GUY CARPENTER

Distribution and Value of Reserves – Paid and Incurred

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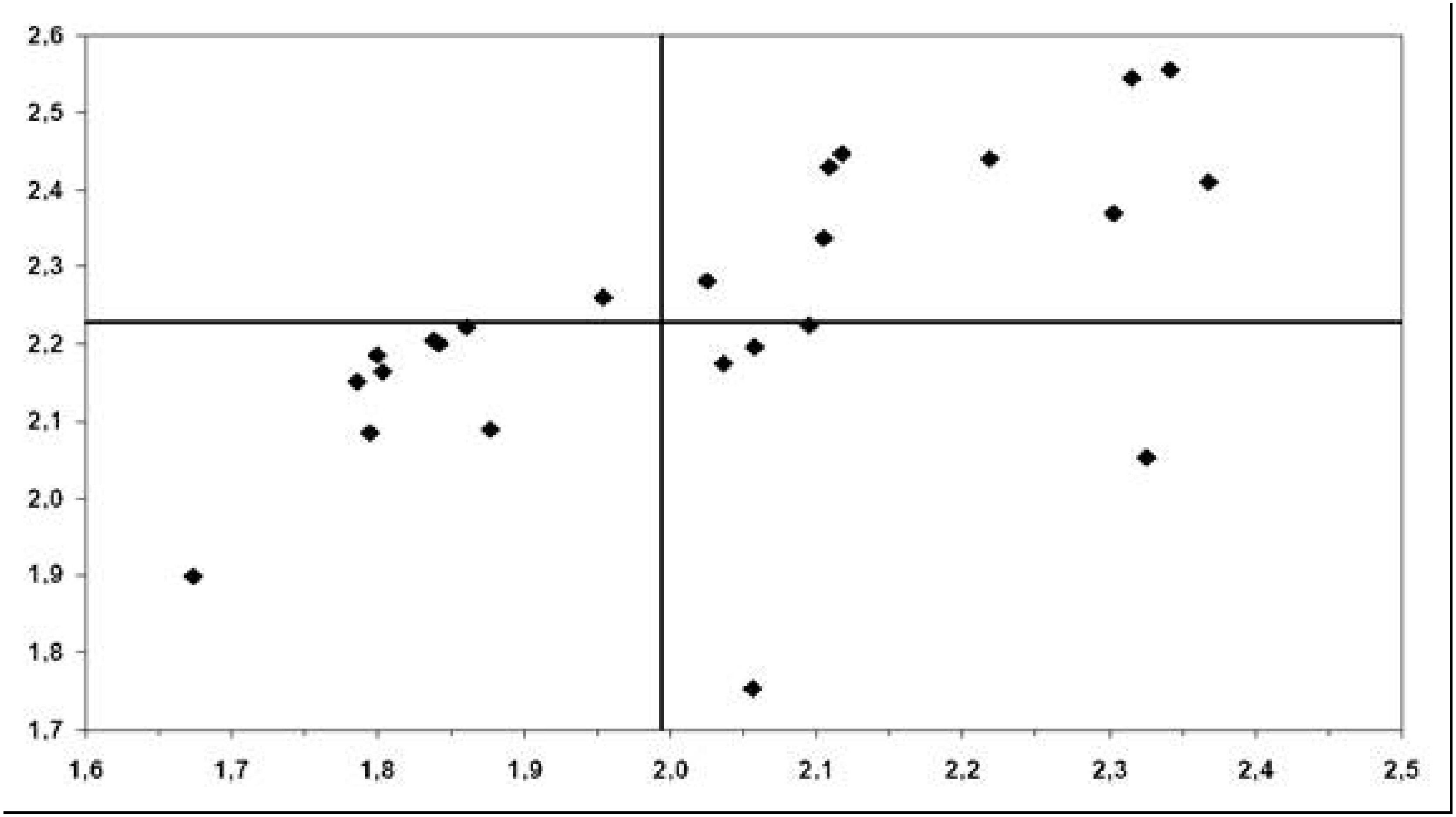
Paid and Incurred Indications

- Often very different
- Quarg and Mack found a way to do a simultaneous paid and incurred chain ladder
- Marker-Mohl 1980 suggested predicting incremental paid from previous case reserves
- Here do development factors as regressions but use paid, incurred, or case reserves as predictors for any of these

Paid Development Factors as a Function of Previous Incurred to Paid Ratio [from Quarg and Mack]



Example Cumulative Paid Triangle

	0	1	2	3	4	5	6
0	576	1804	1970	2024	2074	2102	2131
1	866	1948	2162	2232	2284	2348	
2	1412	3758	4252	4416	4494		
3	2286	5292	5724	5850			
4	1868	3778	4648				
5	1442	4010					
6	2044						

- Paid losses usually increase left to right
- Year 3 highest, year 6 next

Incurred (which has some decreases, in orange)

	0	1	2	3	4	5	6
0	978	2104	2134	2144	2174	2182	2174
1	1844	2552	2466	2480	2508	2454	
2	2904	4354	4698	4600	4644		
3	3502	5958	6070	6142			
4	2812	4882	4852				
5	2642	4406					
6	5022						

- Year 3 pretty high
- Year 6 way high for incurred

Other Modeling Issues

Over-parameterization —»

- Columns near end usually have small factors not significantly different from 1.0
- More of a problem for row-column factors

Diagonal effects —»

- Cells on a diagonal are all same calendar year
- Claims department activity for the year or outside economic events could make some diagonals low and some high
- Not accounting for this can distort other parameters
- Projecting calendar-year effects can increase ranges

Residuals —»

- Not always normal - variance proportional to mean^k
- In exponential family $k = 0, 1, 2, 3$ or $1 < k < 2$ give:
- Normal, Poisson, Gamma, Inverse Gaussian, Tweedie
- Won't stick to exponential family
- Normal regression good starting point
  - Usually coefficients and their significance not too far off

Regression: An Art and a Science

Building models —»

- Find explanatory variables
- Use statistical criteria for evaluating models
- Automated searches to optimize criteria may invalidate some of them

Criteria —»

- Significant variables
  - Coefficient should be at least as high as its standard error and usually at least twice as high
- Standard error of regression useful for comparison of models
  - Reduces with better fits
  - Increases with more parameters

Variables —»

- Often better to look at first differences or ratios, especially in time series
  - Can separate among variables that generally move together
- In reserving, use incremental data

Exploratory Analysis – Incremental Paid Data Lags 1 and 2

Predictive ability of previous cumulative

Correlation with previous:	Incurred	Paid	Unpaid
Paid at Lag 1	88%	84%	70%
Paid at Lag2	68%	57%	92%

- Previous unpaid as predictor of incremental paid

Paid at Lag	1	2	3	4	5	6
Factor	1.95	0.67	0.33	0.33	0.28	0.36

- After lag 2, incremental paid about 1/3 of previous unpaid
- Single coefficient may be significant but 4 separate not

Preliminary Results

Three variable model —»

Paid Incrs	Prev Inc	Prev Unpd	Prev Unpd
1228	978	0	0
1082	1844	0	0
2346	2904	0	0
3006	3502	0	0
1910	2812	0	0
2568	2642	0	0
166	0	300	0
214	0	604	0
494	0	596	0
432	0	666	0
870	0	1104	0
54	0	0	164
70	0	0	304
164	0	0	446
126	0	0	346
50	0	0	120
52	0	0	248
78	0	0	184
28	0	0	100
64	0	0	224
29	0	0	80

- **Coefficients: 0.818 0.696 0.325**
- **Standard errors: 0.033 0.131 0.264**
- **1st 2 significant, 3rd not really**

- **Sum of Residuals and # of Positive Residuals by Diagonal**

Diagonal	1	2	3	4	5	6
Sum	427.5	-470.2	-236.8	200.9	-437.3	532.8
# > 0	1	0	1	3	1	5

- **Diagonals seem to come in offsetting pairs**
- **Strong diagonal effects may distort coefficients in model**
- **Diagonal dummy variables may account for this**
- **Scale dummies by independent variables**

Including Diagonal Dummies

Adding Diagonal Pair Dummies —»

Paid	Incrd	Unpd	Unpd	d 6 - 5	d 4 - 3	d 1- 2
1228	978	0	0	0	0	978
1082	1844	0	0	0	0	-1844
2346	2904	0	0	0	-2904	0
3006	3502	0	0	0	3502	0
1910	2812	0	0	-2812	0	0
2568	2642	0	0	2642	0	0
166	0	300	0	0	0	-300
214	0	604	0	0	-604	0
494	0	596	0	0	596	0
432	0	666	0	-666	0	0
870	0	1104	0	1104	0	0
54	0	0	164	0	-164	0
70	0	0	304	0	304	0
164	0	0	446	-446	0	0
126	0	0	346	346	0	0
50	0	0	120	0	120	0
52	0	0	248	-248	0	0
78	0	0	184	184	0	0
28	0	0	100	-100	0	0
64	0	0	224	224	0	0
29	0	0	80	80	0	0

Parameter	Estimate	St dev
Incurred 0	**0.8286**	**0.0107**
Unpaid 1	**0.6619**	**0.0406**
Unpaid 2 - 5	**0.3342**	**0.0808**
Diag 6 – 5	**0.1378**	**0.0155**
Diag 4 – 3	**0.0326**	**0.0138**
Diag 2 -0.2384	**0.0355**	
Diag 1 0.4270	**0.0656**	

- **SE down from 207 to 73 with 3 dummies, and to 63 with 4 of them**
- **All variables now significant**

Unpaid Model

Parameter	Estimate	St dev
Paid Cum 0	**0.8215**	**0.1036**
Paid Incr 1	**-0.5436**	**0.0864**
Constant 1	**522.68**	**96.860**
Paid Cum 1	**0.0766**	**0.0098**
Unpaid 2 - 5	**0.6615**	**0.0983**
Diag 3 0.0800	**0.0281**	

Distribution of Residuals

p – Distributions: Variance ∝ Mean^p

Normal – p —»

- Take $\sigma^2 = k\mu^p$
- $f(x) = (2\pi k\mu^p)^{-1/2} \exp[-(x-\mu)^2/(2k\mu^p)]$

Gamma – p —»

- Gamma $F(x,\theta,\alpha) = \Gamma(x/\theta\ ;\ \alpha)$ with incomplete gamma Γ.
- This has mean $\alpha\theta$ and variance $\alpha\theta^2$.
- To make mean a parameter, set $F(x,\mu,\alpha) = \Gamma(x\alpha/\mu;\ \alpha)$. Then the variance is μ^2/α.
- For gamma-p, take $F(x;\mu,k,p) = \Gamma[x/(k\mu^{p-1});\ \mu^{2-p}/\ k]$, which has mean μ and variance $k\mu^p$, with skewness = 2CV.

Lognormal – p —»

- Lognormal $F(x;\mu,\sigma) = N\left(\frac{\ln(x)-\mu}{\sigma}\right)$ This has mean $e^{\mu+\sigma^2/2}$ and variance $e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$
- Now reparameterize with three parameters *p, m* and *s*:

$$F(x;m,s,p) = N\left(\frac{\ln\left((x/m)\sqrt{1+s^2m^{p-2}}\right)}{\sqrt{\ln\left(1+s^2m^{p-2}\right)}}\right)$$

- This has mean *m*, variance s^2m^p, and skewness $3CV+CV^3$, where $CV = sm^{p/2-1}$. Here m has been replaced by $\ln\left(\frac{m}{\sqrt{1+s^2m^{p-2}}}\right)$ and s^2 by $\ln(1+s^2m^{p-2})$.

Fits of p-Distributions by MLE

Paid regression —»

	p	– Ln L	Skew
Lognormal-p	1.50	111.94	> 3CV
Gamma-p	1.57	111.23	2CV
ZMCSP-p	1.60	110.52	CV
Normal-p	1.61	109.88	0
Weibull	2	108.76	-0.50

Unpaid regression —»

	p	–Ln L	Skew
ZMCSP-p	1.96	113.30	CV
Normal-p	2.03	112.93	0
Weibull	2	111.88	-0.38

Weibull Parameters and Completing the Square

Paid	Estimate	Unpaid	Estimate
Incurred 0	**0.7811**	Paid Cum 0	**0.7358**
Unpaid 1	**0.6854**	Paid Incr 1	**-0.4275**
Unpaid 2 – 5	0.3306	Constant 1	**388.41**
Diag 6–5+4–3	**0.0339**	Paid Cum 1	**0.0908**
Diagonal 2	**-0.1873**	Unpaid 2 – 5	0.7234
Diagonal 1	**0.3971**	Diagonal 3	**0.0525**

Incurred	0	1	2	3	4	5	6
0	978	2104	2134	2144	2174	2182	2174
1	1844	2552	2466	2480	2508	2454	2460
2	2904	4354	4698	4600	4644	4652	4658
3	3502	5958	6070	6142	6158	6169	6177
4	2812	4882	4852	4863	4871	4877	4881
5	2642	4406	4646	4665	4679	4690	4697
6	5022	6182	6656	6685	6707	6722	6733

Runoff Ranges

Simulating Runoff Ranges

Basic approach —»

- Simulate possible parameter sets
  - Parameter uncertainty
  - Regression coefficients and Weibull shape parameter could all be different
  - Need distribution of possible parameters
- For each parameter set, simulate possible runoff
  - Process uncertainty
  - Simulate from coefficients and Weibull parameters

Distribution of possible parameters —»

- MLE estimates are asymptotically normal with variances from steepness of loglikelihood function
  - For small samples, lognormal may be better
- Or could bootstrap parameters by resampling residuals of fit and re-fitting

Parameter Variance

Fisher Information —»

- Asymptotically minimum variance of parameters given the sample
- Starts with matrix of 2nd derivatives of NLL wrt all the parameters
  - First derivatives should be zero at the minimum
  - 2nd should be positive – lower if flat near minimum
  - Mixed partials can be positive, negative, or neither
- Matrix inverse of information matrix is estimate for covariance matrix of the parameters
  - Higher variance if flat near minimum

Simulation of possible parameters —»

- Simulate normal copula with correlation matrix from Fisher information
- Invert simulated probabilities to get possible parameters
- Even if using lognormal marginals, rank correlation and Kendall's tau are preserved

The Model in Detail

Weibull regression —»

- $F(y_{w,u}) = 1 - \exp[-(y_{w,u}/b_{w,u})^c]$
- $b_{w,u} = \Sigma_j \beta_j x_{w,u,j}$
  - In triangle, $y_{w,u}$ is cell in w^{th} row, u^{th} column
  - In design matrix, it is an element in the 1st column
  - $x_{w,u,j}$ are the covariates for that element
- $NLL = -\Sigma \log f(y_{w,u}) = \Sigma[(y_{w,u}/b_{w,u})^c - \log(c y_{w,u}^{c-1}/b_{w,u}^{c})]$

Mixed 2nd partials —»

- $\partial b_{w,u}/\partial \beta_j = x_{w,u,j}$
- So by chain rule, $$\partial NLL \big/ \partial \beta_i = -\sum_{w,u} x_{w,u,i} \, \partial \log f(y_{w,u}) \big/ \partial b_{w,u}$$
- Similarly, $$\partial^2 NLL \big/ \partial \beta_i \partial \beta_k = -\sum_{w,u} x_{w,u,i} x_{w,u,k} \, \partial^2 \ln f(y_{w,u}) \big/ \partial {b_{w,u}}^2$$
- Only dependence on covariates is $x_{w,u,i} x_{w,u,k}$
- . $$\partial^2 \ln f(y) \big/ \partial b^2 = \frac{c}{b^2}\left[1-(1+c)\left(\frac{y}{b}\right)^c\right]$$

Parameters, Errors, Correlations

Paid	Inc 0	Unpd 1	Unpd 2-5	Diag 6543	Diag 2	Diag 1	c
Param	**0.832**	**0.730**	**0.352**	**0.036**	**-0.200**	**0.423**	**7.427**
Std dev	**0.050**	**0.052**	**0.016**	**0.014**	**0.069**	**0.176**	**1.392**
Ratio	**16.70**	**14.08**	**22.31**	**2.49**	**-2.87**	**2.40**	**5.33**

Unpaid	Pd Cum 0	Pd Inc 1	Const	Pd Cum 1	Unpd 2-5	Diag 3	c
Param	**0.793**	**-0.461**	**418.5**	**0.098**	**0.780**	**0.057**	**6.037**
Std dev	**0.145**	**0.100**	**102.9**	**0.008**	**0.042**	**0.022**	**1.148**
Ratio	**5.48**	**-4.62**	**4.07**	**11.64**	**18.55**	**2.52**	**5.26**

Correlation Matrices Paid and Unpaid

1	0.17	0.00	-0.12	-0.24	-0.28	0.11
0.17	1	0.00	-0.19	-0.62	-0.05	0.14
0.00	0.00	1	0.19	0.01	0.00	0.26
-0.12	-0.19	0.19	1	0.13	0.03	-0.03
-0.24	-0.62	0.01	0.13	1	0.07	-0.08
-0.28	-0.05	0.00	0.03	0.07	1	-0.03
0.11	0.14	0.26	-0.03	-0.08	-0.03	1

1	-0.86	0.00	0.02	0.01	-0.03	0.06
-0.86	1	-0.49	0.00	-0.01	-0.05	-0.03
0.00	-0.49	1	-0.02	-0.01	0.07	-0.03
0.02	0.00	-0.02	1	0.07	-0.29	0.29
0.01	-0.01	-0.01	0.07	1	-0.04	0.22
-0.03	-0.05	0.07	-0.29	-0.04	1	-0.09
0.06	-0.03	-0.03	0.29	0.22	-0.09	1

Simulating Runoff

Procedure —»

Probability	Percentile
0.4%	3639
1.0%	3848
5.0%	4356
10.0%	4640
25.0%	5093
50.0%	5607
75.0%	6105
90.0%	6557
95.0%	6818
99.0%	7344
99.6%	7632

- Simulate parameters from lognormals with correlation matrices
- Simulate paid and unpaid from Weibull model
  - Mean = 5600, Std dev = 750
  - Result very close to normal
- Paid and unpaid parameters assumed independent
  - Bootstrap may get correlations
  - E.g., 2 – 5 previous unpaid in paid and unpaid regressions may be negatively correlated
    - The less is paid, the more there is unpaid
- May or may not increase runoff ranges

Adding Systematic Risk

Other Systematic Risk

Runoff risk sources —»

- (i) Frequency or severity trends that are different from v is built into the projection model
- (ii) Year-to-year fluctuation in trends
- (iii) New claim types arising, like asbestos, mold, etc., with retroactive application to prior accident years
- (iv) Changes in the portfolio that lead to changes in loss emergence patterns
- (v) Changes in the economy that affect claim filing and emergence
- (vi) Misestimation and changes in payout patterns

Gluck's model —»

(Stochastic factor to apply to paid loss in accident year w for lag d)

- $H_{w,d} = B_{w+d} D^{w+d-n} E_{w+d} [P_{dC^{(w+d)}} - P_{(d-1)C^{(w+d-1)}}]$;
- B_{w+d} is a mean ≥ 1 factor to represent (iii) – (v) for calendar year *w+d* in the simulation;
- D is a mean 1 draw for all calendar years in the simulation to represent (i); *n* is last diagonal in data;
- E_{w+d} is from an AR-1 model to represent (ii)
- For (vi), P_i is portion of losses expected paid by lag *i* ; $C^{(w+d)}$ is mean 1 random draw of acceleration factor for calendar year w+d, and so $P_{dC^{(w+d)}}$ is the accelerated (or decelerated) cumulative payment portion.

AR – 1 Model for Trend Fluctuations

- Let X_i's be independent $N(0, \sigma^2)$ random draws
- $\rho \in [0,1]$ is the autocorrelation coefficient
- Let $t_1 = X_1$, and $t_{i+1} = \rho t_i + X_{i+1}$. Then

$$E_{w+d} = \exp\left(\sum_{j=1}^{w+d-n} t_j\right)$$

Value of Reserves

Theories of Value

In general

(i) Standard deviation load
(ii) Percentile of distribution
(iii) CAPM
(iv) CLPM?
  a) If an asset is more risky its value is less
  b) If a liability is more risky its value is greater
(v) Transformed distribution mean
(vi) Market value of disposal
(vii) Undiscounted mean

Examples

- Australia: 75th percentile with special rules if highly skewed
- Fair value: market value of disposal to counterparty with similar credit risk
  - So value decreases with credit risk?
  - Never gets to zero?
  - Accounting theory is in a crisis about how close book should be to market

Additive Valuations

- Accountants like to add and subtract
  - Short-term liabilities + long-term liabilities = all liabilities
- Market values should be additive
  - If sum of values > value of sum, reinsure a lot of individual lines, then retrocede for a profit with no risk
    - Basically prices for pieces reflects pooling that will be done
- Favors means of transformed distributions
  - Transform puts more weight on adverse outcomes
  - CAPM can be so expressed
- But any homogeneous degree-1 loading (e.g., standard deviation, VaR can be allocated additively using the Euler method
  - Derivative of total risk wrt to line volume

Probability Transform

Original Wang Transform (NN) —»

- S(x) is survival probability 1 – F(x)
- $S^*(x) = \Phi[\lambda + \Phi^{-1}(S(x))]$
- Gets normal percentile of S(x), increases it, and then gets probability of the higher percentile
- Increases probability of higher percentiles

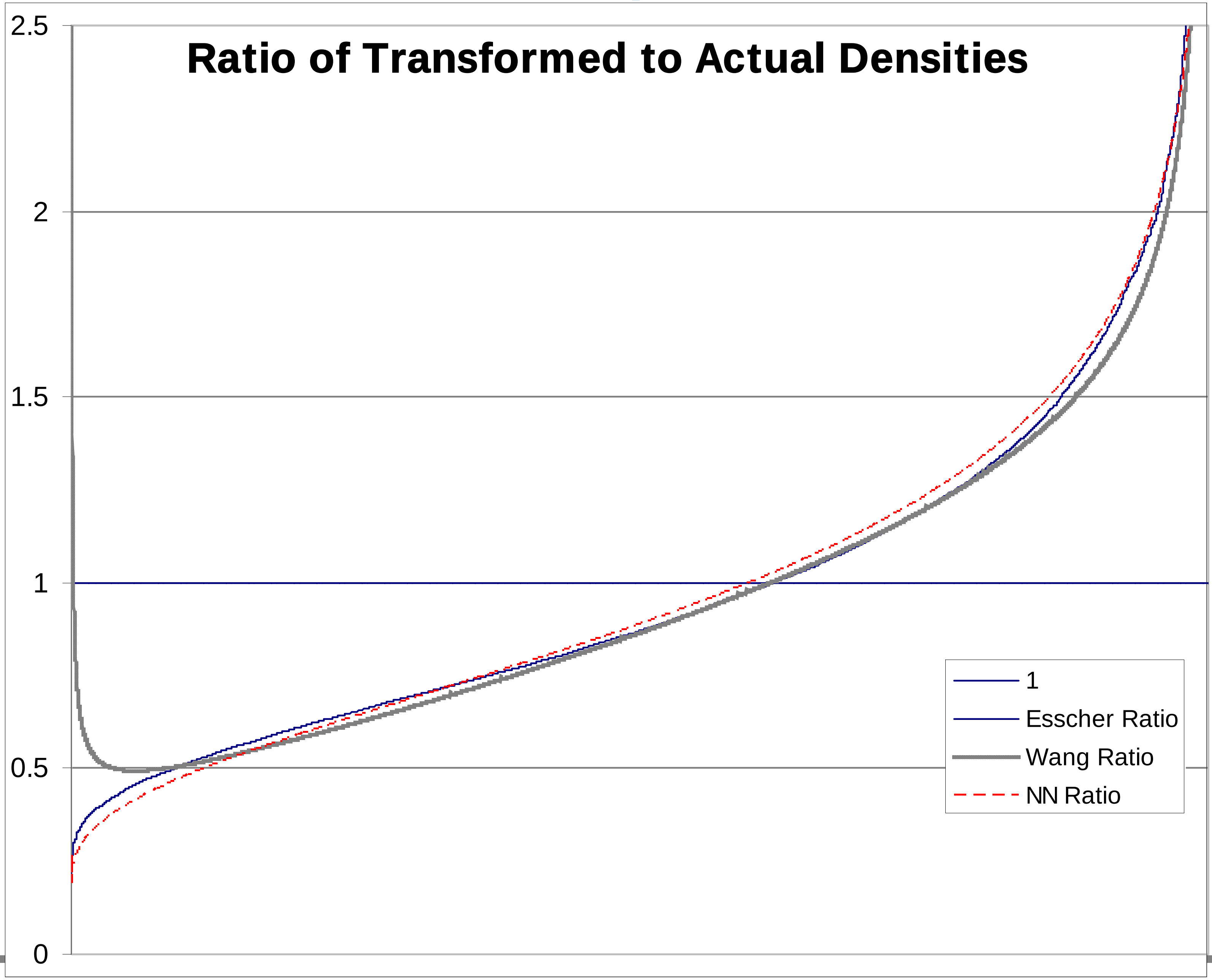
New form (John Major) —»

- $S^*(x) = Q_v[\lambda + \Phi^{-1}(S(x))]$
- Q_v is t-distribution with v dof
  - Puts more weight in both tails, especially for low v
  - Sometimes too much weight on left tail

Esscher transform (on density) —»

- $g^*(x) = g(x)\exp(cx/EX)/ E[\exp(cX/EX)]$
- For Poisson frequency / severity model, do that on severity
- Frequency $\lambda^* = \lambda\, E[\exp(cX/EX)]$

Transforms for Reserve Example



Summary

Summary

- Paid and incurred triangles can be used together
- Exploratory analysis gives you a way to start regression process
- Regression software helps identify useful models
- Can then try other distributions for residuals
- Information matrix gives parameter uncertainty
  - Bootstrapping is an alternative
- Simulating parameters then losses can estimate the runoff distribution
- Systematic risk should be considered to get realistic distribution of actual runoff
- Additive methods helpful for risk pricing once you have the distribution

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