A better Bootstrap: the Mack method, the Extended Link Ratio Family (ELRF) and the Probabilistic Trend Family (PTF) modelling Frameworks

Prof Ben Zehnwirth
Visiting Professor, Department of Applied Finance & Actuarial Studies, Faculty of Business and Economics, Macquarie University (NSW)
&
Managing Director Insureware Pty Ltd

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A better Bootstrap, Mack, and the ELRF and PTF modelling Frameworks

- Bootstrap technique- a powerful diagnostic tool for testing a model;
- The Bootstrap is a technique not a model;
- When is the Bootstrap technique needed or necessary?
- Bootstrap samples (are supposed to) replicate the statistical features of the real loss development (array);
- Two Families of models:
  - **Extended Link Ratio Family** (ELRF) that includes Mack, Murphy and extensions/derivatives thereof;
  - **Probabilistic Trend Family** (PTF) that fit a distribution to every cell, equivalently fit the trends in the three directions and the quality of the volatility about the trend structure

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Summary- Link Ratio Methods including Mack and relatives thereof

- Link ratio methods - Mack & Murphy & quasi-Poisson GLM are structure-less, information free, no descriptors of the features in the data. Give incorrect calendar period liability stream;
- On updating, estimates of mean ultimates may be grossly inconsistent;
- Bootstrap samples generated from Mack method are easily distinguishable from the real data;
- Mack, equivalently, volume weighted average (CL) link ratios do not distinguish between development and accident periods! It's the same arithmetic irrespective of the statistical features in the data;

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Summary

- PTF (and MPTF) modeling framework for building single-/multi-triangle models that can capture trend structure and volatility in real data- the latter also the three types of correlations
- Identified model in PTF framework describes the trend structure and volatility succinctly (four pictures). All assumptions tested and validated.
- Model satisfies axiomatic trend properties of every real datset
- Real loss triangle can be regarded as sample path from fitted probabilistic model. Can't tell the difference between real and simulated triangles. Also **Bootstrap samples are indistinguishable from the real data**

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Summary

- Two LOBs written by the same company rarely have the same trend structure (including in the calendar year direction) and often process (volatility) correlation is either zero or very low. Reserve distribution correlation is often zero and if significant quite low.
- No two companies are the same in respect of trend structure, and process (volatility) correlation is often zero (for the 'same' LOB).
- No company is the same as the industry, unless it is a very large proportion of the industry.
- **All the above are demonstrated with real life data**.

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Summary- Advantages of the PTF and MPTF modelling frameworks

- Readily obtain percentiles , V@R and T-V@R tables for total reserve and aggregates, by calendar year and accident year for the aggregate of multiple LOBs and each LOB, conditional on explicit auditable assumptions
- Measurement of the three types of correlations (relationships) between LOBs
- Obtain consistent estimates of prior year ultimates, and SII and IFRS 4 metrics on updating
- Calendar year liability stream distributions (and their correlations) are critical for risk capital allocation and cost of capital calculations; and SII and IFRS 4 metrics (What do they depend on?)
- Pricing future underwriting years
- **No two companies are the same in respect of volatility and correlations**

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Variability and Uncertainty

- different concepts; not interchangeable

  "Variability is a phenomenon in the physical world to be measured, analyzed and where appropriate explained. By contrast uncertainty is an aspect of knowledge."
  – Sir David Cox

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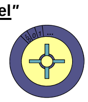
Example: Coin vs Roulette Wheel

Coin	**"Roulette Wheel"**
100 tosses *fair* coin (*#H*?) Mean = 50 Std Dev = 5 CI [50,50]	No. 0,1, …, 100 Mean = 50 Std Dev = 29 CI [50,50] 
In 95% of experiments with the coin the number of heads will be in interval [40,60].	In 95% of experiments with the wheel, observed number will be in interval [2, 97].

Where do you need more risk capital?

Introduce uncertainty into our knowledge - if coin or roulette wheel are mutilated then conclusions could be made only on the basis of observed data

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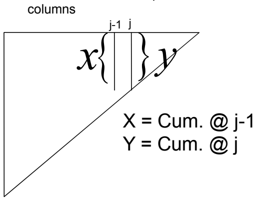
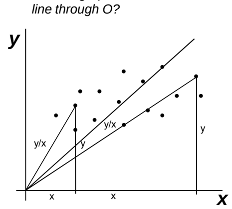
ELRF (Extended Link Ratio Family) Modelling Framework- Regression formulation of link ratios and extensions. Includes Mack, Murphy.

x is cumulative at dev. j-1 and y is cumulative at dev. j

- Link Ratios are a comparison of columns



- We can graph the ratios of Y:X - *line through O?*



Using ratios => $E(Y|x) = \beta x$

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Mack (1993) $\delta = 1$

is a regression formulation of volume weighted average link ratios

$$y = bx + \varepsilon \quad : \quad V(\varepsilon) = \sigma^2 x^\delta$$

Minimize $\quad \Sigma\, w\,(y - bx)^2$

where $\quad w = 1 / x^\delta$

$$1.\quad \delta = 1, \quad \hat{b} = \frac{\Sigma\, x \frac{y}{x}}{\Sigma\, x} = \frac{\Sigma\, y}{\Sigma\, x}$$

Chain Ladder Ratio (***Volume Weighted Average***)

$$2.\quad \delta = 2, \quad \hat{b} = \frac{1}{n} \Sigma \frac{y}{x}$$

Arithmetic Average

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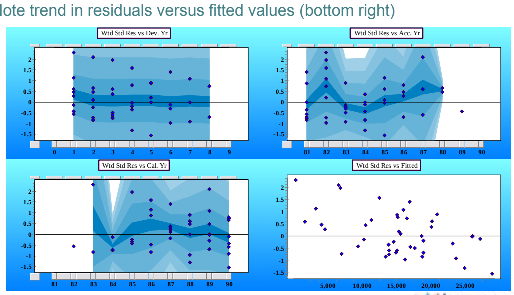
IL(C) Data

Mack (=volume weighted average) weighted standardized residuals

- Note trend in residuals versus fitted values (bottom right)



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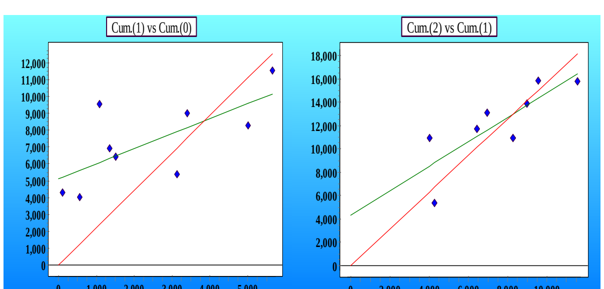
IL(C) Data

Need intercepts- best link ratios are not through origin- hence method over fits big values and under fits small values



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Intercept (Murphy (1994))

$$y = a + bx + \varepsilon \quad : \quad V(\varepsilon) = \sigma^2 x^\delta$$

Since ***y*** already includes ***x***: ***y = x + p, ie p = y - x***

$$p = a + (b-1)x + \varepsilon \; : \; V(\varepsilon) = \sigma^2 x^\delta$$

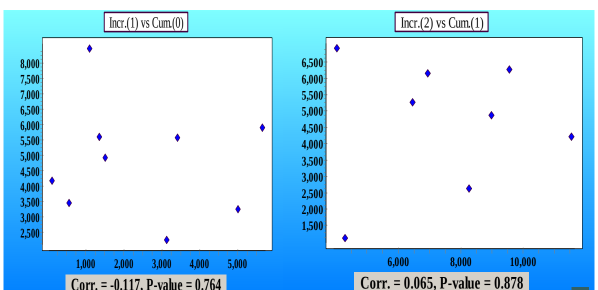
↑ Incremental at j   ↑ Cumulative at j -1

Is b -1 significant ?   Venter (1996)

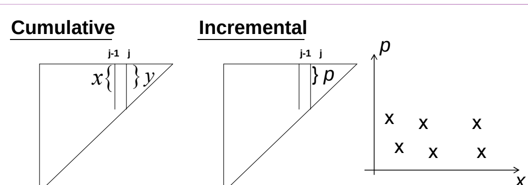
IL(C) data

Link Ratios=1 in presence of an intercept. **Zilch Predictive power**

Incremental incurred not correlated to previous period cumulatives!



Abandon Link Ratios - No predictive power



$$p = a + (b-1)x + \varepsilon \; : \; V(\varepsilon) = \sigma^2 x^\delta$$

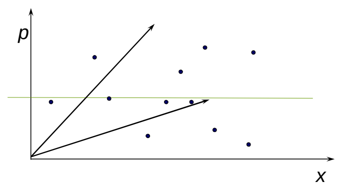
Case (i) $b > 1 \quad a = 0$

Case (ii) $b = 1 \quad a \neq 0$ Link ratio b has no predictive power

$$\hat{a} = \text{Ave(Incrementals)}$$

Is assumption E(p | x) = a + (b-1) x tenable?

- Note: If corr(x, p) = 0, then corr((b-1)x, p) = 0
- If x, p uncorrelated, no ratio has predictive power
- Ratio selection by actuarial judgment can't overcome zero correlation.



- Corr. often close to 0
- Sometimes not
  - Does this imply ratios are a good model?
  - Ranges?

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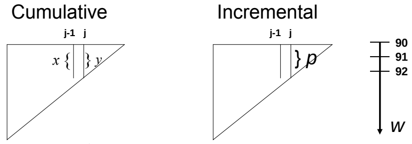
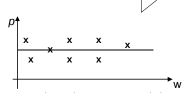
Extended Link Ratio Family (ELRF) Modelling Framework

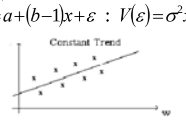


Condition 1:



$$p = a + (b-1)x + \varepsilon \ : \ V(\varepsilon) = \sigma^2 x^{\delta}$$

Condition 2:



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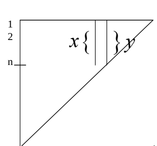
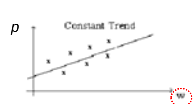
Now Introduce Trend Parameter For Incrementals





$$p = a_0 + a_1 w + (b-1)x + \varepsilon$$

a_0 = Intercept
a_1 = Trend
b = Ratio

p vs acci. yr, and previous cumulative

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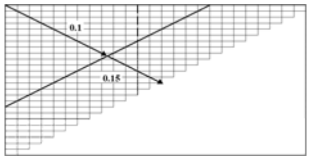
The Probabilistic Trend Family (PTF) Modelling Framework Study in later slides

Condition 3:
Incremental



Review 3 conditions:

Condition 1: Zero trend

Condition 2: Constant trend, positive or negative

Condition 3: Non-constant trend

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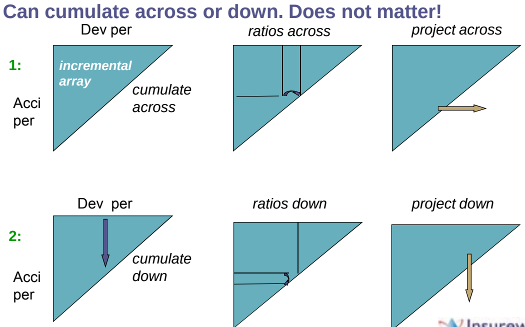
Mack=Chain Ladder (volume weighted average) treats accident years like development years

Can cumulate across or down. Does not matter!



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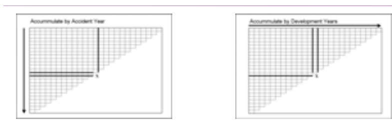
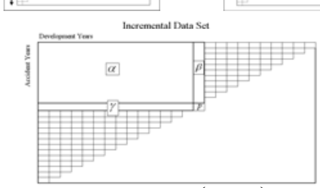
Mack does not distinguish between accident years and development years





$$p = \gamma\left(\frac{\alpha+\beta}{\alpha} - 1\right) = \frac{\gamma\beta}{\alpha}$$

$$p = \beta\left(\frac{\alpha+\gamma}{\alpha} - 1\right) = \frac{\beta\gamma}{\alpha}$$

The standard deviations are different because of different conditioning

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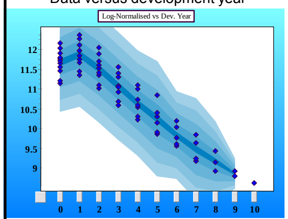
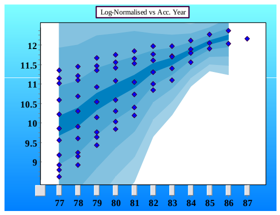
Dataset ABC- Worker's Comp large company

Data versus development year



Data versus accident year



Very different structure. So CL (Mack) ignores this information that sticks out!

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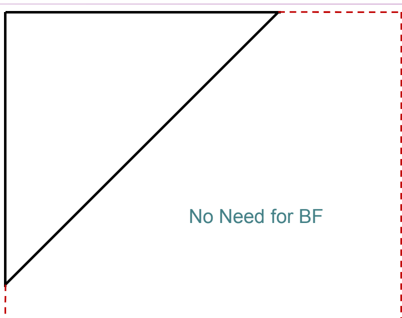
The Probabilistic Trend Family (PTF)Modelling Framework
Here I will use the highlighter to illustrate rudimentary concepts



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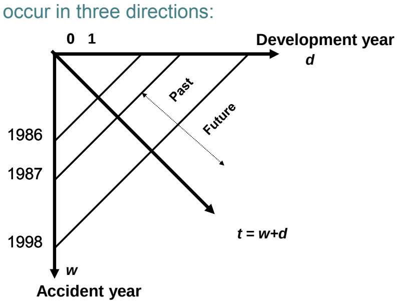
The PTF Modelling Framework
Trend axioms satisfied by every real incremental triangle

- Trends occur in three directions:



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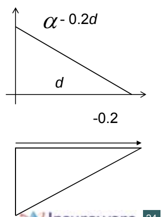
M3IR5 Data- Deterministic data with a single development period trend

0	1	2	3	4	5	6	7	8	9	10	11	12	13
100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	7427
100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080	9072	
100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534	11080		
100000	81873	67032	54881	44933	36788	30119	24660	20190	16530	13534			
100000	81873	67032	54881	44933	36788	30119	24660	20190	16530				
100000	81873	67032	54881	44933	36788	30119	24660	20190					
100000	81873	67032	54881	44933	36788	30119	24660						
100000	81873	67032	54881	44933	36788	30119							
100000	81873	67032	54881	44933	36788								
100000	81873	67032	54881	44933									
100000	81873	67032	54881										
100000	81873	67032											
100000	81873												
100000													

alpha = 11.513

PAID LOSS = EXP(alpha - 0.2d)



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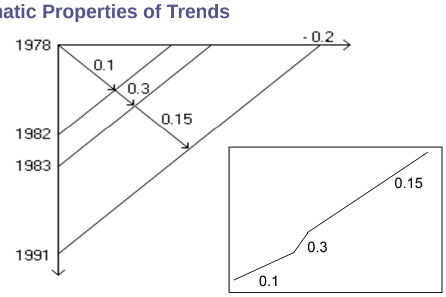
Probabilistic Modelling

We introduce three calendar year trends

Axiomatic Properties of Trends



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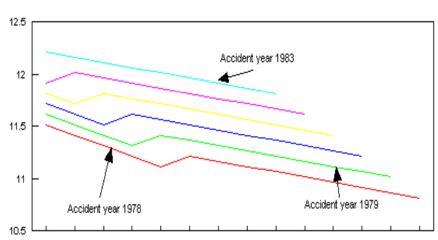
Resultant development year trends (and accident year trends)



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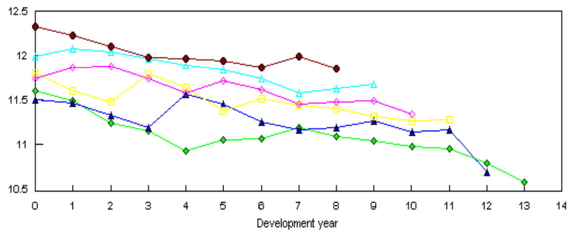
Trends + randomness



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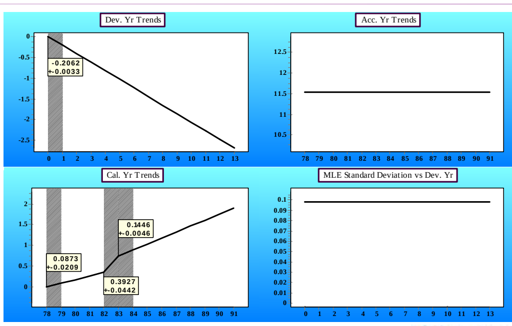
MODEL DISPLAYS- four integral graphs
Graph bottom right represents process variability



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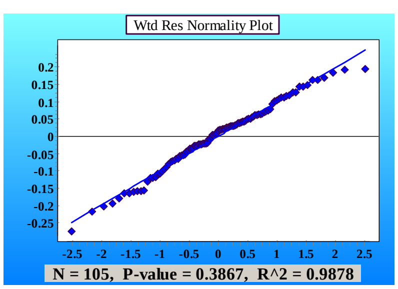
Normal distribution about trend structure
- integral part of model



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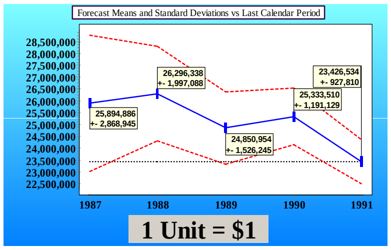
Validation analyses- removal of years



At end of 1991 Reserve dsn mean=23.4, SD=0.928, and at end 1987 mean=25.9, SD=2.87

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Forecast lognormals for each cell

- All assumptions are explicit
- Process variability and parameter uncertainty included

1984	205,644	220,996	169,549	166,858	15,289
1985	224,587	211,182	198,582	186,737	175,603
	221,660	247,187	207,918	18,780	17,816
1986	259,547	244,060	229,502	215,816	202,951
	220,334	234,427	23,094	21,896	20,799
1987	299,956	282,062	265,241	249,428	234,563
	271,278	28,430	26,939	25,576	24,325
1988	346,664	325,989	306,553	288,281	271,105
	35,037	33,181	31,483	29,927	28,496
1989	400,654	376,764	354,306	333,193	313,345
	40,913	38,797	36,858	35,076	33,433
1990	463,061	435,456	409,506	385,110	362,175
	47,859	45,440	43,218	41,171	39,280
1991	535,200	503,303	473,316	445,126	418,623
	56,078	53,304	50,750	48,391	46,206
	1995	1996	1997	1998	1999
al. Per.	2,506,808	2,278,761	2,052,087	1,824,784	1,594,672
Total	122,636	119,405	115,402	110,321	103,865

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Simulate from forecast correlated lognormals Percentiles (Quantiles) and V@R statistics

- All assumptions are explicit
- Process variability and parameter uncertainty included

Quantile Statistics and Value

%	Sample			Kernel		
	Quantile	# S.D.'s	V-a-R	Quantile	# S.D.'s	V-a-R
99.995	26.970	3.820	3.544	27.145	4.008	3.718
99.99	26.937	3.783	3.510	27.065	3.922	3.639
99.98	26.866	3.707	3.439	26.970	3.820	3.544
99.97	26.803	3.640	3.377	26.904	3.748	3.477
99.96	26.773	3.607	3.347	26.850	3.690	3.423
99.95	26.755	3.587	3.328	26.802	3.639	3.376
99.94	26.749	3.581	3.323	26.759	3.592	3.333
99.93	26.703	3.532	3.277	26.719	3.549	3.293
99.92	26.691	3.519	3.265	26.682	3.508	3.255
99.91	26.587	3.406	3.160	26.646	3.469	3.219
99.9	26.567	3.385	3.141	26.611	3.432	3.185
99.8	26.299	3.096	2.872	26.353	3.154	2.927
99.7	26.152	2.937	2.725	26.201	2.991	2.775
99.6	26.049	2.827	2.623	26.096	2.877	2.670

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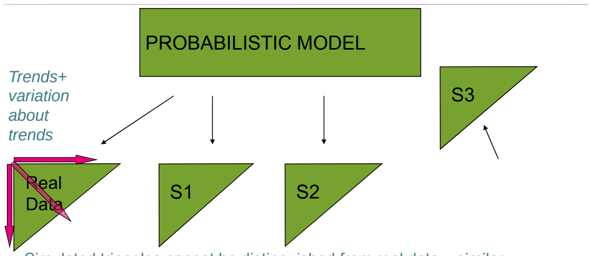

Simulated triangles cannot be distinguished from real data – similar trends, trend changes in same periods, same amount of random variation about trends

Models project past volatility into the future

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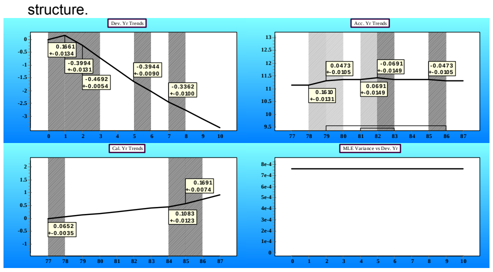
Dataset ABC: The PTF model

The optimal PTF identified model. Note the model fits a normal distribution to each cell. The means are related via the trend structure.



Note major calendar year trend shift

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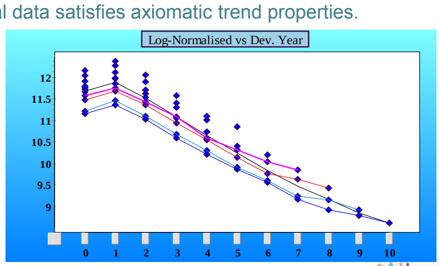
Dataset ABC

- As you move down the accident years the "kick-up" is one development period earlier
- Real data satisfies axiomatic trend properties.



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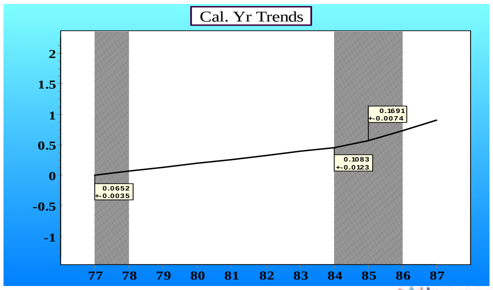
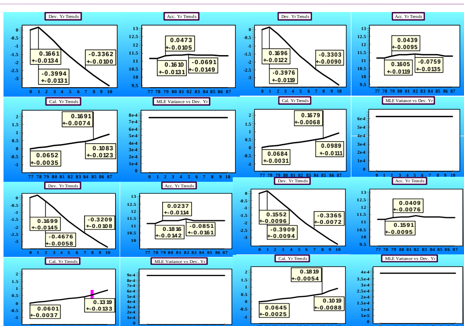
Dataet ABC
PTF-Calendar Year Trends

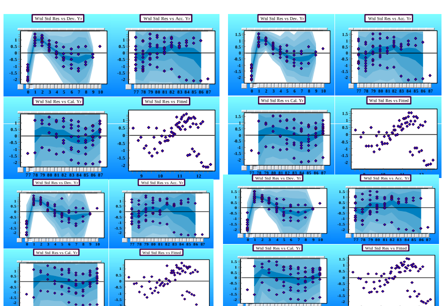
Have control on future assumptions



Dataset ABC: Three simulated triangles from the fitted model, and the real data triangle? Which is real data?

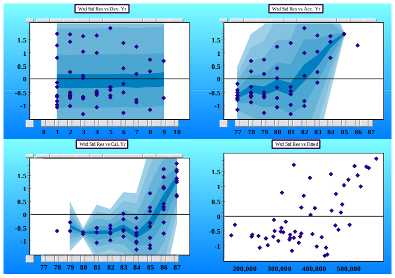


Dataset ABC: Three simulated, one real. Residuals of fitting only one parameter in each direction. Which is the eal data? Simulated triangles have the same statistical features as the real data! We will use Bootstrap echnique later to do same thing.



Dataset ABC- Wtd Standardized Residuals of Mack method (CL link ratios)

It is impossible for any link ratio method including Mack (=CL ratios) to capture and describe trends in any direction, let alone the calendar years.



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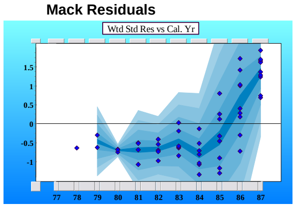
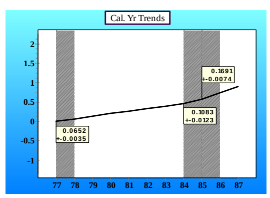
Dataset ABC

ELRF- Mack (volume weighted average link ratios) Residuals versus calendar year. Cannot capture calendar year trend structure. No control on assumptions going forward either, and averager calendar year trend captured cannot be discerned.

Mack Residuals



Calendar Year trends in incrementals



Left) Residuals after applying Mack method to the loss array for Dataset ABC. Note the sharp trend after 1984. Mack under fits recent calendar years and overfits earlier years. (Right) Probability Trend Family model picks up the change in trend structure in this direction, the other two directions and the volatility.

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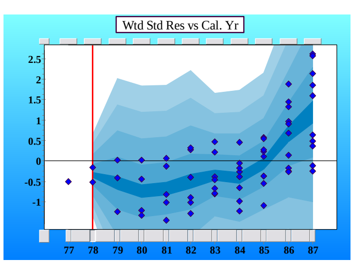
Dataset ABC- Removing the three calendar year trends. That is setting the trend to zero for all calendar years in the PTF modelling framework

Looks a bit like the Mack residuals (but on a log scale)



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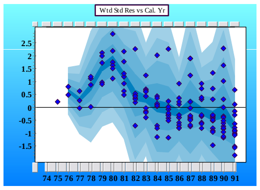
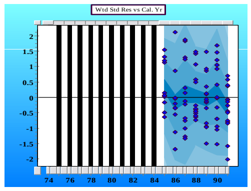
Dataset Mack (CL ratios) reserve too high by a factor of 2!

Reserve = 901,941T +- 108,577T

Reserve = 489,017T +_40,316T





Data trend minus trend estimated by Mack is negative

An ELRF model that better captures calendar year trend in recent cys

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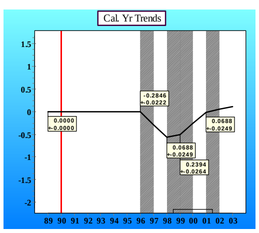
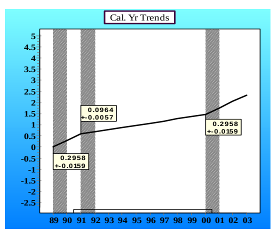
The power of the PTF modelling framework

COMPANY XYZ: CREs versus Paids.
When was the company sold?

CREs



Paids



Insureware 43

The Bootstrap Technique- it is not a model!
The Bootstrap can be used as a powerful diagnostic tool

According to François Morin:

"Bootstrapping utilizes the sampling-with-replacement technique on the residuals of the historical data",

and

"Each simulated sampling scenario produces a new realization of "triangular data" **that has the same statistical characteristics as the actual data**." (Emphasis added)

- François Morin , Integrating Reserve Risk Models into Economic Capital Models, CLRS Seminar, Washington D.C. 2008

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This is worth repeating

- "Each simulated sampling scenario produces a new realization of "triangular data" **that has the same statistical characteristics as the actual data**." (Emphasis added)
- **This only true if the model has the same statistical features as the data!**
- **Bootstrap samples are generated from a model**

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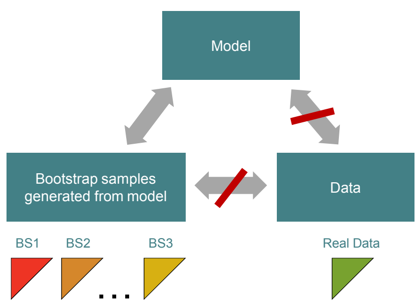
Bootstrap Samples



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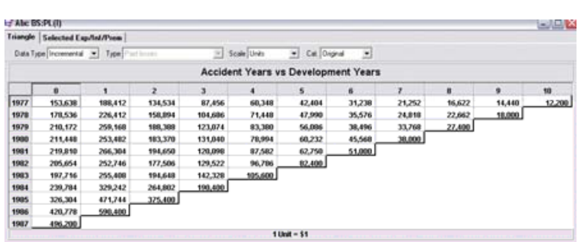
Do you Bootstrap a triangle?
The observations in a triangle are not iid



Accident Years vs Development Years

	0	1	2	3	4	5	6	7	8	9	10
1977	153,638	188,412	134,534	87,456	60,348	42,404	31,238	21,252	16,622	14,440	12,200
1978	178,536	226,412	158,894	104,686	71,448	47,990	35,576	24,818	22,662	18,000	
1979	210,172	259,168	188,388	123,074	83,380	56,086	38,496	33,768	27,400		
1980	211,448	253,482	183,370	131,040	78,994	60,232	45,568	38,000			
1981	219,810	266,304	194,650	120,098	87,582	62,750	51,000				
1982	205,654	252,746	177,506	129,522	96,706	82,400					
1983	197,716	255,408	194,648	142,328	105,600						
1984	239,784	329,242	264,802	190,400							
1985	326,304	471,744	375,400								
1986	420,778	590,400									
1987	496,200										

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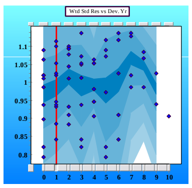
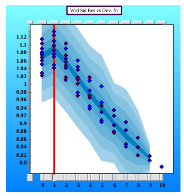
Bootstrapping the data is like assuming each fitted value is zero. That is, a residual = observation

Would anybody want to do that? Why not?

A bootstrap sample



Data



You can easily tell the difference between the BS sample and the real data.
So we need a better model

The Residuals

- These are the differences between the observed values and the fitted values:

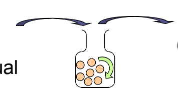
$$e_i = Y_i - \hat{Y}_i, \qquad i = 1 \ldots N.$$

- **The residuals represent the trends in the data minus the trends estimated by the model.**

Bootstrapped Dataset

$$Y_i = \hat{Y}_i + e_i$$

Data = Fit + residual



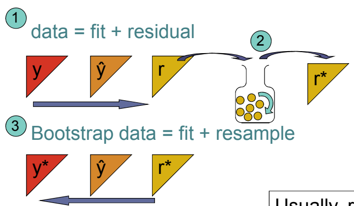
e_i^*

- Working backwards from the bootstrapped residuals $\{e_1^*, \ldots, e_n^*\}$ we form a bootstrap dataset

$$Y_i^* = \hat{Y}_i + e_i^*$$

Bootstrap sample = Fit + re-sample residual (scaled)

Bootstrap sample for a loss development array



resample whole array of "Std residuals"

Usually, r's scaled to constant variance at step (2) then rescaled at step (3)

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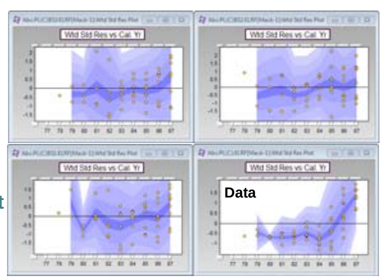
Mack and the bootstrap (Dataset ABC)
The bootstrap as a diagnostic tool

- Mack fitted to the real data contains structure by calendar year

Bootstrap samples from the Mack method lose this structure as it has been randomized!



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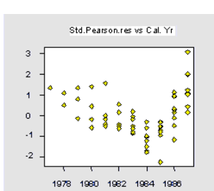
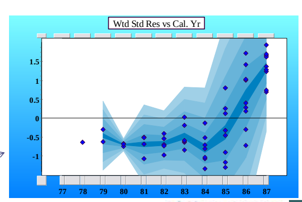
Log-Linear Poisson Residuals versus Mack Residuals- very different. It is not the same model!



The Log-Linear Poisson residuals for **Dataset ABC** also show obvious structure in the calendar direction.



Mack residuals

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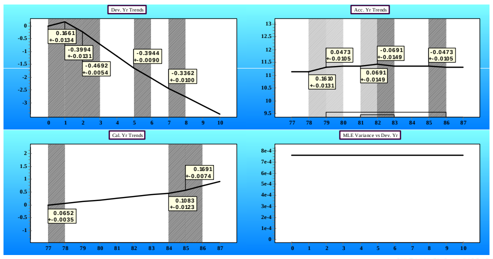
Dataset ABC: The optimal identified PTF model

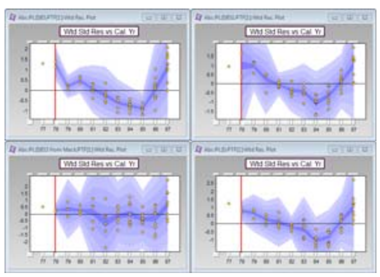
- The optimal PTF model for ABC (again)



Mack bootstrap sample versus bootstrap samples from the identified PTF model (ABC)- The bootstrap technique as a diagnostic tool

<u>Statistical</u> CL applied to four datasets: Real, a Mack bootstrap sample, and two bootstrap samples from the identified PTF model?

No prize for guessing the odd man out!



Residuals of fitting the model with a single parameter in each direction for three datasets: real and two BSs from the identified optimal PTF model

- Which display is the real data? Impossible to tell!