 \\ 0.23 & -0.81 & -0.02 & 0.18 & 0.08 & -1.89 & 0.13 & -0.16 & 1.12 & 0.58 \\ -0.21 & -1.74 & -0.07 & -0.79 & -0.01 & -1.60 & 0.58 & -0.40 & 0.58 & 6.71 \end{pmatrix}.$$

Clearly, $\mu_S = 134.13$ and $\sigma_S = 6.726$.

Example - cont.

Also, we readily have that (X_k, S) is distributed $t_2(\boldsymbol{\mu}_{k,S}, B_{k,S}, q)$, where, for, $k = 1, \ldots, n$ and

$$A_k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \overbrace{1}^{k\text{-th}} & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}'$$

the new parameters are

$$\boldsymbol{\mu}_{k,S} = A_k' \boldsymbol{\mu} \text{ and } B_{k,S} = A_k' B A_k.$$

And according to the WIPM, we obtain that:

$$GS_p^\lambda[X_k, S] = \mu_k + \frac{\beta_{k,S}}{\beta_S^2} \left(GS_p^\lambda[S] - \mu_S \right).$$

Example - cont

The diversification effect due to the GS r.m. is stronger than the one due to the Expected Shortfall r.m.

		Lines of business								
		1	2	3	$\cdots$	8	9	10	Total	DIV
$q = 1.5$	SDS_0^λ	NaN	NaN	NaN	$\cdots$	NaN	NaN	NaN	NaN	NaN
	ES_p	30.35	45.62	1.21	$\cdots$	6.15	6.23	14.05	145.78	0.15
	SDS_p^λ	NaN	NaN	NaN	$\cdots$	NaN	NaN	NaN	NaN	NaN
	GS_p^λ	30.63	46.08	1.24	$\cdots$	6.25	6.34	14.31	146.48	0.16
$q = 2$	SDS_0^λ	25.88	38.16	0.87	$\cdots$	4.56	4.47	9.75	134.61	0.01
	ES_p	28.56	42.62	1.07	$\cdots$	5.51	5.52	12.32	141.30	0.10
	SDS_p^λ	28.73	42.91	1.09	$\cdots$	5.57	5.59	12.49	141.73	0.10
	GS_p^λ	30.40	45.70	1.22	$\cdots$	6.17	6.25	14.10	145.91	0.15
$q = \infty$	SDS_0^λ	25.88	38.16	0.87	$\cdots$	4.56	4.47	9.75	134.61	0.01
	ES_p	29.11	43.54	1.12	$\cdots$	5.71	5.74	12.85	142.68	0.11
	SDS_p^λ	29.20	43.70	1.12	$\cdots$	5.74	5.77	12.94	142.91	0.12
	GS_p^λ	29.21	43.71	1.12	$\cdots$	5.75	5.77	12.95	142.91	0.12

Table 1: Risk measures for the Student-t risks with varying degrees of freedom q, the parameter choices $p = 0.75$ and $\lambda_p = 0.0714$, and the diversification per unit of risk (DIV).

Example - cont

Again, the diversification affect is stronger due to the GS r.m.

k	1	2	3	$\cdots$	8	9	10
$w_{k,S}$	0.10	0.46	0.01	$\cdots$	-0.03	-0.01	0.07
$\mathrm{ES}_p[X_k,\ S]$	26.85	43.21	0.88	$\cdots$	4.19	4.26	10.37
$\mathrm{ES}_p[X_k]$	30.45	45.62	1.21	$\cdots$	6.15	6.23	14.05
$\mathrm{GS}_p^{\lambda}[X_k,\ S]$	26.92	43.53	0.89	$\cdots$	4.17	4.25	10.42
$\mathrm{GS}_p^{\lambda}[X_k]$	30.63	46.08	1.24	$\cdots$	6.25	6.34	14.31

Table 2: Economic versus actuarial pricing using the ES and GS r.m.'s when $p = 0.75$, $q = 1.5$ and $\lambda_p = 0.0714$.

Conclusions

- CAPM-like pricing is feasible to achieve for non-elliptical loss r.v.'s.

Simple prices via the WIPM can be derived for loss r.v.'s that are:

- dependent or independent;
- symmetric or positively skewed;
- heavy- or light-tailed;
- distributed according to a variety of probability laws of interest in insurance, e.g., gamma, Pareto, inverse Gaussian, compound Poisson with gamma secondary distribution, etc.

Pricing via the WIPM is simple based on such well-known r.m.'s as:

- Expected Shortfall; and more generally
- the class of distorted r.n.'s,

as well as based on the new Gini Shortfall.