Mack Model

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Mack Model

- Mack developed a distribution-free method for computing the variances of chain ladder age-to-age and age-to-ultimate factors
- The method is applied to weighted average development factors
- While no distribution assumptions are required to estimate factor variances, they are needed to compute percentiles of ultimate loss and reserve distributions

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Chain Ladder Assumptions

[1] E[c(w,d+1)| c(w,1),…,c(w,d)] = c(w,d)F(d)

[2] {c(i,1),…,c(i,n)} & {c(j,1),…,c(j,n)} are independent for $i \neq j$

[3] Var[c(w,d+1)| c(w,1),…,c(w,d)] = $\alpha_d^2 \sigma^2$ - or – Var[c(w,d+1)/ c(w,d)| Data] = α_d^2/c(w,d) for proportionality constants α

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Mack Mean

- Under these assumptions, the best estimate of the age-to-age factor is a weighted average

$$E[F(d)] = \sum_w \frac{c(w,d)}{\sum_w c(w,d)} \times \frac{c(w,d+1)}{c(w,d)} = \frac{\sum_w c(w,d+1)}{\sum_w c(w,d)}$$

- The Ultimate estimate is:

$$E[c(w,n)|D] = c(w,d) \times F(d) \times F(d+1) \times \ldots \times F(n-1)$$

where D is known data c(w,1),…,c(w,d)

Variance of a Development Age

- Since the mean is weighted, the variance is also weighted:
- Variance associated with one age-to-age factor or column of losses, σ_d^2:

$$\sigma_d^2 = \frac{1}{N-d-1} \sum_{j=1}^{N-d} c(j,d) \left(\frac{c(j,d+1)}{c(j,d)} - F(d) \right)^2$$

Exercises using Mack Data

- Compute age-to-age factor triangle for the exercise data
- Compute the weighted average age-to-average factors for each column
- Compute the weighted variances for age 1 for the factors in the exercise triangle

Answer: Variance of Column 1

Computation of σ_1^2 for Development Age 1

Accident Year (w)	$c(w,1)$	$F(w,1)$	$[F(w,1) - F(1)]^2$	Weighted Deviation
(1)	(2)	(3)	(4)	(5)
			$[(3) - 2.334]^2$	(2) x (4)
1981	5,012	1.650	0.47	2,345.0
1982	106	40.425	1,450.90	153,795.4
1983	3,410	2.637	0.09	313.3
1984	5,655	2.043	0.08	477.3
Weighted Average [F(1)]		2.334	Σ (5) =	156,930.9
			$\sigma_1^2 = \Sigma(5) / (N-d-1) =$	52,310.3

Mack Age-to-Age Variance

- The estimated variance of the ultimate for accident year w is the *M.S.E.* of the ultimate for the accident year
- The estimate of the variance of the reserve for each year w $[R(w,n) = R_w = c(w,n) - c(w,d)]$ equals the estimated variance of the ultimate
- Note $c(w,d)$ (the diagonal losses) are constants and makes no contribution to the variance

Mack Variance of a Row (AY)

- The Mack formula for the variance of the reserve estimate for accident year w is:

$$Var[R(w,d)] = U(w)^2 \sum_{d=n+1-w}^{n-1} \frac{\sigma_d^2}{F(d)^2} \left(\frac{1}{E[c(w,d)]} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)} \right)$$

Process Variance (variance of the column of observed development factors)

Parameter Variance (variance of the calculated weighted average development factor)

Example

Computation of *Standard Error* of Ultimate for AY 1985 (w=5)

d	$F(d)$	σ_d^2	$\frac{\sigma_d^2}{F(d)^2}$	$E[c(5,d)]$	$\Sigma_j[c(j,d)]$	$\frac{1}{E[c(1,d)]} + \frac{1}{\Sigma_j[c(j,d)]}$	x $\frac{Sigma_d^2}{F(d)^2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			(3) / (2)^2			1/ (5) + 1/ (6)	(4) x (7)
4	1.147	161.2	122.6	4,920	11,805	0.0002880	0.04
3	1.378	2,887.3	1,519.8	3,569	16,303	0.0003415	0.52
2	1.401	161.2	82.2	2,549	21,546	0.0004388	0.04
1	2.334	52,310.3	9,603.8	1,092	14,183	0.0009863	9.47
			$U(w) =$	5,642		$\Sigma_d =$	10.06
						$Var[R(w,d)] = U(w)^2 \Sigma_d =$	320,349,084
						$SE = [U(w)^2 \Sigma_d]^{1/2} =$	17,898

Exercise

- Compute the standard error for 1984
  - Compute σ_d^2 for ages 2 through 4 for the exercise data (use minimum of 2 & 3 for 4)
  - Assume age 5 is the ultimate valuation.

Answer

Computation of Standard Error of Ultimate for AY 1984

d	$F(d)$	σ_d^2	$\frac{\sigma_d^2}{F(d)^2}$	$E[c(5,d)]$	$\Sigma_j[c(j,d)]$	$\frac{1}{E[c(1,d)]} + \frac{1}{\Sigma_j[c(j,d)]}$	x $\frac{Sigma_d^2}{F(d)^2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			(3) / (2)^2			1/ (5) + 1/ (6)	(4) x (7)
4	1.147	161.2	122.6	22,306	11,805	0.0001295	0.02
3	1.378	2,887.3	1,519.8	16,183	16,303	0.0001231	0.19
2	1.401	161.2	82.2	11,555	21,546	0.0001330	0.01
			$U(w) =$	25,582		$\Sigma_d =$	0.21
						$Var[R(w,d)] = U(w)^2 \Sigma_d =$	140,015,175
						$SE = [U(w)^2 \Sigma_d]^{1/2} =$	11,833

Mack Total Variance

- The total reserve is the sum of the reserve random variable for each accident year

$$R_{tot} = R_1 + R_2 + \ldots + R_w + \ldots + R_N$$

- The variance of a sum equals the sum of the variances plus twice the covariances

Mack Total Variance

- The Mack formula for the variance of the total reserve estimate is:

$$SE(R_{tot})^2 = \sum_{w=2}^{N} \left\{ SE(R_w)^2 + \underbrace{U(w) \left(\sum_{i=w+1}^{N} c(i,n) \right) \sum_{d=n+1-w}^{n-1} \left(\frac{2\sigma_d^2 / F(d)^2}{\sum_{j=1}^{N-d} c(j,d)} \right)}_{\text{Covariance term}} \right\}$$

Calculation Pointers

- It is easiest to set up a set of triangles to perform the calculations
  - Create a triangle of weighted (by *c*(*w*,*d*)) squared deviations of development factors from their mean
  - Create a row of column sums of cumulative losses (excluding the diagonal)
  - Create a projected runoff triangle that computes each estimate of cumulative losses, *c*(*w*,*d*), for all future periods
  - Create a triangle of inverses of projected runoff plus inverse of sum of cumulative losses

Using Mack Parameters

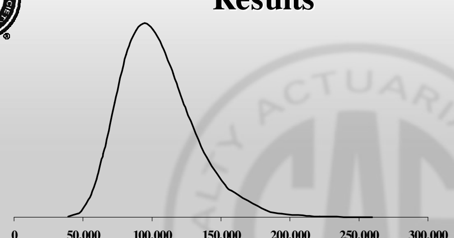
- We have a mean and a variance for reserve amounts. Now what?
- Assumptions must be made to derive confidence intervals or probability distributions (Mack recommends lognormal, can use others)
  - Use mean and variance of reserve amounts to derive method-of-moments parameters for a distribution
  - Use this distribution to estimate percentiles and other statistics for reserve amounts

Group Exercise

- Compute the variance of the total reserve amount using the Mack data
- Assume total reserve amount follows a lognormal distribution and compute the parameters μ & σ. Compute the 75th percentile of the reserve (IBNR) amount.

Refer to Mack Model workbook for results

Results



Mean reserve	104,270
Standard Error	26,909
75th Percentile (lognormal)	119,823
75th Percentile (gamma)	120,945

Open issues

- Covariance term for the oldest year
  - Multiply ultimate for year by
    1. Sum of ultimates for all subsequent years
    2. Times the factor variance (σ_d^2) for last age-to-age factor
    3. Divide by square of last age-to-age factor
  - For Other years, need a sum of the ratio computed in 2 and 3
- Tail Factors? Recursion formula is useful
- Assumption testing

Testing Assumption 1

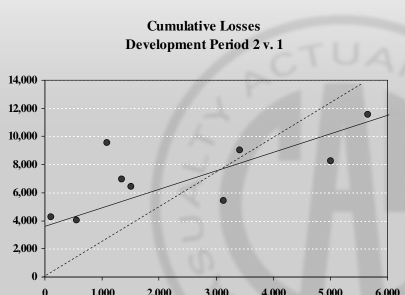
[1] $E[c(w,d+1) | c(w,1),\ldots,c(w,d)] = c(w,d)F(d)$

- Graph two consecutive cumulative development periods.
  - Does it look like a linear relationship?
  - Does it look like the intercept is zero?
- Fit a regression of the form

$$c(w,d+1) = \alpha + \beta c(w,d)$$

Test α for statistical significance

Testing Assumption 1 (cont)



Testing Assumption 2

[2] {c(i,1),…,c(i,n)} & {c(j,1),…,c(j,n)} are independent for i≠j

- Calculate correlation coefficients (matrix) between all accident years
- Test correlations for statistical significance:

$$t(n-2) = \rho\left[\frac{(n-2)}{1-\rho^2}\right]^{1/2}$$

Testing Assumption 2 (cont)

Correlation Coefficients between Incremental Developments

	1	2	3	4	5	6	7	8
1		-0.117	-0.234	-0.104	-0.858	0.632	0.843	-0.962
2			0.395	0.703	0.694	-0.620	-0.314	-0.071
3				-0.145	0.386	-0.283	0.306	-0.777
4					0.562	-0.692	-0.683	0.966
5						-0.786	-0.988	0.816
6							0.381	-0.839
7								-0.824
8								

T values (~ t(n-2))

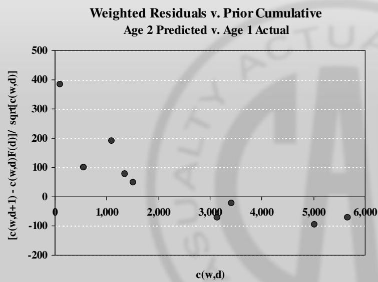
	1	2	3	4	5	6	7	8
1		-0.602	-1.202	-0.511	-8.023	3.824	7.188	-15.680
2			2.153	4.836	4.622	-3.706	-1.515	-0.317
3				-0.716	2.007	-1.382	1.473	-5.520
4					3.259	-4.491	-4.280	16.594
5						-5.967	-29.219	6.320
6							1.887	-6.888
7								-6.502
8								

Testing Assumption 3

[3] $Var[c(w,d+1) | c(w,1),\ldots,c(w,d)] = \alpha_w^2\sigma^2$ for unknown proportionality constants α

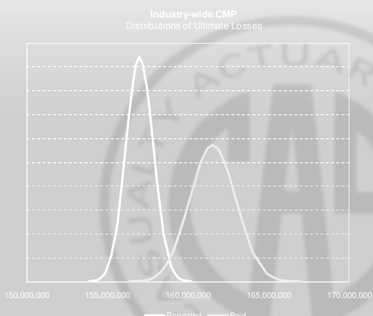
- Graph weighted residuals v. cumulative losses
  $[c(w,d+1) - c(w,d)F(d)] / c(w,d)^{1/2}$ v. $c(w,d)$

Testing Assumption 3 (cont.)



A Final Thought

What do you do when two estimators give you two different answers?



Questions on Mack Model?