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Statistical Weights on Reserve Methods

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Agenda

- Introduction
- Methodology
- Parameter risk mitigation
- Case study
- Discussion and conclusions
- Q&A

1. Introduction

- Reserve by definition is management's best point estimate of future liability
- A weighted average of various reserve methods is often used to reach a point estimate
- The weights are subjective and from actuarial judgment
- Sometimes, the weight selections are arbitrary

1. Introduction

- Why statistical weights?
  - Reduce reserve variability and projection errors
  - Objectivity and clarity
  - Easier to explain
  - Complement and supplement actuarial judgments

4

1. Introduction

- Three sources of projection errors
  - Process risks due to the random nature of claim generation, reporting, and settlement
  - Parameter risks due to small sample of data
  - Model risks due to unknown underlying models or distributions
  - Weighted average reduces parameter and model risks through diversification

5

1. Introduction

- Jing, Lebens and Lowe (2009)
  - Minimize variance of error distribution
  - Pioneer the study of statistical weights
- This study extends their work
  - Introduces a bias term into the optimization
  - Allows the weights to vary by accident years
  - Adds practical constraints, such as non-negative weights and decreasing B-F weights with age

6

2. Methodology

To minimize mean-square projection error (MSE)

- Classical performance measure of a statistical estimation
- Two components of MSE: a bias term and a variance term
- The variance term can be further decomposed into a process variance and an estimation variance

2. Methodology

Mean-square error

If

reserve estimate has an even chance to be 0.9 and 1.1,

the future liability has an even chance to be 0.85 and 1.25,

and the correlation between reserve estimate and future liability is 0.25

then:

$$Bias(\hat{L}, L) = mean(L) - mean(\hat{L}) = 0.05$$

$$Var(L - \hat{L}) = 0.06$$

$$MSE = E(L - \hat{L})^2 = 0.0625 = Bias(\hat{L}, L)^2 + Var(L - \hat{L})$$

2. Methodology

Ultimate Projection Errors

- Cannot be observed till many years later
- Estimable by development age
  - CLDF is not observable, but can be estimated by multiplying LDFs from each development year
  - Similarly, Ultimate projection errors can be estimated by adding projection errors of incremental losses from each development year

2. Methodology

Out-of-sample projection to avoid over-fitting

AY	DY1	DY2	DY3	DY4	DY5
1	33,663	102,929	148,601	169,559	181,455
2	29,222	95,725	139,421	162,819	
3	30,192	95,302	135,137		
4	22,077	73,907			
5	22,719				

- Use only the historical data before a point of time to predict the future loss at that point of time
- If we use 3.195 as LDF, it is not out-of-sample projection.

3.195=(102929+95725+95302+73097)/(33663+29222+30192+22077)

2. Methodology

Projection Error Estimation: loss development

AY	DY1	DY2	DY3	DY4	DY5
1	33,663	102,929	148,601	169,559	181,455
2	29,222	95,725	139,421	162,819	
3	30,192	95,302	135,137		
4	22,077	73,907			
5	22,719				

- LDF1-2= 3.158=(102929+95725+95302)/(33663+29222+30192)
- Projected incremental loss=22077*(3.158-1)=47647
- Absolute projection error at DY2 =(73,907-22077)-47647=4184
- Relative error=4184/EP=1.95%

2. Methodology

Projection Error Estimation: BF

AY	DY1	DY2	DY3	DY4	DY5
1	33,663	102,929	148,601	169,559	181,455
2	29,222	95,725	139,421	162,819	
3	30,192	95,302	135,137		
4	22,077	73,907			
5	22,719				

- Projected incremental Loss at DY2=EP*LR*(1/CLDF2-1/CLDF1)=49110
- CLDF1=5.893; CLDF2=1.866; LR=65%;
- Absolute error at DY2 =(73907-22077)-49110=2720
- Relative error=4184/EP=1.32%

2. Methodology

- Theoretical framework
  - Notations

i, accident year
j, development year
m, method index, 1 is loss-development, 2 is BF
D, *data index*, 1 is paid; 2 is incurred
${}_dQ_{i,j}$, cumulative loss of d^{th} dataset
${}_dq_{i,j}$, incremental loss of d^{th} dataset
P_i , earned premium for calendar year i
${}_{d,m}\varepsilon_{i,j}$, projection error on the incremental loss
${}_{d,m}w_i$, weight of reserve estimate associated with (i, d, m)

2. Methodology

- Theoretical framework (cont'd)
  - Cumulative future loss for accident year i:

    $${}_dL_i = \sum_{j=N-i+2}^{N} {}_dq_{i,j}$$

  - Ultimate loss for accident year i: $U_i = {}_dQ_{i,N-i+1} + {}_dL_i$
  - Projection of ${}_dq_{i,j}$ by m^{th} method using d^{th} triangle:

    $${}_{d,cl}\hat{q}_{i,j} = {}_dQ_{i,j-1} * ({}_d\hat{d}_{i,j-1} - 1)$$

    $${}_{d,bf}\hat{q}_{i,j} = P_i * \hat{LR}_i * (1/{}_d\hat{D}_j - 1/{}_d\hat{D}_{j-1})$$

    where ${}_d\hat{d}_{i,j-1}$ is the projected loss development factor,

    $\hat{LR}_i$ is expected loss ratio for accident year i ,

    ${}_d\hat{D}_j$ is estimated cumulative LDF.

2. Methodology

- Theoretical framework (cont'd)
  - Projection error on the incremental loss

    ${}_{d,m}\varepsilon_{i,j} = {}_{d,m}\hat{q}_{i,j} - {}_dq_{i,j}$ and ${}_{d,m}\varepsilon_{i,j} = P_i * ({}_{d,m}b_j + {}_{d,m}e_{i,j})$

    where ${}_{d,m}b_j = E({}_{d,m}\varepsilon_{i,j} / P_i)$ and ${}_{d,m}e_{i,j}$ is i.i.d. and $E({}_{d,m}e_{i,j}) = 0$
  - Future loss projection for ${}_dL_i$: ${}_{d,m}\hat{L}_i = \sum_{j=N-i+2}^{N} {}_{d,m}\hat{q}_{i,j}$ and

    Ultimate loss projection: ${}_{d,m}\hat{U}_{i,N-i+1} = {}_dQ_{i,N-i+1} + {}_{d,m}\hat{L}_i$.

    The projection error of ultimate loss

    $${}_{d,m}\hat{U}_{i,N-i+1} - U_i = \sum_{j=N-i+2}^{N} {}_{d,m}\hat{q}_{i,j} - \sum_{j=N-i+2}^{N} {}_dq_{i,j} = P_i({}_{d,m}B_i + {}_{d,m}e_i)$$

    where the projection bias of ultimate loss

    ${}_{d,m}B_i = \sum_{j=N-i+2}^{N} {}_{d,m}b_j$ and the sum of future random

    disturbance terms ${}_{d,m}e_i = \sum_{j=N-i+2}^{N} {}_{d,m}e_{i,j}$.

2. Methodology

- Theoretical framework (cont'd)
  - Projection error of the weighted average

    $$ULWA_i - U_i = \sum_{d,m} {}_{d,m}w_i * P_i * {}_{d,m}B_i + \sum_{d,m} {}_{d,m}w_i * P_i * {}_{d,m}e_i$$

  - The variance term of the ultimate loss projection error

    $$\text{var}(U_i) = \text{var}(\sum_{d,m} ({}_{d,m}w_i * P_i \sum_{j=N-i+2}^{N} {}_{d,m}e_{i,j})) = \sum_{j=N-i+2}^{N} W_i{}' * \Omega_j * W_i$$

  - Optimization to minimize the variance of the projection error for accident year i:

    $$\underset{w}{Min} \sum_{j=N-i+2}^{N} W_i{}' * \Omega_j * W_i \text{, subject to } sum(w_i) = 1$$

  - Optimization to assure non-negative weights and introduce the bias terms of MSE:

    $$\underset{w}{Min}(W_i{}' * B_i)^2 + \sum_{j=N-i+2}^{N} W_i{}' * \Omega_j * W_i \text{, subject to } w_i > 0 \text{ and } sum(w_i) = 1$$

  - Optimization to solve statistical weights for multiple accident years

    $$\underset{w}{Min}(\sum_{i=1}^{N} \sum_{j=N-i+2}^{N} W_i{}' * B_i)^2 + \sum_{i=1}^{N} \sum_{j=N-i+2}^{N} (W_i{}' * \Omega_j * W_i),$$

    subject to ${}_{d,m}w_i > 0$, $sum({}_{d,m}w_i) = 1$, and ${}_{d,M}w_i \geq {}_{d,M}w_{i-1} \geq \ldots \geq {}_{d,M}w_1$.

3. Parameter Risk Mitigation

- Parameter risk: many parameters (bias and var-cov terms) to be estimated with small triangular data
- The underlying sample size of claims used to construct the triangle is very credible
- To mitigate parameter risk, we propose a few sampling techniques to recreate many pseudo triangles, each representing a possible and reasonable realization of losses

3. Parameter Risk Mitigation

- Resampling technique could be used to mitigate parameter risk and improve the credibility on parameter estimation.
  - Bootstrapping (resampling with replacement): randomly pick 1/n of policies to construct a triangle, and repeat it n time.
  - Randomization (resampling without replacement): randomly split data into n groups, and construct n triangles
  - Stratified bootstrapping
  - Stratified Randomization
- Resampling technique requires data at policy and claim level.

3. Parameter Risk Mitigation

Random LDF Selections

AY	DY1	DY2
1	33,663	102,929
2	29,222	95,725
3	30,192	95,302
4	22,077	73,907
5	22,719	74,457
6	18,424	61,632

- Using immediate past three observations, we will only have one error for cell (AY4, DY2)
- Relaxing out-of-time projection, we can have 10 errors for cell (AY4, DY2), $10 = \binom{3}{5} = \frac{5 * 4 * 3}{3!}$.

4. Case Study

- Data for study
  - 14-year paid and incurred loss triangles of liability of private passenger auto
  - Reasons for auto liability data
    - A relatively long-tail line compared to property lines
    - A commonly used line in reserve literature
    - A relative stable line so that the result is relatively robust
  - Data is resampled and scaled to block proprietary information

4. Case Study

- Paid and incurred triangles
  - Age-to-age link raitos
  - Weighted 3-year LDF
  - Cumulative LDF (CLDF)
  - Incomplete ratios (1-1/CLDF)

Paid loss triangle

AY	DY 1 to 2	DY 2 to 3	DY 3 to 4	DY 4 to 5
1	3.058	1.444	1.141	1.070
2	3.276	1.456	1.168	1.067
…	…	…	…	…
10	3.680	1.504	1.215	1.092
11	3.365	1.544	1.226	
12	3.621	1.407		
13	3.118			
3-year LDF	3.366	1.478	1.216	1.085
CLDF	6.886	2.046	1.384	1.138
Incomplete Ratio	85.5%	51.1%	27.8%	12.1%

4. Case Study

- Chain-ladder and B-F Ultimate Loss Projections
  - The expected loss ratio in the B-F method for a specific AY usually varies by time.
  - For simplicity, we use expected LR when it first appeared.

Earned Premium and Expected LR

AY	EP	B-F LR
4	244,050,443	67.4%
5	235,873,906	65.3%
6	229,066,664	63.3%
7	214,159,547	59.7%
8	206,333,044	58.5%
9	198,979,451	58.3%
10	231,228,078	58.7%
11	279,796,492	58.6%
12	318,455,838	58.5%
13	321,007,995	57.9%
14	312,647,327	56.8%

4. Case Study

- Out-of-sample loss development factors
  - When calculating the LDFs, only the information available before the point of time is used.
  - This guarantees "true projections" of the LDFs.

Out-of-sample LDFs for Incurred triangle

AY	DY 1 to 2	DY 2 to 3	DY 3 to 4	DY 4 to 5
1	**0.929**	1.051	0.994	0.998
2	**0.870**	1.050	1.018	1.007
3	**0.866**	1.069	1.022	1.000
4	**0.811**	1.073	1.006	1.005
5	**0.875**	1.114	1.013	1.004
6	**0.838**	1.102	1.025	0.992
7	**0.866**	1.107	1.019	1.004
8	0.980	1.112	1.026	1.005
9	1.038	1.126	1.014	1.004
10	1.063	1.089	1.034	1.023
11	0.946	1.137	1.022	
12	0.989	1.077		
13	0.982			

4. Case Study

- Projection error

B-F projection error as % of premium using paid triangle

AY	DY2	DY3	DY4	DY5
4	3.4%	-0.3%	-0.3%	0.7%
…	…	…	…	…
10	-3.0%	0.3%	-1.4%	-1.3%
11	0.4%	-0.1%	-1.0%	
12	-3.1%	1.8%		
13	1.8%			
Average Bias	0.4%	0.0%	-0.5%	-0.2%

Variance-covariance matrix at DY2

Method	Covariance				Correlation			
	IL CL	IL B-F	PD CL	PD B-F	IL CL	IL B-F	PD CL	PD B-F
IL CL	0.001916	0.001993	0.000312	0.000537	1.000	0.996	0.320	0.521
IL B-F	0.001993	0.002090	0.000325	0.000499	0.996	1.000	0.319	0.464
PD CL	0.000312	0.000325	0.000497	0.000213	0.320	0.319	1.000	0.406
PD B-F	0.000537	0.000499	0.000213	0.000555	0.521	0.464	0.406	1.000

4. Case Study

- Variance, covariance, and correlation
  - Variances of incurred-loss projections at DY2 are much higher than those of paid-loss projections.
  - Variances of projection errors decrease as age matures.
  - The correlations between the chain-ladder and B-F methods increase with age.
  - Study primarily focuses on relatively young accident years.

4. Case Study

- Practical considerations
  - Ultimate loss projections from four models are very close for third prior year and before.
  - Coefficients of variation of four projections are all less than 1%.
  - The weights before 3rd prior AY are not important.
  - Only calculate statistical weights for the latest 3 accident years.
  - The bias and variance after DY5 are ignored because their impacts are not material.

4. Case Study

- Subjective factors
  - Subjective selections – “intervention points”
  - Determination of bias
    - Actuary’s judgment based on knowledge and statistical/actuarial analysis.
  - Determination of variance-covariance matrix
    - Rationale based on historical claim practice
  - Constraints
  - Therefore, statistical weights are not “purely statistical”.

4. Case Study

- Optimization setup
  - Two sets of weights
    - Zero biases: minimizing the variance of error distribution
    - Sample averages are used as bias estimates.
  - Variance-covariance matrices
    - No actuarial interventions
    - Directly calculated from sample data
  - Constraints
    - Force decreasing B-F weights with age for both paid and incurred triangles

4. Case Study

- Statistical weights for the latest AY only

$$\underset{{}_{d,m}w_i}{Min} \sum_{j=N-i+2}^{N} W_i'*\Omega_j*W_i \quad subject \ to$$

$$sum(w_i)=1 \quad and \quad w_i>0$$

Method	Paid C-L	Paid B-F	Incurred C-L	Incurred B-F
Weight	31.6%	59.7%	0.0%	8.7%

4. Case Study

- Statistical weights assuming zero biases
  - The weights on B-F methods decrease as age matures (built-in feature of the optimization).

$$\sum_{i=1}^{N} \sum_{j=N-i+2}^{N} (W_i'*\Omega_j*W_i) \quad subject \quad to$$

$${}_{d,m}w_i > 0$$

$$sum({}_{d,m}w_i)=1$$

$${}_{d,bf}w_i \geq {}_{d,bf}w_{i-1} \geq \ldots \geq {}_{d,bf}w_1$$

AY	Paid		Incurred	
	Chain-ladder	B-F	Chain-ladder	B-F
14	31.7%	61.7%	0.0%	6.6%
13	0.0%	55.0%	38.4%	6.6%
12	42.6%	0.0%	57.4%	0.0%

4. Case Study

- Statistical weights with biases

$$\underset{{}_{d,m}w_i}{Min}\left(\sum_{i=1}^{N}\sum_{j=N-i+2}^{N} W_i'*B_i\right)^{\wedge}2+\sum_{i=1}^{N}\sum_{j=N-i+2}^{N}(W_i'*\Omega_j*W_i) \quad subject \ \ to$$

$${}_{d,m}w_i > 0$$

$$sum({}_{d,m}w_i)=1$$

$${}_{d,bf}w_i \geq {}_{d,bf}w_{i-1} \geq .. \geq {}_{d,bf}w_1$$

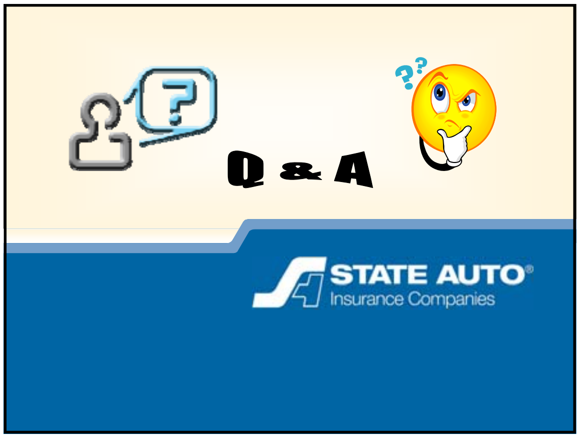
AY	Paid		Incurred	
	Chain-ladder	B-F	Chain-ladder	B-F
14	13.4%	86.6%	0.0%	0.0%
13	0.0%	62.9%	37.1%	0.0%
12	0.0%	0.0%	100.0%	0.0%

4. Case Study

- Findings for this specific case
  - B-F is given more weight than chain-ladder in the less mature accident years.
  - The weights on incurred triangle are much smaller than those on paid triangle because of the changes in setting case reserve.
  - Weight on chain-ladder increases for relatively mature years.
  - Biases may impact the weight calculation significantly: the weight on Paid B-F on the latest accident year increases --- paid B-F has the lowest biases.

5. Discussion and Conclusions

- A statistical method on the weights would complement and supplement reserve actuaries' experience.
- Weights (or method) selection in practice is an art and science.
- Our work extends previous research from two perspectives: introduction of a bias term and practical constraints.
- This study is not to replace the actuarial judgment on weights with statistical estimations, but to provide actuaries a statistical tool to make better decisions when assigning the weights.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