A Stochastic Reserving Today (Beyond Bootstrap)

Presented by
Roger M. Hayne, PhD., FCAS, MAAA

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Reserves in a Stochastic World

- At a point in time (valuation date) there is a range of possible outcomes for a book of (insurance) liabilities. Some possible outcomes may be more likely than others
- Range of possible outcomes along with their corresponding probabilities are the distribution of outcomes for the book of liabilities – i.e. reserves are a distribution
- The distribution of outcomes may be complex and not completely understood
- Uncertainty in predicting outcomes comes from
  - Process (pure randomness)
  - Parameters (model parameters uncertain)
  - Model (selected model is not perfectly correct)

Stochastic Models

- In the actuarial context a stochastic model could be considered as a mathematical simplification of an underlying loss process with an explicit statement of underlying probabilities
- Two main features
  - Simplified Statement
  - Explicit probabilistic statement
- In terms of sources of uncertainty two of three sources may be addressed
  - Process
  - Parameter
- Within a single model, the third source (model uncertainty) usually not explicitly addressed

Basic Traditional Actuarial Methods

- Traditional actuarial methods are simplifications of reality
  - Chain ladder
  - Bornhuetter-Ferguson or it’s close relative Cape Cod
  - Berquist-Sherman Incremental Average
  - Others
- Usually quite simple thereby “easy” to explain
- Traditional reserve approaches rely on a number of methods
- Practitioner “selects” an “estimate” based on results of several traditional methods
- No explicit probabilistic component

Traditional Chain Ladder

- If C_{ij} denotes incremental amount (payment) for exposure year i at development age j
- Deterministic chain ladder

$$C_{ij+1} = f_j \sum_{k=1}^{j} C_{ik}$$

- Parameters f_j usually estimated from historical data, looking at link ratios (cumulative paid at one age divided by amount at prior age)
- Forecast for an exposure year completely dependent on amount to date for that year so notoriously volatile for least mature exposure period

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Traditional Bornhuetter-Ferguson

- Attempts to overcome volatility by considering an additive model
- Deterministic Bornhuetter-Ferguson

$$C_{ij} = f_j e_i$$

- Parameters f_j usually estimated from historical data, looking at link ratios
- Parameters e_i, expected losses, usually determined externally from development data but “Cape Cod” (Stanard/Buhlmann) variant estimates these from data
- Exposure year amount not completely dependent on to-date number

Traditional Berquist-Sherman Incremental

- Attempts to overcome volatility by considering an additive model
- Deterministic Berquist-Sherman incremental severity

$$C_{ij} = E_i \alpha_j \tau_j^i$$

- Parameters E_i exposure measure, often forecast ultimate claims or earned exposures
- Parameters α_j and τ_j usually estimated from historical data, looking at incremental averages
- Berquist & Sherman has several means to derive those estimates
- Often simplified to have all τ_j equal

Curve Extrapolation

- All models previously discussed have
  - A relatively large number of parameters
  - Are confined solely to data observed
- Extrapolation curves have been used to overcome these problems
- Consider a surface based on a curve fit discussed by Tom Wright

$$C_{ij} = \exp\left(\alpha_1 + \alpha_2 j + \alpha_3 j^2 + \alpha_4 \ln(j) + \tau i\right)$$

  - The α parameters define a flexible curve in the development direction that is either unimodal or monotonic.
  - The τ parameter provides for a uniform accident year trend

A Stochastic Framework

- Instead of incremental paid, consider incremental average $A_{ij} = C_{ij}/E_{ij}$
- The amounts are averages of a (large?) sample, assumed from the same population
- Law of large numbers would imply, if variance is finite, that distribution of the average is asymptotically normal
- Thus assume the averages have Gaussian distributions (next step in stochastic framework)
- Note here we have not specified which of the above traditional methods we are considering

A Stochastic Incremental Model – Cont.

- Now that we have an assumption about the distribution (Gaussian) and expected value all needed to specify the model is the variance in each cell
- In stochastic chain ladder frameworks the variance is assumed to be a fixed (known) power of the mean

$$\mathrm{Var}\left(C_{ij}\right) = \sigma \mathrm{E}\left(C_{ij}\right)^{k}$$

- We will follow this general structure, however allowing the averages to be negative and the power to be a parameter fit from the data, reflecting the sample size for the various sums

$$\mathrm{Var}\left(A_{ij}\right) = e^{\kappa - e_i}\left(\mathrm{E}\left(A_{ij}\right)^2\right)^{p}$$

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An Observation on the Methods

- Each of the four traditional methods can be expressed as a function of a number of parameters

$$C_{ij} = g_{ij}\left(\boldsymbol{\theta}\right)$$

- Here $\boldsymbol{\theta}$ represents a vector of the parameters with different lengths for different models
- Instead of specifying a particular method now we will talk in terms of a general method where the incremental amounts can be expressed as a function of a vector of parameters
- For the stochastic version we assume

$$\mathrm{E}\left(A_{ij}\right) = g_{ij}\left(\boldsymbol{\theta}\right)$$

Parameter Estimation

- Number of approaches possible
- If we have an a-priori estimate of the distribution of the parameters we could use Bayes Theorem to refine those estimates given the data
- Maximum likelihood is another approach
- In this case the negative log likelihood function of the observations given a set of parameters is given by

$$\ell\left(A_{11}, A_{12}, \ldots, A_{n1}; \boldsymbol{\theta}, \kappa, p\right) =$$

$$\sum \frac{\kappa - e_i + \ln\left(2\pi\left(g_{ij}\left(\boldsymbol{\theta}\right)^2\right)^p\right)}{2} + \frac{\left(A_{ij} - g_{ij}\left(\boldsymbol{\theta}\right)\right)^2}{2e^{\kappa - e_i}\left(g_{ij}\left(\boldsymbol{\theta}\right)^2\right)^p}$$

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Distribution of Outcomes Under Model

- Since we assume incremental averages are independent once we have the parameter estimates we have estimate of the distribution of future outcomes given the parameters

$$R_i \sim \mathrm{N}\left(E_i \sum_{j=n-i+2}^{n} g_{ij}\left(\hat{\boldsymbol{\theta}}\right), E_i^2 \sum_{j=n-i+2}^{n} e^{\hat{\kappa}-e_i}\left(g_{ij}\left(\hat{\boldsymbol{\theta}}\right)^2\right)^{\hat{p}} \right)$$

$$R_T \sim \mathrm{N}\left(\sum_{i=1}^{m} E_i \sum_{j=n-i+2}^{n} g_{ij}\left(\hat{\boldsymbol{\theta}}\right), \sum_{i=1}^{m} E_i^2 \sum_{j=n-i+2}^{n} e^{\hat{\kappa}-e_i}\left(g_{ij}\left(\hat{\boldsymbol{\theta}}\right)^2\right)^{\hat{p}} \right)$$

- This is the estimate for the average future forecast payment per unit of exposure, multiplying by exposures
- This assumes parameter estimates are correct – does not account for parameter uncertainty

Parameter Uncertainty

- Some properties of maximum likelihood estimators
  - Asymptotically unbiased
  - Asymptotically efficient
  - Asymptotically normal
- We implicitly used the first property in the distribution of future payments under the model
- Define the Fisher information matrix as the expected value of the Hessian matrix (matrix of second partial derivatives) of the negative log-likelihood function
- The variance-covariance matrix of the limiting Gaussian distribution is the inverse of the Fisher information matrix typically evaluated at the parameter estimates

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The Information Matrix

- Key to calculating the variance-covariance matrix for the parameter estimates is calculating the Fisher Information Matrix
- Recall the negative log likelihood function is a function of the parameters $\boldsymbol{\theta}$, κ, and p.

$$\ell\left(A_{11},A_{12},\ldots,A_{n1};\boldsymbol{\theta},\kappa,p\right)=$$

$$\sum\frac{\kappa-e_i+\ln\left(2\pi\left(g_{ij}\left(\boldsymbol{\theta}\right)^2\right)^p\right)}{2}+\frac{\left(A_{ij}-g_{ij}\left(\boldsymbol{\theta}\right)\right)^2}{2e^{\kappa-e_i}\left(g_{ij}\left(\boldsymbol{\theta}\right)^2\right)^p}$$

- So the Hessian and hence its expected value is a function of the parameters κ and p, as well as the partial derivatives of g_{ij} with respect to the θ parameters otherwise independent of g_{ij}

Incorporating Parameter Uncertainty

- If we assume
  - The parameters have a multi-variate Gaussian distribution with mean equal to the maximum likelihood estimators and variance-covariance matrix equal to the inverse of the Fisher information matrix
  - For fixed parameters the losses have a Gaussian distribution with the mean and variance the given functions of the parameters
- The posterior distribution of outcomes is rather complex
- Can be easily simulated:
  - First randomly select parameters from a multi-variate Gaussian Distribution
  - For these parameters simulate losses from the appropriate Gaussian distributions

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Parameterization – Cape Cod

- Simple parameterization for the Cape Cod above overspecifies the model
- We use the following (similar to England & Verall)

$$g_{ij}(\boldsymbol{\theta}) = \begin{cases} \theta_1 \text{ if } i = j = 1 \\ \theta_1\theta_i \text{ if } j = 1 \text{ and } i > 1 \\ \theta_1\theta_{m+j-1} \text{ if } i = 1 \text{ and } j > 1 \\ \theta_1\theta_i\theta_{m+j-1} \text{ if } i > 1 \text{ and } j > 1 \end{cases}$$

- θ_1 is the upper left corner incremental
- θ_i for i = 2, …, n is change in incremental from accident year *i-1* to age *i*
- θ_i for i = n+1, …, m+n-1is change from age i – n to accident year i – n +1

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Parameterization – Berquist-Sherman & Surface Models

- Actually a special case of the Cape Cod
- Replace the accident year change parameters by trend

$$g_{ij}(\boldsymbol{\theta}) = \theta_j e^{i\theta_{n+1}}$$

- θ_j for j = 1, …, n is the accident year 0 average incremental cost at age j
- θ_{n+1} is the natural log of the annual trend in the data
- Parameterization of surface model is unchanged from above

$$g_{ij}(\boldsymbol{\theta}) = \exp\left(\theta_1 + \theta_2 j + \theta_3 j^2 + \theta_4 \ln(j) + i\theta_5\right)$$

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Parameterization – Chain Ladder

- Basic requirements for expected values
  - Ratio of cumulative averages from one age to the next same for all accident years
  - The expected amount to date (on the diagonal) is observed amount to date
- In our parameterization we label the amount to date for accident year *i* as P_i and the age of accident year *i* to date as n_i
- Also in our parameterization we can think of the parameters θ_j as the portion of the total amounts emerging at age *j*
- The incremental percentages can be negative or larger than 1
- We force the percentage for the last age to be the complement of the remainder resulting in $n - 1$ parameters.

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Parameterization – Chain Ladder (Continued)

$$g_{ij}\left(\boldsymbol{\theta}\right)=\begin{cases} P_1\theta_j \text{ if } j<n \text{ and } i=1 \\ P_1\left(1-\sum_{k=1}^{n-1}\theta_k\right) \text{ if } j=n \text{ and } i=1 \\ \dfrac{P_i\theta_j}{\sum_{k=1}^{n_i}\theta_k} \text{ if } j<n \text{ and } i\neq 1 \\ \dfrac{P_i}{\sum_{k=1}^{n_i}\theta_k}\left(1-\sum_{k=1}^{n-1}\theta_k\right) \text{ if } j=n \text{ and } i\neq 1 \end{cases}$$

Example Commercial Auto Liab. Paid Data

Cumulative Average Paid Loss & Defense & Cost Containment Expenses per Estimated Ultimate Claim

Accident	Months of Development										Count
Year	12	24	36	48	60	72	84	96	108	120	Forecast
2001	670	1,480	1,939	2,466	2,838	3,004	3,055	3,133	3,141	3,160	39,161
2002	768	1,593	2,464	3,020	3,375	3,554	3,602	3,627	3,646		38,672
2003	741	1,616	2,346	2,911	3,202	3,418	3,507	3,529			41,801
2004	862	1,755	2,535	3,271	3,740	4,003	4,125				42,263
2005	841	1,859	2,805	3,445	3,950	4,186					41,481
2006	848	2,053	3,076	3,861	4,352						40,214
2007	902	1,928	3,004	3,881							43,599
2008	935	2,104	3,182								42,118
2009	759	1,585									43,479
2010	723										49,492

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Results

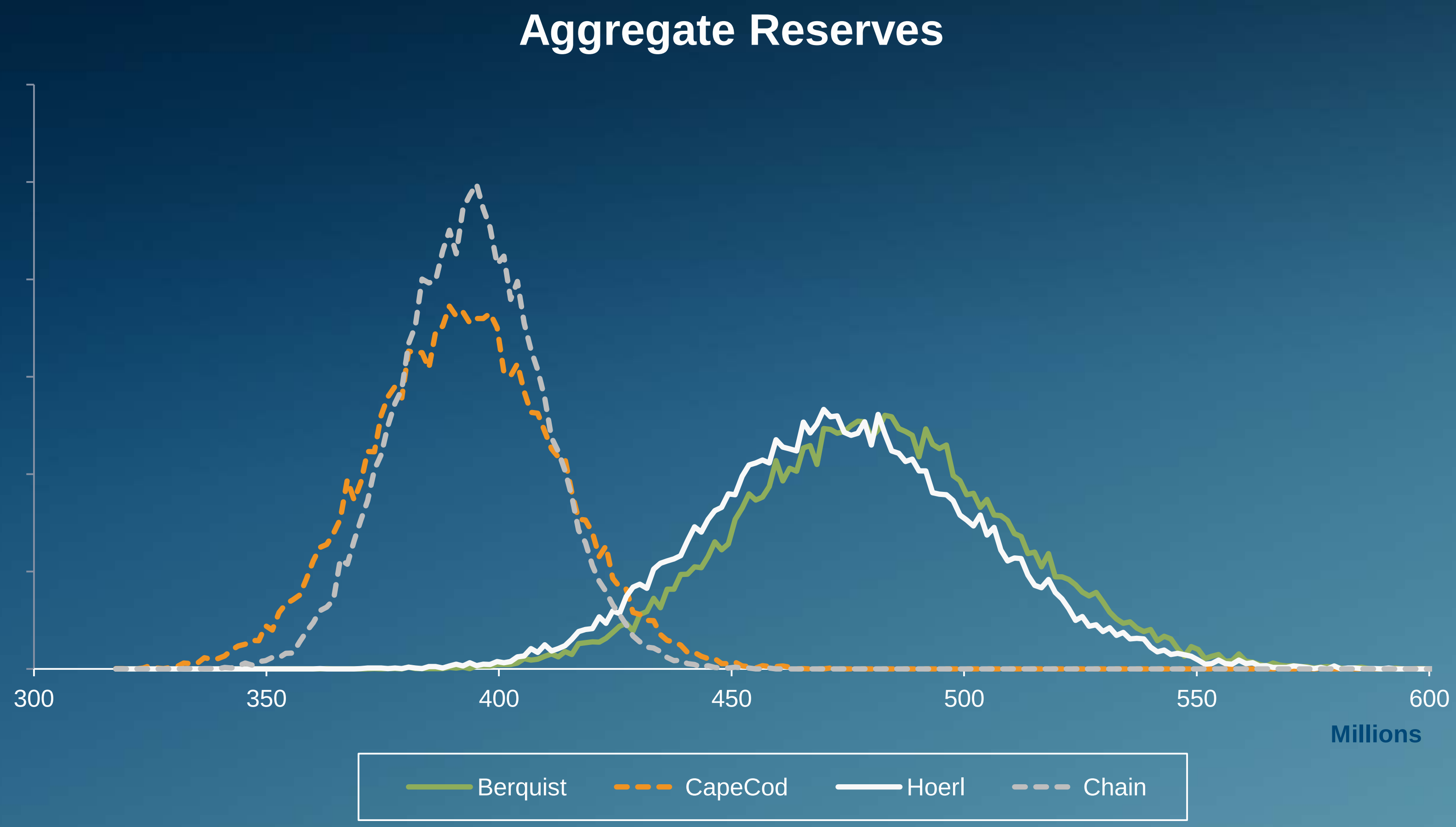
Model	Expected Reserves (000,000)
Berquist Incremental Severity	$480
Cape Cod	391
Generalized Hoerl Curve	474
Chain Ladder	393

- Some difference in expected reserves
- Is the difference random?
- Is the difference significant?
- How do you know?
- Stochastic models help answer these questions

Process vs. Parameter Uncertainty

Model	Total Reserve Process Std. Dev. (000)	Total Reserve Total Std. Dev. (000)
Berquist Incremental Severity	$15,997	$29,405
Cape Cod	9,435	20,101
Generalized Hoerl Curve	16,115	29,454
Chain Ladder	9,447	15,557

Reserve Forecasts by Model



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What Happened?

Standardized Residuals

0.22	0.13	-2.73	-0.58	0.08	-0.57	-0.97	1.88	-0.49	0.31
0.90	-0.07	0.89	-0.55	-0.41	-0.45	-1.22	-1.32	0.87	
0.30	0.04	-0.76	-0.76	-1.64	0.19	0.32	-1.61		
1.18	-0.16	-0.63	0.78	0.72	1.02	1.54			
0.60	0.54	0.51	-0.55	0.93	0.17				
0.31	1.67	0.80	0.58	0.42					
0.46	-0.14	0.89	1.18						
0.39	0.60	0.50							
-1.52	-2.43								
-2.29									

Berquist

Berquist

0.86	1.25	-3.37	0.16	0.92	-0.31	-0.76	1.99	-0.14	0.30
0.78	-0.37	1.66	-0.81	-0.61	-0.71	-1.30	-1.22	0.71	
0.75	0.83	-0.17	-0.34	-1.79	0.53	0.51	-1.34		
0.67	-1.11	-1.41	0.84	0.65	0.82	1.29			
-0.19	0.16	0.58	-1.23	1.01	-0.19				
-1.38	1.24	0.30	-0.08	-0.24					
-0.74	-1.35	0.96	1.33						
-0.65	0.17	0.52							
0.58	-0.48								
0.03									

Cape Cod

Cape Cod

0.20	-0.03	-2.85	-0.45	0.13	-0.61	-1.16	1.73	-0.39	1.92
0.90	-0.23	0.82	-0.40	-0.31	-0.51	-1.32	-0.63	0.30	
0.32	-0.11	-0.85	-0.60	-1.44	0.01	-0.35	-0.83		
1.23	-0.30	-0.71	0.99	0.78	0.71	0.44			
0.67	0.46	0.49	-0.35	1.00	0.01				
0.39	1.66	0.81	0.83	0.53					
0.57	-0.23	0.93	1.49						
0.52	0.58	0.54							
-1.47	-2.66								
-2.27									

Hoerl

0.91	1.30	-3.32	0.20	0.95	-0.29	-0.75	2.02	-0.14	0.31
0.83	-0.31	1.72	-0.76	-0.57	-0.68	-1.29	-1.21	0.72	
0.77	0.86	-0.14	-0.32	-1.78	0.55	0.52	-1.34		
0.61	-1.16	-1.46	0.79	0.61	0.80	1.28			
-0.21	0.13	0.56	-1.24	0.99	-0.19				
-1.38	1.23	0.30	-0.08	-0.24					
-0.74	-1.35	0.95	1.32						
-0.67	0.14	0.50							
0.55	-0.51								
0.00									

Chain Ladder

Hoerl

Chain Ladder

Some Observations

- The data imply that the variance for payments in a cell are roughly proportional to the mean to the 0.85 power for both Cape Cod and Chain Ladder, roughly to the mean for the Hoerl model and to the mean to the 1.30 power for the Berquist model.
- Total standard deviation well above process, often more than double, meaning parameter uncertainty is significant
- Comparison of forecasts among models underlines the importance of model uncertainty
- Still more work to be done to get a handle on model uncertainty – possibly greater than the other two sources

More Observations

- We chose a relatively simple models for the expected value
- Nothing in this approach makes special use of the structure of the models
- Models do not need to be linear nor do they need to be transformed to linear by a function with particular properties
- Variance structure is selected to parallel stochastic chain ladder approaches (overdispersed Poisson, etc.) and allow the data to select the power
- The general approach is also applicable to a wide range of models
- This allows us to consider a richer collection of models than simply those that are linear or linearizable

Some Cautions

- MODEL UNCERATINTY STILL NEEDS TO BE CONSIDERED thus distributions are distributions of outcomes under a specific models and must not be confused with the actual distribution of outcomes for the loss process
- An evolutionary Bayesian approach can help address model uncertainty
  - Apply a collection of models and judgmentally weight (a subjective prior)
  - Observe results for next year and reweight using Bayes Theorem
- We are using asymptotic properties, no guarantee we are far enough in the limit to assure these are close enough
- Actuarial “experiments” not repeatable so frequentist approach (MLE) may not be appropriate