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The Prediction Error of Bornhuetter/Ferguson

by

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Notation

v_i = premium volume of accident year i, $1 \leq i \leq n$,

C_{ik} = cumulative loss amount of acc. year i after dev. year k, $1 \leq k \leq n$,

$C_{i,n+1-i}$ = currently known loss amount (paid or incurred),

$S_{ik} = C_{ik} - C_{i,k-1}$ = incremental loss amount, $C_{i0} := 0$,

$S_{i,n+1}$ = incremental loss amount after development year n (tail),

$U_i = S_{i1} + \ldots + S_{in} + S_{i,n+1}$ = ultimate loss amount of acc. year i,

$R_i = S_{i,n+2-i} + \ldots + S_{i,n+1} = U_i - C_{i,n+1-i}$ outstanding losses of acc. year i.

The B/F method

$$\hat{R}_i^{BF} = \hat{U}_i \left(1 - \hat{z}_{n+1-i}\right)$$

where $\hat{U}_i = v_i \hat{q}_i$, $\hat{q}_i$ = 'initial' estimate of the ultimate loss ratio,

$\hat{z}_k \in [0; 1]$ estimated expected loss part known after DY k,

i.e. $(z_1, z_2, \ldots, z_n, 1)$ = cumulative development pattern.

$\hat{q}_i$ is usually taken from pricing and later on adjusted somehow,

$\hat{z}_k$ is usually obtained from the chain ladder factors $\hat{f}_k$ via

$$\hat{z}_n = \hat{f}_\infty^{-1}, \ \hat{z}_{n-1} = \left(\hat{f}_n \hat{f}_\infty\right)^{-1}, \ldots, \ \hat{z}_1 = \left(\hat{f}_2 \cdot \ldots \cdot \hat{f}_\infty\right)^{-1}.$$

What does ‘initial’ mean?

$\hat{q}_i$ is called ‘initial’ or ‘prior’ estimate of the ultimate loss ratio

as opposed to the ‘posterior’ estimate $\left(C_{i,n+1-i} + \hat{R}_i^{BF}\right)/v_i$.

$$\frac{\hat{U}_i}{v_i} = \hat{q}_i \neq \frac{C_{i,n+1-i} + \hat{R}_i^{BF}}{v_i} = \frac{C_{i,n+1-i} + \hat{U}_i(1-\hat{z}_{n+1-i})}{v_i} \iff C_{i,n+1-i} \neq \hat{U}_i \hat{z}_{n+1-i} ,$$

i.e. prior $\neq$ posterior <==> $C_{i,n+1-i} \neq$ its expectation

Comparison to related methods

$\hat{R}_i^{CL} = C_{i,n+1-i}\left(\hat{f}_{n+2-i} \cdot \ldots \cdot \hat{f}_\infty - 1\right)$ Chain Ladder

depends strongly on $C_{i,n+1-i}$, e.g. $C_{n1} = 0$ ==> $\hat{R}_n^{CL} = 0$,

is the higher, the higher $C_{i,n+1-i}$ is.

$\hat{R}_i^{EL} = \hat{U}_i - C_{i,n+1-i}$ Expected Loss

depends strongly on $C_{i,n+1-i}$. Is the lower, the higher $C_{i,n+1-i}$ is.

$\hat{R}_i^{BF}$ does not at all depend on $C_{i,n+1-i}$,

treats the deviation $C_{i,n+1-i} - \hat{U}_i\hat{z}_{n+1-i}$ as purely random.

Why to not use the CL pattern?

The use of the CL factors implies

that the outstanding losses are a multiple of the already known loss.

This contradicts to the fundamental B/F idea

of independence between $C_{i,n+1-i}$ and $\hat{R}_i^{BF}$.

Moreover, with the use of the CL pattern,

B/F cannot really claim to be a standalone reserving method.

How to correctly estimate the B/F pattern?

$$\hat{R}_i^{BF} = \hat{U}_i\left(1-\hat{z}_{n+1-i}\right) \implies \hat{z}_{n+1-i} = 1-\frac{\hat{R}_i^{BF}}{\hat{U}_i} = \frac{\hat{U}_i - \hat{R}_i^{BF}}{\hat{U}_i} \approx \frac{C_{i,n+1-i}}{\hat{U}_i}$$

This suggests the estimate $\hat{z}_k = \sum_{i=1}^{n+1-k} C_{ik} \Big/ \sum_{i=1}^{n+1-k} \hat{U}_i$ as weighted avg. of $\frac{C_{ik}}{\hat{U}_i}$.

This may lead to inversions $\hat{z}_k > \hat{z}_{k+1}$ (due to varying no. of summands).

Therefore, the increments $\hat{y}_k = \sum_{i=1}^{n+1-k} S_{ik} \Big/ \sum_{i=1}^{n+1-k} \hat{U}_i$ should be used

and $\hat{z}_k := \hat{y}_1 + \ldots + \hat{y}_k$.

This is the genuine B/F development pattern!

A stochastic model underlying the B/F method

(BF1) All increments S_{ik}, $1 \le i \le n$, $1 \le k \le n+1$, are independent.

(BF2) $E(S_{ik}) = x_i y_k$ with $y_1 + \ldots + y_{n+1} = 1$.

(BF3) $Var(S_{ik}) = x_i s_k^2$.

Justification of (BF2):

$E(R_i) = E(S_{i,n+2-i} + \ldots + S_{i,n+1}) = x_i(1 - z_{n+1-i})$ with $z_k := y_1 + \ldots + y_k$.

Justification of (BF3):

$E(U_i) = x_i$ shows that x_i can be seen as measure of volume of acc. year i.

The GLM-like $Var(S_{ik}) = c(x_i y_k)^\zeta$ does not work for $y_k < 0$.

Implications for $\hat{U}_i = \hat{x}_i$

$\hat{U}_i = \hat{x}_i$ has to be an estimate for $E(U_i)$ and not for $E(U_i \mid C_{i,n+1-i})$.

It is wrong to use the posterior $\hat{U}_i^{post}$ from last year as prior $\hat{U}_i$!

Even $\hat{U}_i < C_{i,n+1-i}$ is possible (accidental large claim) !!

But this does not mean that $\hat{U}_i$ cannot change over the years:

Originally, $\hat{U}_i$ is gained from pricing based on the preceding acc. years.

These years develop as well as the inflation index; new acc. years emerge.

==> $\hat{U}_i$ should be obtained by re-pricing the business

with an only marginal influence of accident year i.

How to estimate the parameters y_k and s_k^2 ?

Raw estimates $\hat{\hat{y}}_k = \sum_{i=1}^{n+1-k} S_{ik} \Big/ \sum_{i=1}^{n+1-k} \hat{U}_i, \quad k \le n,$

$$\hat{s}_k^2 = \frac{1}{n-k} \sum_{i=1}^{n+1-k} \left(S_{ik} - \hat{U}_i \hat{y}_k\right)^2 \Big/ \hat{U}_i, \quad k \le n\text{-}1.$$

Main problems: $\hat{y}_{n+1}, \ \hat{s}_n^2, \ \hat{s}_{n+1}^2$

Minimize $\sum_{i=1}^{n} \sum_{k=1}^{n+1-i} \frac{\left(S_{ik} - \hat{U}_i \hat{y}_k\right)^2}{\hat{U}_i \hat{s}_k^2}$ under $\hat{y}_1 + \ldots + \hat{y}_n = 1 - \hat{y}_{n+1}$.

See the paper for details.

The prediction error

$$\mathrm{msep}\left(\hat{R}_i^{BF}\right) = E\left(\left(\hat{R}_i^{BF} - R_i\right)^2 \middle| S_{i1},\ldots,S_{i,n+1-i}\right)$$

$$= \mathrm{Var}(R_i) + \mathrm{Var}(\hat{R}_i^{BF}) = \text{process error} + \text{parameter error}$$

to be used for RBC calc., premium loading, confidence interval for R_i .

Process error: $\mathrm{Var}(R_i) = x_i\left(s^2_{n+2-i} + \ldots + s^2_{n+1}\right)$,

$$\hat{\mathrm{V}}\mathrm{ar}(R_i) = \hat{U}_i\left(\hat{s}^2_{n+2-i} + \ldots \hat{s}^2_{n+1}\right) .$$

The parameter error

to be used to assess the significance of differences to other methods

and to construct a confidence interval for $E(R_i)$.

$$\mathrm{Var}\left(\hat{R}_i^{BF}\right) = \mathrm{Var}\left(\hat{U}_i(1-\hat{z}_{n+1-i})\right)$$

$$\approx \underset{\substack{\uparrow \\ \text{s.e.}(\hat{U}_i)}}{\mathrm{Var}\left(\hat{U}_i\right)} \cdot \left(1-\hat{z}_{n+1-i}\right)^2 + \hat{U}_i^2 \cdot \underset{\substack{\uparrow \\ \text{s.e.}(\hat{z}_{n+1-i})}}{\mathrm{Var}(\hat{z}_{n+1-i})}$$

$\left(\text{s.e.}(\hat{z}_k)\right)^2 < \left(\text{s.e.}(\hat{y}_1)\right)^2 + \ldots + \left(\text{s.e.}(\hat{y}_k)\right)^2$ as $\hat{y}_k$ negatively correlated.

$$\left(\text{s.e.}(\hat{y}_k)\right)^2 \approx \frac{\hat{s}_k^2}{\sum_{j=1}^{n+1-k} \hat{U}_j}$$

The main problem: How to assess s.e.$(\hat{U}_i)$?

* s.e.$(\hat{U}_i)$ has nothing to do with last year's parameter error s.e.$\left(\hat{R}_i^{BF}\right)$.

  s.e.$(\hat{U}_1)$ has nothing to do with s.e.$(\hat{y}_{n+1})$.

* The variability $\frac{v_i}{n-1}\sum_{j=1}^{n} v_j\left(\frac{\hat{U}_j}{v_j}-\hat{q}\right)^2$ with $\hat{q}=\frac{\sum_j \hat{U}_j}{\sum_j v_j}$

  overestimates Var$(\hat{U}_i)$ because it contains the premium cycle too.

* s.e.$(\hat{U}_i)$ is only slightly changing over the years.

  s.e.$(\hat{U}_i) > 0$ even at the end of the development!

Assessing s.e.($\hat{U}_i$) from a confidence interval

From the Normal distribution we know

$$\text{Prob}\,(\,\mu - 2\sigma \;<\; X \;<\; \mu + 2\sigma\,) \;=\; 95\% \,.$$

Similarly,

$$E(U_i) \in \left(\hat{U}_i - 2\cdot \text{s.e.}(\hat{U}_i),\quad \hat{U}_i + 2\cdot \text{s.e.}(\hat{U}_i)\right) \quad \text{with 95\% probability.}$$

Thus, from a 95% confidence interval around $\hat{U}_i$, we can deduce s.e.($\hat{U}_i$):

$$\text{e.g. } \hat{U}_i/v_i = 80\% \in (60\%,\ 100\%) \implies \text{s.e.}(\hat{U}_i/v_i) = 10\% \,.$$

N.B. We need a confidence interval for $E(U_i)$ and not for U_i !

The same approach can also be applied for s.e.($\hat{z}_k$).

Numerical Example

from Th. Mack, Parameter Estimation for BF, CAS Forum 2006

AY		BF	CL		BF	CL
2002	Reserves	139	133	Prediction	22	42
2003		149	72	Error	22	54
2004		155	353		23	265
2002	Estimation	15	17	Process	15	38
2003	Error	16	13	Error	15	52
2004		17	122		16	235

Conclusion

On basis of the above implications

– regarding the B/F development pattern and
– the assessment of the initial estimate $\hat{U}_i$

the B/F method is a really stand-alone reserving method

(and not just a manipulated chain ladder)

and has an underlying stochastic model

from which a formula for the prediction error can be derived.