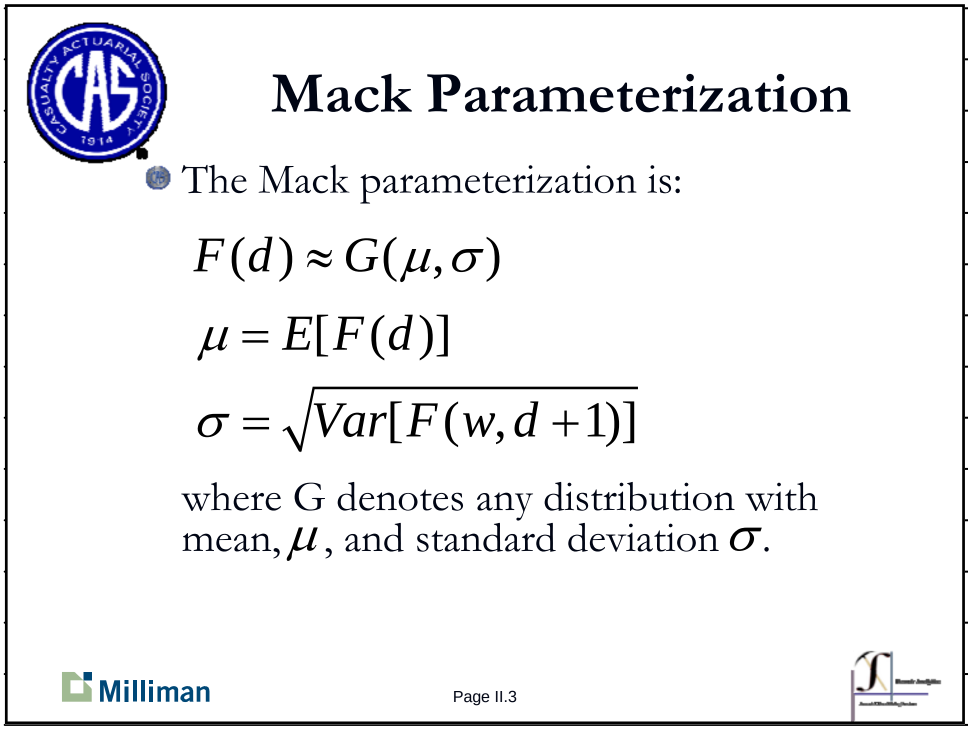
Mack Model

Mack Method

- Mack developed a distribution-free method for computing the Variances of chain ladder age-to-age and age-to-ultimate factors
- The method is applied to weighted average development factors
- While no distribution assumptions are required to estimate factor Variances, they are needed to compute percentiles of ultimate loss and unpaid distributions



Mack Parameterization

- The Mack parameterization is:

$$F(d) \approx G(\mu, \sigma)$$

$$\mu = E[F(d)]$$

$$\sigma = \sqrt{Var[F(w, d+1)]}$$

where G denotes any distribution with mean, μ, and standard deviation σ.

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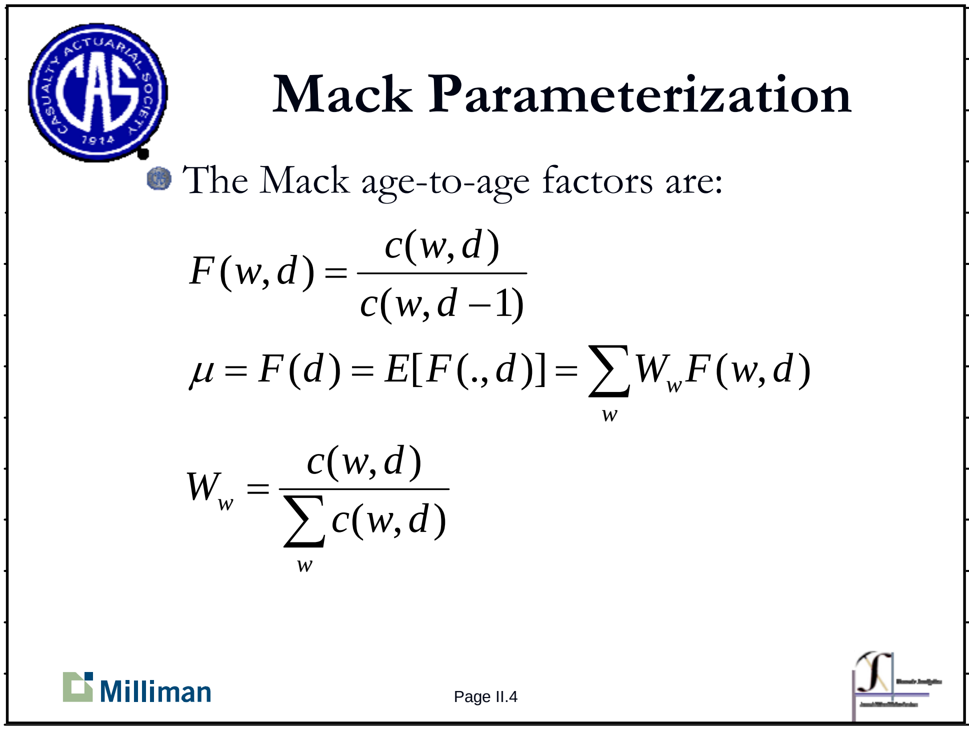

Mack Parameterization

- The Mack age-to-age factors are:

$$F(w,d) = \frac{c(w,d)}{c(w,d-1)}$$

$$\mu = F(d) = E[F(.,d)] = \sum_{w} W_w F(w,d)$$

$$W_w = \frac{c(w,d)}{\sum_{w} c(w,d)}$$

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Mack Means

- Under these assumptions, the best estimate of the age-to-age factor is a weighted average

$$E[F(d)] = \sum_w \frac{c(w,d)}{\sum_w c(w,d)} \times \frac{c(w,d+1)}{c(w,d)} = \frac{\sum_w c(w,d+1)}{\sum_w c(w,d)}$$

- The Ultimate estimate is:

$$E[c(w,n)|D] = c(w,d) \times F(d) \times F(d+1) \times \ldots \times F(n-1)$$

where D is known data

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Un-weighted Variance

- Without weighting, the age-to-age differences become:

$$\text{Squared Deviation} = [F(w,d) - \mathrm{F'(d)}]^2 = \left(\frac{c(w,d)}{c(w,d\text{ -}1)} - \mathrm{F'(d)} \right)^2$$

- The un-weighted variance is thus:

$$\text{Unweighted Variance} = \sum \frac{[F(w,d) - \mathrm{F'(d)}]^2}{N - d - 1}$$

$$\mathrm{F'(d)} = \text{Unweighted Mean}$$

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Weighted Variance

- Since the mean is weighted, the variance is also weighted:

$$\text{Weighted Variance} = \sum W_w [F(w,d) - F(d)]^2$$

Where:

$$W_w = \frac{c(w,d)}{\sum_w c(w,d)}$$

$$F(d) = \text{weighted mean}$$

Weighted Variance

- A factor with a smaller cumulative prior loss contributes more noise to the estimate and is given lower weight, thus:
  - Variance of factor *F(d)* is inversely proportional to *c(w,d)*:

$$Var(F(d)) \propto {}^{1}\!/_{c(w,d)}$$

Exercises using Mack Data

- Compute Age-to-Age factor triangle for the Mack data
- Compute the average age-to-age factors for each column
- Compute the weighted average age-to-average factors for each column
- Compute the unweighted variances up to age 7 for the factors in the Mack triangle
- Compute the weighted variances up to age 7 for the factors in the Mack triangle
- How might you use the means and variances you computed to model unpaid claim variability?

Mack Variance

- Mack uses the following relationship to calculate the variance of losses:

$$Var[c(w,d+1) \mid c(w,d)] = c(w,d)\hat{\sigma}_d^2$$

Mack Variance

- Variance associated with one age-to-age factor or column of losses, $Sigma_d^2$:

$$\sigma_d^2 = \frac{1}{N-d-1}\sum_{j=1}^{N-d} c(j,d)\left(\frac{c(j,d+1)}{c(j,d)} - F(d)\right)^2$$

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Example

Computation of $Sigma_1^{\,2}$ for Development Age 1

Accident Year	$c(w,1)$	$F(w,1)$	$[F(w,1) - F(1)]^2$	Weighted Deviation
(1)	(2)	(3)	(4)	(5)
			$[(3) - 2.999]^2$	(2) x (4)
1981	5,012	1.650	1.82	9,127.9
1982	106	40.425	1,400.64	148,468.2
1983	3,410	2.637	0.13	447.9
1984	5,655	2.043	0.91	5,168.7
1985	1,092	8.759	33.18	36,227.4
1986	1,513	4.260	1.59	2,403.5
1987	557	7.217	17.79	9,909.3
1988	1,351	5.142	4.59	6,203.0
1989	3,133	1.722	1.63	5,112.0
Average [$F(1)$]		2.999	Sum =	223,067.8
			$Sigma_1^{\,2}$ = Sum / (N-d-1) =	27,883.5

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Variance of Ultimates

- We want variance of future payments or future incurred loss changes
- $MSE[c(w,n)] = E[\{c(w,n) – E[c(w,n)]\}^2 \mid D]$ where D is data
- Iterative rule of expectations
- $MSE[c(w,n)] = Var[c(w,n) \mid D] + \{E[c(w,n) \mid D] – E[c(w,n)]\}^2$
- Mean squared error = process variance of Ultimate + Parameter variance of estimate of ultimate
- Does not take into account changes in underlying model in the future.

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Variance of Ultimate

- Iterative computation to get variance of ultimate

$$Var[c(w,n)] = E[c(w,n-1)]\sigma_{n-1}{}^2 + E[c(w,n-1)]^2 F(n-1)^2 =$$
$$c(w,n-k+1)F(n-k+1)...F(n-2)\sigma_{n-1}{}^2 +$$
$$[E[c(w,n-2)]^2 F(n-2)^2 F(n-1)^2 + E[c(w,n-2)]F(n-1)^2\sigma_{n-2}{}^2]$$

- Variance of unpaid = variance of ultimate

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Mack Variance

- The Mack formula for the Variance of the unpaid estimate for accident year *w* is:

$$Var[R(w,d)] = U(w)^2 \sum_{d=n+1-w}^{n-1} \frac{\sigma_d^2}{F(d)^2} \left(\underbrace{\frac{1}{E[c(w,d)]}}_{\text{Process Variance}} + \underbrace{\frac{1}{\sum_{j=1}^{N-d} c(j,d)}}_{\text{Parameter Variance}} \right)$$

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Mack Variances

- There is a $Sigma_d^2$ for each development age
  - $Sigma_d^2$ is a weighted Variance
- There is a Standard Error ($Sigma_w^2$) for each Accident (or Policy) year

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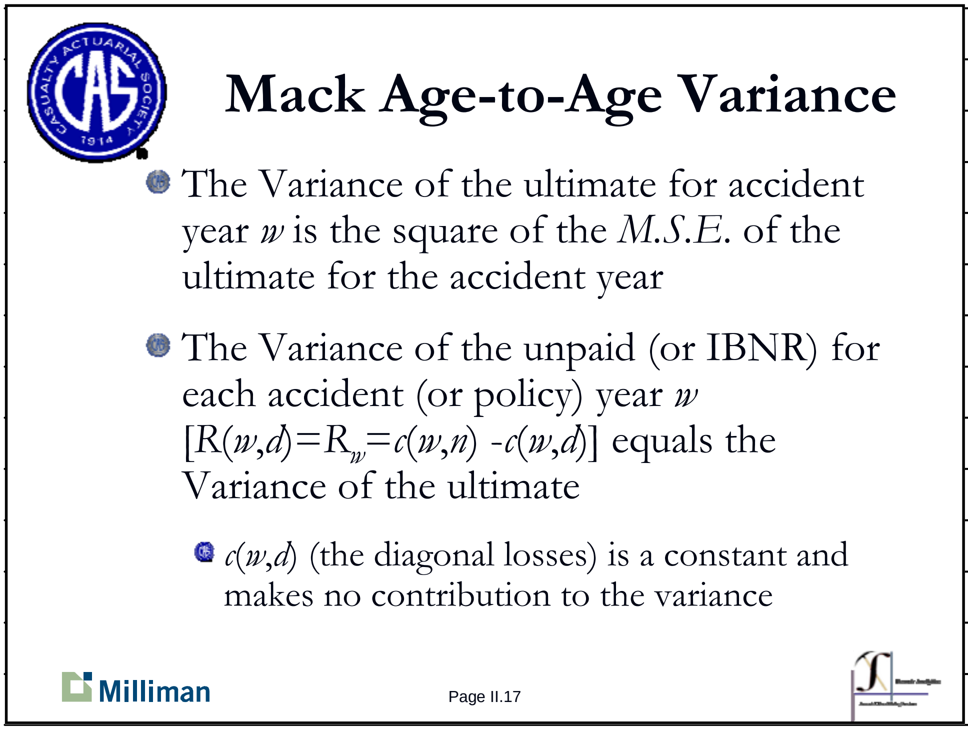

Mack Age-to-Age Variance

- The Variance of the ultimate for accident year *w* is the square of the *M.S.E.* of the ultimate for the accident year
- The Variance of the unpaid (or IBNR) for each accident (or policy) year *w* $[R(w,d)=R_w=c(w,n)-c(w,d)]$ equals the Variance of the ultimate
  - $c(w,d)$ (the diagonal losses) is a constant and makes no contribution to the variance

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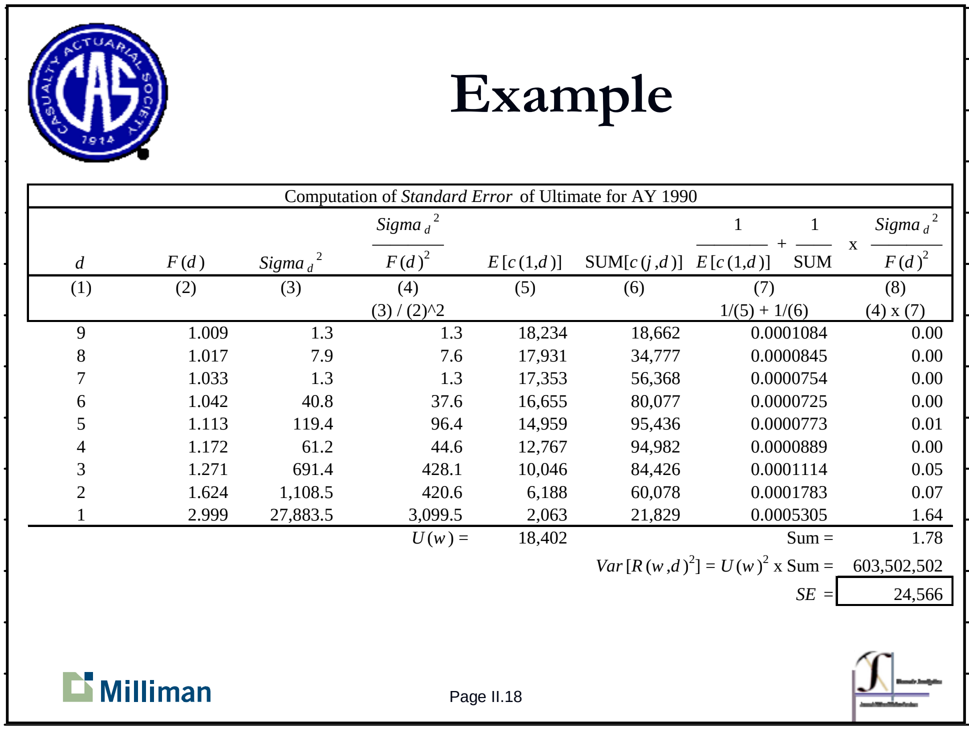

Example

Computation of *Standard Error* of Ultimate for AY 1990

d	$F(d)$	$Sigma_d^2$	$\frac{Sigma_d^2}{F(d)^2}$	$E[c(1,d)]$	$SUM[c(j,d)]$	$\frac{1}{E[c(1,d)]}+\frac{1}{SUM}$	$\times \frac{Sigma_d^2}{F(d)^2}$
(1)	(2)	(3)	(4) (3) / (2)^2	(5)	(6)	(7) 1/(5) + 1/(6)	(8) (4) x (7)
9	1.009	1.3	1.3	18,234	18,662	0.0001084	0.00
8	1.017	7.9	7.6	17,931	34,777	0.0000845	0.00
7	1.033	1.3	1.3	17,353	56,368	0.0000754	0.00
6	1.042	40.8	37.6	16,655	80,077	0.0000725	0.00
5	1.113	119.4	96.4	14,959	95,436	0.0000773	0.01
4	1.172	61.2	44.6	12,767	94,982	0.0000889	0.00
3	1.271	691.4	428.1	10,046	84,426	0.0001114	0.05
2	1.624	1,108.5	420.6	6,188	60,078	0.0001783	0.07
1	2.999	27,883.5	3,099.5	2,063	21,829	0.0005305	1.64
			$U(w) =$	18,402		Sum =	1.78
						$Var[R(w,d)^2] = U(w)^2$ x Sum =	603,502,502
						$SE =$	24,566

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Exercise

- Assume age 10 is the ultimate valuation.
- Compute $Sigma_d^2$ for ages 2 through 9 for the Mack data (use minimum of 7 & 8 for 9)
- Compute the unpaid (IBNR) for 1989
- Compute the Standard Error for 1989

Mack Total Variance

- The total unpaid is the sum of the unpaid random variable for each accident year
- $R_{tot} = R_1 + R_2 + \ldots + R_N$
- The Variance of a sum equals the sum of the Variances plus twice the Covariances

Mack Total Variance

- The Mack formula for the Variance of the total unpaid estimate is:

$$SE(R_{tot})^2 = \sum_{w=2}^{N} \left\{ SE(R_w)^2 + U(w) \left(\sum_{i=w+1}^{N} c(i,n) \right) \sum_{d=n+1-w}^{n-1} \left(\frac{2\sigma_d^2 / F(d)^2}{\sum_{j=1}^{N-d} c(j,d)} \right) \right\}$$

Calculation Pointers

- It is easiest to set up a set of triangles to perform the calculations
  - First create a row of column sums of cumulative losses x the last observation
  - Create a triangle of weighted square deviations of development factors from their mean
  - Create a projected runoff triangle that computes each estimate of cumulative losses, $c(w,d)$, for all future periods
  - Create a triangle of inverses of projected runoff plus inverse of sum of cumulative losses
  - A spreadsheet showing the calculation for the Mack data is provided

Covariance Earliest Year

- Multiply ultimate for year by
  1. Sum of ultimates for all subsequent years
  2. Times the factor Variance (Sigma_d^2) for last age-to-age factor
  3. Divide by square of last age-to-age factor
- For Other years
  - Need a sum of the ratio computed in 2. and 3

Using Mack Parameters

- We have a mean and a variance for unpaid (or IBNR) amounts. Now what?
- To get confidence intervals or probability distribution, assumptions must be made
- Assume unpaid (or IBNR) amounts follow a probability distribution, say the Gamma
- Use mean and variance of unpaid (or IBNR) amounts to derive parameters for distribution
- Use this distribution to estimate percentiles and other statistics for unpaid (or IBNR) amounts

Exercise

- Assume you are only interested in the unpaid amounts (*i.e.*, IBNR) for years 1981 through 1983
- Compute the Variance of the total unpaid amounts for the three years
- Assume total unpaid amount for the three years follows a Gamma distribution. Using the Mean and Variance of the unpaid (IBNR) amount compute the Gamma parameters. Then compute the 95th percentile of the unpaid (IBNR) amount.

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Questions on Mack Model?

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