Surplus stability analysis and control strategy in general insurance

Rong Li [1], Athanasios A.Pantelous[2] and **Lin Yang**[1]

[1]Deparmtent of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou, China
[2]Department of Econometrics and Business Statistics, Monash University, Wellington Road, Clayton VIC 3800, Australia

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Introduction

- Application of robust control theory in insurance.
- Actuaries should delineate in the modelling process the stochastic and uncertain nature of different parameters involved in the process.
- The stability and stabilization processes under risk and uncertainty is a challenge issue that insurance companies are particularly interested with under Solvency II/ C-ROSS. The concept of stability was initiated in engineering.
- In actuarial practice, the actuary wants to calculate a minimum premium, sufficient enough to cover the claims, and moreover, increases the expected surplus sufficiently to cover the unexpected losses.

Introduction

- The Solvency II/ C-ROSS framework and other national regulations emphasize on the capital requirement and risk management issues.
- Previous research in this area assumes that the Premium-Reserve process is described as only one standard regime system.
- Regime switching and time delay are always exist the real world situation.

Motivation

- Regime switching models attempt to capture the long-term structural changes in the various variables involved in the models, particularly in finance and economics, their importance is indisputable
- An arbitrary regime switching system environment employed in this research that further increases the flexibility of the parameters involved, and hopefully allows us to model more representatively the real market dynamics.

Motivation

- The purpose of this study is to develop a sound regime-switching approach for the Premium-Reserve system in insurance, which form the basis of good surplus management and premium rating policy.
- It can provide a new approach for financial strength analysis, solvency margin supervision, and management of premium rating policy.
- It is too challenging to model all the real world characteristics, but it is desirable to capture the essential influential factors in an appropriate way.

Previous Relevant Research - feedback control in insurance model

Probably the first actuarial publication where the control theory has been involved was the famous paper by De Finetti (1957). He proposed for the classical risk theory problem a control action based on a pre-defined level of the surplus (accumulated) reserve. They suggest a premium refund whenever the surplus exceeds a certain limiting level.

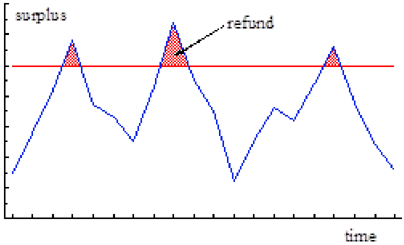

Figure: De Finetti's (1957) approach to control of surplus.

Previous Relevant Research - feedback control in insurance model

- Balzer and Benjamin (1980), Balzer (1982), Matrin-Löf (1983, 1994), and Rantala (1986, 1988) presented some new concepts of the linear deterministic control theory into actuarial problems.
- Zimbidis and Haberman (2001, IME) considered a discrete-time equation to describe the development of the reserve process for an insurance system having time-invariant delays.
- Pantelous and Papageorgiou (2013, EAJ) proposed a new approach, which uses the recent claim experience and a negative feedback mechanism of the known surplus value
- Pantelous and Yang (2014, IME) showed the robust H_∞ stabilization of the discrete-time premium-reserve system with norm-bound parameter uncertainties and time-varying delay in a stochastic framework for a first time.

Previous Relevant Research - feedback control in insurance model

- Pantelous and Yang (2015, RISK) used the robust H_∞ stabilization of the discrete-time reserve-premium system with predefined risky investment strategy in a stochastic framework.
- Yang et al. (2016, ASTIN) proposed the robust H_∞ stabilization and control for the Markovian regime switching Premium-Reserve system in discrete-time framework.
- In this research, some elements from the robust control theory are used in order to investigate the robust stability, stabilization and H_∞ control for the arbitrary switching accumulated reserve process in a stochastic discrete-time framework. This work extends the recent results proposed by Pantelous and Yang (2014, 2015) and Yang et al.(2016).

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Assumptions

- **Assumption 1**: We assume that there is a binding agreement between the insurer and the insured indicating that all contracts will remain long term. This assumption is strong but necessary in our model.
- **Assumption 2**: The relationship among the relative operation costs, the desired profit margin and corresponding premium can be expressed by the equation

$$\textbf{Operation Costs} + \textbf{Profit Margin} = (1 - e)P_t.$$

We assume the insurer calculates a fair (and as much as possible a competitive premium s/he can) gross premium covering the expected claims, the respective administration expenses and the rational profit margin.

Assumptions

- **Assumption 3**: Let $\{\sigma_t; t \geq 0\}$ be an arbitrary switching signal with state space $\mathcal{S} = \{1, 2 \ldots, N\}$, i.e., σ_t is a piecewise constant function of time and the transition probability is unknown (or even not existed). Although, σ_t is unknown a prior, we assume its instantaneous value is available in real time. This assumption guarantees that the switching sequence is not known a priori.

Assumptions

- **Assumption 4**: Positive integer τ_i represents the time delay when the system operates in the regime i. Then we denote

$$\tau_{\max} = \max\{\tau_i, i \in \mathcal{S}\},$$

$$\tau_{\min} = \min\{\tau_i, i \in \mathcal{S}\}.$$

We consider a mode-dependent delay, τ_{σ_t}, which is upper and lower bounded, i.e. $\tau_{\min} \leq \tau_{\sigma_t} \leq \tau_{\max}$ with $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. So, considering a specific time-delay interval, at the end of each year $[t, t+1)$, we have the exact information up to the end of the year $t - \tau_{\sigma_t}$. The value for τ_i can be estimated using past experience and statistical data. Moreover, the national and international regulatory policy might be also applied for defining the upper bound of this interval.

Assumptions

- **Assumption 5**: The portfolio of m individual insurance lines (or products or policies) can be either *independent* or *dependent*. The different lines are dependent when there is interaction among the different accumulated reserve accounts.
- **Assumption 6**: The state of the insurer is described by one variable only, namely accumulated reserve or risk capital. Similarly controller in premiums is the only control variable.

Parameter Uncertainties

- Time-varying parameter uncertainties $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are admissible if both 1 and 2 hold.

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}], \tag{1}$$

$M_i, N_{1i}, N_{2i}, N_{3i}$ are known real constant matrices and
$F_t : \mathbb{N} \to \mathbb{R}^{s\times j}$ is an unknown time-varying matrix function satisfying

$$F_t^T F_t \leq I, \quad \forall t \in \mathbb{N}, \tag{2}$$

Formula of premium

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t, \qquad (3)$$

where $\underline{U}_t = K_{1i}\underline{R}_t$, $\underline{U}_t \in \mathbb{R}^m$ is the control input.
The equation above means that the premium $\underline{P}_t$ at time $t+1$ is $\underline{\hat{C}}_{t+1}$ plus a correction which depends linearly on the past accumulated reserve $\underline{R}_{t-\tau_t}$ and the current accumulated reserve $\underline{R}_t$ values through $\underline{U}_t$. $\underline{U}_t \in \mathbb{R}^m$ is the control input that has been added in the original system. However, for simplicity, the state feedback controller is considered to depend on the latest value of $\underline{R}_t$: $\underline{U}_t = K_{1i}\underline{R}_t$, where the matrix K_{1i} should be determined by solving an appropriate LMI (convex optimization) problem.

Arbitrary regime switching reserve-premium system

Let $\underline{R}_t = (R_{1,t} R_{2,t} \cdots R_{m,t})^T$ be the vector expression of the accumulated reserves, where $R_{i,t}$ is the accumulated reserve of i^{th} insurance line at time t. The **accumulated reserve**, $\underline{R}_t$, evolves according to :

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\underline{P}_{t+1} - \underline{C}_{t+1}. \quad (4)$$

The system Θ has N system regimes. J_{σ_t} is the investment return matrices in time t for the risk-free asset. Switching signal σ_t is a piecewise constant function of time which takes value i in the finite set $\mathcal{S} = [1 \quad 2 \cdots N]$. The states represent different system regimes. We assume that the switching signal σ_t is governed by arbitrary jump process.

Arbitrary regime switching reserve-premium system

Then, we consider the following discrete time arbitrary regime switching linear Premium-Reserve system with a state feedback controller:

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \qquad (\Theta)$$

where $\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1}$.
$\underline{w}_{t+1}$ is one of the disturbance to system which is caused by the error between estimated claim value and actual incurred value.
The accumulated reserve at time $t+1$ depends linearly on the previous state, on the previous control action and on the present process disturbance $\underline{w}_{t+1}$.

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Robust stability

In this subsection, we consider the uncertain discrete time system Θ with state feedback controller $\underline{U}_t = 0$ and disturbance $\underline{w}_{t+1} = 0$. It means that the actual incurred claims is exactly the same with the estimation.

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \qquad (\Theta_1)$$

Robust stability

Definition

The uncertain discrete time-delay system Θ_1 is said to be robustly stochastically stable if there exists a scalar $c > 0$ such that for all admissible uncertainties

$$\mathbb{E}[\sum_{t=0}^{\infty} |\underline{R}_t|^2 | \underline{R}_0, \sigma_0] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2, \tag{5}$$

when $\underline{w}_{t+1} = 0$, where $\underline{R}_t$ denotes the reserve at time t under initial condition.

Remark This definition means that the total value of the accumulated reserve process in the system is bounded by a finite number, i.e. for any "admissible" input the reaction of $\underline{R}_t$ is also bounded in the expected value sense.

Robust stability

Theorem

The system Θ_1 is robust stochastically stable for any time-varying delay τ_{σ_t} satisfying $\tau_{\max} > \tau_{\min} \geq 0$, if there exist matrices P_i, $Q > 0$, L_i, S_i, $\epsilon_i > 0$, such that the following conditions hold for $\forall (i,j) \in \mathcal{S} \times \mathcal{S}$:

$$\begin{bmatrix} \Lambda_1 & \Lambda_2 & eL_i^T E_i & L_i^T M_i & (N_{1i}+N_{2i})^T \\ \Lambda_2^T & \Lambda_3 & eS_i^T E_i & S_i^T M_i & 0 \\ eE_i^T L_i & eE_i^T S_i & -Q & 0 & -N_{2i}^T \\ M_i^T L_i & M_i^T S_i & 0 & -\epsilon_i I & 0 \\ N_{1i}+N_{2i} & 0 & -N_{2i} & 0 & -\epsilon_i^{-1} I \end{bmatrix} < 0, \tag{6}$$

where

$$\Lambda_1 = L_i^T[J_i - eE_i - I] + [J_i - eE_i - I]^T L_i + P_j - P_i, \tag{7}$$

$$\Lambda_2 = P_j - L_i^T + [J_i - eE_i - I]^T S_i, \tag{8}$$

$$\Lambda_3 = P_j + \tau_{\max}^2 Q - S_i - S_i^T. \tag{9}$$

Robust H_∞ stabilization

Here, we consider the uncertain discrete time system Θ with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} \neq 0$. It means that the actual incurred claims is not the same as the estimator. We use the following LMI condition (9) to find a feasible state H_∞ controller to control this process.

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \qquad (\Theta)$$

H_∞ stability definition

Definition

The uncertain stochastic discrete time-delay system Θ is said to be robustly stable with disturbance attenuation level γ if it is robustly stable and the (10) is satisfied,

$$||\underline{z}_t|\underline{R}_0, \sigma_0||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}, \tag{10}$$

for all nonzero $\underline{w}_t \in l_{e_2}([0, \infty); \mathbb{R}^m)$, and is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$, where $\gamma > 0$ is a given scalar and $\underline{z}_t = C\underline{R}_t$ is the control output observation.

Remark This definition means that the magnitude of movement in output observer due to system disturbance is bounded by γ. In practice, it means the worst impact of disturbance in claim process on the accumulated reserve level is bounded when the system is robustly stable.

LMI stabilization criterion

Theorem

The system Θ is robustly stabilizable with noise attenuation level γ, if there exist matrices X_i, Y_i, $\tilde{Q} > 0$, B_i, and D_i, such that the following conditions hold $\forall (i,j) \in \mathcal{S} \times \mathcal{S}$:

$$\begin{bmatrix} -X_i & \Pi_1 & 0 & 0 & X_i C^T & 0 & \Pi_2 & X_i + D_i^T & \tau_{\max} D_i^T \\ \Pi_1^T & \Pi_3 & e\tilde{Q}E_i & I & 0 & M_i & 0 & B_i^T & \tau_{\max} B_i^T \\ 0 & eE_i^T\tilde{Q} & -\tilde{Q} & 0 & 0 & 0 & -\tilde{Q}N_{2i}^T & 0 & 0 \\ 0 & I & 0 & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 \\ CX_i^T & 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 \\ 0 & M_i^T & 0 & 0 & 0 & -\epsilon_i I & 0 & 0 & 0 \\ \Pi_2^T & 0 & -\tilde{Q}^T N_{2i} & 0 & 0 & 0 & -\epsilon_i^{-1} I & 0 & 0 \\ X_i^T + D_i & B_i & 0 & 0 & 0 & 0 & 0 & -X_j & 0 \\ \tau_{\max} D_i & \tau_{\max} B_i & 0 & 0 & 0 & 0 & 0 & 0 & -\tilde{Q} \end{bmatrix} < 0 \tag{11}$$

In this case, an appropriate robust stabilizing state feedback controller can be chosen as $\underline{U}_t = Y_i X_i^{-1} \underline{R}_t$.

What does this result provide?

- It provide a sufficient condition for the stabilization of the reserve-premium system Θ.
- LMI criterion can be solved conveniently by some toolbox in Matlab.
- If LMI criterion is solved, then we can construct appropriate feedback controllers for each regime.
- Most important thing is that it provide a solution to such a stabilization problem while considering all the parameter uncertainties, regime switching impact and random disturbance in the premium-reserve system.

Numerical Example

- We simulate a business insurance company which runs 3 different insurance lines.
- We assume there are two different system regimes for the system Θ and arbitrary regime switching state space $\mathcal{S} = [1, 2]$.
- For the time delay, we assume that the mode-dependent delay are $\tau(1) = 3$ for Regime 1 and $\tau(2) = 1$ for Regime 2. Therefore, $\tau_{\text{min}} = 1$ and $\tau_{\text{max}} = 3$.
- We assume that the accumulated reserve account for each product lines is \$0 at time $t = 0$ respectively. The value of the accumulated reserve accounts at $t = 0$ is given by the following matrix,

$$\underline{R}_0 = \begin{bmatrix} R_{0,1} \\ R_{0,2} \\ R_{0,3} \end{bmatrix} = \begin{bmatrix} \$0 \\ \$0 \\ \$0 \end{bmatrix},$$

Numerical Example

- We assume that the corresponding rate of income is given from the following matrix:

$$J_1 = \begin{bmatrix} 1.021 * w_{11} & 1.021 * w_{12} & 1.021 * w_{13} \\ 1.021 * w_{21} & 1.021 * w_{22} & 1.021 * w_{23} \\ 1.021 * w_{31} & 1.021 * w_{32} & 1.021 * w_{33} \end{bmatrix}.$$

$$J_2 = \begin{bmatrix} 1.039 * w_{11} & 1.039 * w_{12} & 1.039 * w_{13} \\ 1.039 * w_{21} & 1.039 * w_{22} & 1.039 * w_{23} \\ 1.039 * w_{31} & 1.039 * w_{32} & 1.039 * w_{33} \end{bmatrix}.$$

- The weight ratios $w_{n,m}$ which demonstrates the solvency relation between each product have the following values:
  $w_{11} = 0.86$, $w_{12} = 0.07$ and $w_{13} = 0.07$,
  $w_{21} = 0.10$, $w_{22} = 0.87$ and $w_{23} = 0.03$,
  $w_{31} = 0.08$, $w_{32} = 0.09$ and $w_{33} = 0.83$.

Numerical Example

- We assume the value in the parameter matrix E:
  For Regime 1

$$E_1 = \begin{bmatrix} 0.13 * w_{11} & 0.13 * w_{12} & 0.13 * w_{13} \\ 0.13 * w_{21} & 0.13 * w_{22} & 0.13 * w_{23} \\ 0.13 * w_{31} & 0.13 * w_{32} & 0.13 * w_{33} \end{bmatrix},$$

  For Regime 2

$$E_2 = \begin{bmatrix} 0.18 * w_{11} & 0.18 * w_{12} & 0.18 * w_{13} \\ 0.18 * w_{21} & 0.18 * w_{22} & 0.18 * w_{23} \\ 0.18 * w_{31} & 0.18 * w_{32} & 0.18 * w_{33} \end{bmatrix},$$

Numerical Example

- For parameter e, we let $e = 0.8$, which means $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the operation cost and give company a reasonable profit margin.
- $\gamma = 21.8$. This is the value represent the the maximum impact level of the disturbance to the accumulated reserves.
- We assume the insurer will change their operating regime since some relative key economic and market factors are not constant. In this simulation, it is assumed the insurer company can switch between 2 regimes.

Numerical Example

We can calculate the system feedback controller for each regime by applying previous theorem. After LMI condition is solved, the result are shown below

If system is in Regime 1:

$$K11 = Y_1 X_1^{-1} = \begin{bmatrix} 0.6783 & -0.9012 & -0.8369 \\ -0.7034 & 0.8533 & -0.6048 \\ 1.1697 & 1.2032 & 2.6863 \end{bmatrix}$$

If system is in Regime 2:

$$K12 = Y_2 X_2^{-1} = \begin{bmatrix} 1.0600 & -0.5498 & -0.4322 \\ -0.8053 & 0.7840 & -0.6738 \\ 0.8661 & 0.8753 & 2.2867 \end{bmatrix}$$

$$\underline{U_t} = K_{1i} \underline{R_t},$$

Numerical Example

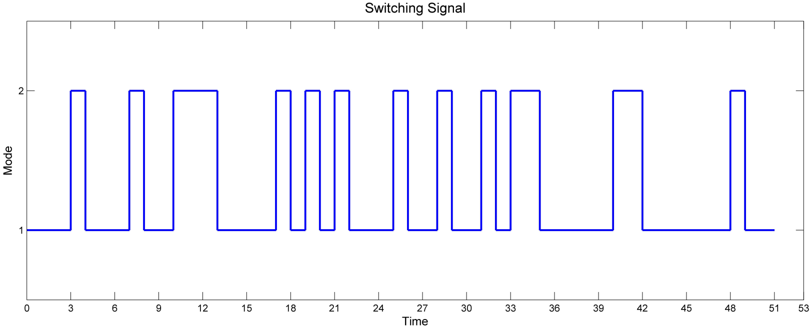

Figure: Switching signal

Numerical Example

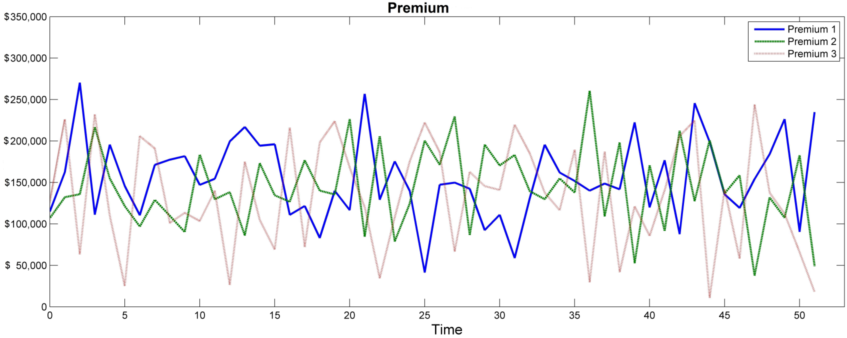

Figure: Process of Accumulated Reserve for individual product line

Numerical Example

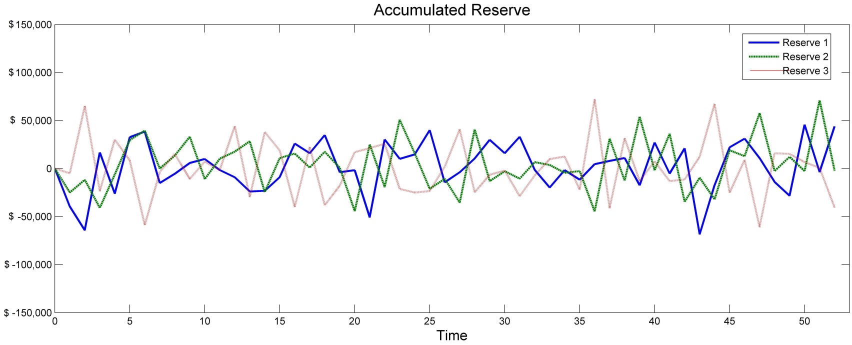

Figure: Sum of 3 Accumulated Reserve account

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Conclusion

- The premium-reserve model in this project consider different types of uncertainties as well as to face the impact of the external disturbances. It enable us to consider all the uncertainties and disturbance within the model.
- It generate a state feedback controller to manipulate the system stability when facing different uncertainties and risks.
- Abrupt changes in structures is considered by arbitrary regime regime switching P-R model. This model extends the models proposed by Zimbidis and Haberman (2001), Pantelous and Papageorgiou (2013) and Pantelous and Yang (2014, 2015) and Yang et al.(2016).

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Thank you!