Milliman

Review of Development Triangle GLM-based Stochastic Reserving

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Overview

Participants should read this BEFORE attending AR-6.

The session assumes some prior knowledge about a number of stochastic reserving concepts, which have been grouped under the following headings for this set of review slides:

- **Part I – Triangle GLMs**
- **Part II – Trend Models**
- **Part III – Reserve Projections**
- **Part IV – Stochastic Reserve Ranges**

Part I – Triangle GLMs

Part I – Triangle GLMs

Learning Objectives.

Readers should (re-)familiarize themselves with the following concepts:

- **2d model for triangles** (levels for exposure and development periods)
- **Aliasing with 2d** (need to drop one level for unique parameterization)
- **Parameters vs. modeled values** (equivalent parameterizations)
- **Linearizing multiplicative model** (taking the logarithm)
- **2d+1 model for triangles** (adding levels for payment periods)
- **Aliasing with 2d+1** (need to drop three levels for unique parameterization)
- **Reference levels** (flexibility for setting up models for incomplete triangles)
- **Design matrix** (linear algebra representation of model)

2d Multiplicative Model.

$$\mu_{ij} = a_i \cdot b_j$$

$a_1 \cdot b_1$	$a_1 \cdot b_2$	$a_1 \cdot b_3$	$a_1 \cdot b_4$	$a_1 \cdot b_5$
$a_2 \cdot b_1$	$a_2 \cdot b_2$	$a_2 \cdot b_3$	$a_2 \cdot b_4$	
$a_3 \cdot b_1$	$a_3 \cdot b_2$	$a_3 \cdot b_3$		
$a_4 \cdot b_1$	$a_4 \cdot b_2$			
$a_5 \cdot b_1$				

Generally we are interested in the class of models that have one parameter for each row and column. The μ_{ij} represent the expected amounts for cells of the development triangle.

Aliasing with 2d.

$$\mu_{ij} = a_i \cdot b_j,$$

where $a_2 = 1$.

$a_1 \cdot b_1$	$a_1 \cdot b_2$	$a_1 \cdot b_3$	$a_1 \cdot b_4$	$a_1 \cdot b_5$
b_1	b_2	b_3	b_4	
$a_3 \cdot b_1$	$a_3 \cdot b_2$	$a_3 \cdot b_3$		
$a_4 \cdot b_1$	$a_4 \cdot b_2$			
$a_5 \cdot b_1$				

As it turns out we do not lose any generality by requiring that one of the parameters is equal to 1. Hence, for an $n \times n$ triangle, the degree of freedom for the 2d model is $2n - 1$.

Part I – Triangle GLMs

Parameters vs. Modeled Values

Different sets of parameters can represent the same set of modelled values μ_{ij}. Note that all these sets of parameters have the same degree of freedom.

$a_1 \cdot b_1$	$a_1 \cdot b_2$	$a_1 \cdot b_3$	$a_1 \cdot b_4$	$a_1 \cdot b_5$
b_1	b_2	b_3	b_4	
$a_3 \cdot b_1$	$a_3 \cdot b_2$	$a_3 \cdot b_3$		
$a_4 \cdot b_1$	$a_4 \cdot b_2$			
$a_5 \cdot b_1$				

a_1	$a_1 \cdot b_2$	$a_1 \cdot b_3$	$a_1 \cdot b_4$	$a_1 \cdot b_5$
a_2	$a_2 \cdot b_2$	$a_2 \cdot b_3$	$a_2 \cdot b_4$	
a_3	$a_3 \cdot b_2$	$a_3 \cdot b_3$		
a_4	$a_4 \cdot b_2$			
a_5				

$a_1 \cdot b_1 \cdot c$	$a_1 \cdot c$	$a_1 \cdot b_3 \cdot c$	$a_1 \cdot b_4 \cdot c$	$a_1 \cdot b_5 \cdot c$
$b_1 \cdot c$	c	$b_3 \cdot c$	$b_4 \cdot c$	
$a_3 \cdot b_1 \cdot c$	$a_3 \cdot c$	$a_3 \cdot b_3 \cdot c$		
$a_4 \cdot b_1 \cdot c$	$a_4 \cdot c$			
$a_5 \cdot b_1 \cdot c$				

Part I – Triangle GLMs

Linearizing Multiplicative Model.

$$\eta_{ij} = \log(\mu_{ij}) = \alpha_i + \beta_j$$

$\alpha_1 + \beta_1$	$\alpha_1 + \beta_2$	$\alpha_1 + \beta_3$	$\alpha_1 + \beta_4$	$\alpha_1 + \beta_5$
$\alpha_2 + \beta_1$	$\alpha_2 + \beta_2$	$\alpha_2 + \beta_3$	$\alpha_2 + \beta_4$	
$\alpha_3 + \beta_1$	$\alpha_3 + \beta_2$	$\alpha_3 + \beta_3$		
$\alpha_4 + \beta_1$	$\alpha_4 + \beta_2$			
$\alpha_5 + \beta_1$				

Taking the logarithm converts all multiplications to additions. Using this log-link function is the reason why many stochastic reserving models only work for positive incremental amounts.

Part I – Triangle GLMs

Linearizing Multiplicative Model.

$$\eta_{ij} = \alpha_i + \beta_j,$$

where $\beta_2 = 0$.

$\alpha_1 + \beta_1$	α_1	$\alpha_1 + \beta_3$	$\alpha_1 + \beta_4$	$\alpha_1 + \beta_5$
$\alpha_2 + \beta_1$	α_2	$\alpha_2 + \beta_3$	$\alpha_2 + \beta_4$	
$\alpha_3 + \beta_1$	α_3	$\alpha_3 + \beta_3$		
$\alpha_4 + \beta_1$	α_4			
$\alpha_5 + \beta_1$				

Aliasing is still an issue. Since $\ln(1) = 0$, the multiplicative parameters that are set to 1, become 0 and are therefore "dropped" from the model. The degree of freedom is still $2n - 1$.

Part I – Triangle GLMs

2d+1 Linear Model.

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_1 = \beta_1 = \gamma_2 = 0.$

γ_1	β_2	$\beta_3 + \gamma_3$	$\beta_4 + \gamma_4$	$\beta_5 + \gamma_5$
α_2	$\alpha_2 + \beta_2 + \gamma_3$	$\alpha_2 + \beta_3 + \gamma_4$	$\alpha_2 + \beta_4 + \gamma_5$	
$\alpha_3 + \gamma_3$	$\alpha_3 + \beta_2 + \gamma_4$	$\alpha_3 + \beta_3 + \gamma_5$		
$\alpha_4 + \gamma_4$	$\alpha_4 + \beta_2 + \gamma_5$			
$\alpha_5 + \gamma_5$				

We add parameters for payment periods. Aliasing is more complicated since the payment period can be calculated from the exposure and development period. The degree of freedom for the 2d+1 model is $3n - 3$.

Reference Levels.

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_1 = \beta_1 = \gamma_3 = 0.$

		β_3	$\beta_4 + \gamma_4$	$\beta_5 + \gamma_5$
	$\alpha_2 + \beta_2$	$\alpha_2 + \beta_3 + \gamma_4$	$\alpha_2 + \beta_4 + \gamma_5$	
α_3	$\alpha_3 + \beta_2 + \gamma_4$	$\alpha_3 + \beta_3 + \gamma_5$		
$\alpha_4 + \gamma_4$	$\alpha_4 + \beta_2 + \gamma_5$			
$\alpha_5 + \gamma_5$				

We can choose which parameters to drop, provided that the reference levels do NOT coincide in the same triangle cell. Changing the reference level can become necessary if we are dealing with incomplete triangles. In this example we cannot drop γ_2, since the 2nd diagonal does not exist.

Design Matrix.

The computational methods for fitting GLMs heavily rely on linear algebra. The information about the structure of the model is encoded in a design matrix. The matrix on the right encodes all the information about the linear predictor on the previous slide. In linear algebra notation the equation becomes

$$\eta_{ij} = X \cdot \pi,$$

where X is the design matrix, and π is the vector of parameters.

	α_2	α_3	α_4	α_5	β_2	β_3	β_4	β_5	γ_4	γ_5
(1,3)	0	0	0	0	0	1	0	0	0	0
(1,4)	0	0	0	0	0	0	1	0	1	0
(1,5)	0	0	0	0	0	0	0	1	0	1
(2,2)	1	0	0	0	1	0	0	0	0	0
(2,3)	1	0	0	0	0	1	0	0	1	0
(2,4)	1	0	0	0	0	0	1	0	0	1
(3,1)	0	1	0	0	0	0	0	0	0	0
(3,2)	0	1	0	0	1	0	0	0	1	0
(3,3)	0	1	0	0	0	1	0	0	0	1
(4,1)	0	0	1	0	0	0	0	0	1	0
(4,2)	0	0	1	0	1	0	0	0	0	1
(5,1)	0	0	0	1	0	0	0	0	0	1

Part II – Trend Models

Part II – Trend Models

Learning Objectives.

Readers should (re-)familiarize themselves with the following concepts:

- **Trend parameters** (first differences: change from level to level)
- **Recovering level parameters** (cumulating from reference levels)
- **Parameters vs. modeled values** (equivalent parameterizations)
- **Trend parameters almost unique** (constant shift when changing ref level)

Part II – Trend Models & Reserve Projections

Trend parameters.

Level parameter for "each" period:

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_r = \beta_s = \gamma_t = 0$.

Note that there are $n-1$ non-zero values for the α_i, the β_j, and the γ_k, while $1 \le i, j, k \le n$.

Period-to-period trend parameters:

$$\ddot{\alpha}_\ell = \alpha_{\ell+1} - \alpha_\ell$$
$$\ddot{\beta}_\ell = \beta_{\ell+1} - \beta_\ell$$
$$\ddot{\gamma}_\ell = \gamma_{\ell+1} - \gamma_\ell$$

where $1 \le \ell \le n-1$.

Note that none of the $\ddot{\alpha}_\ell$, the $\ddot{\beta}_\ell$, and the $\ddot{\gamma}_\ell$ are identically zero.

Recovering level parameters.

The formulas for the level parameters in terms of the trend parameters can be read off from the following expression for the linear predictor:

$$\eta_{ij} = \begin{cases} -\sum_{i}^{r-1} \ddot{\alpha}_\ell \\ +\sum_{r}^{i-1} \ddot{\alpha}_\ell \end{cases} + \begin{cases} -\sum_{j}^{s-1} \ddot{\beta}_\ell \\ +\sum_{s}^{j-1} \ddot{\beta}_\ell \end{cases} + \begin{cases} -\sum_{i+j-1}^{t-1} \ddot{\gamma}_\ell \\ +\sum_{t}^{i+j-2} \ddot{\gamma}_\ell \end{cases}.$$

The curly left braces serve as a reminder that for each of the stacked summations only one will actually contribute terms.

Part II – Trend Models & Reserve Projections

Trend parameters almost unique.

When changing reference levels from r, s, t to r', s', t' all parameter values are shifted by the same value, δ. The transformations are given by

$$\ddot{\alpha}'_\ell = \ddot{\alpha}_\ell - \delta, \qquad \ddot{\beta}'_\ell = \ddot{\beta}_\ell - \delta, \qquad \ddot{\gamma}'_\ell = \ddot{\gamma}_\ell + \delta,$$

and δ is give by

$$0 = (r' + s' - t' - 1)\delta + \begin{cases} -\sum_{r}^{r'-1} \ddot{\alpha}_\ell \\ +\sum_{r'}^{r-1} \ddot{\alpha}_\ell \end{cases} + \begin{cases} -\sum_{s}^{s'-1} \ddot{\beta}_\ell \\ +\sum_{s'}^{s-1} \ddot{\beta}_\ell \end{cases} + \begin{cases} -\sum_{t}^{t'-1} \ddot{\gamma}_\ell \\ +\sum_{t'}^{t-1} \ddot{\gamma}_\ell \end{cases}.$$

Part III – Reserve Projections

Part III – Reserve Projections

Learning Objectives.

Readers should (re-)familiarize themselves with the following concepts:

- **Reserve projection with 2d model** (extend triangle using fitted parameters)
- **Snag with 2d+1 model** (need future payment period parameters)
- **Extrapolating trend parameters** (make sure weights add up to one)

Part III – Reserve Projections

Reserve projection with 2d model.

$$\eta_{ij} = \alpha_i + \beta_j,$$

where $\beta_2 = 0$.

$\alpha_1 + \beta_1$	α_1	$\alpha_1 + \beta_3$	$\alpha_1 + \beta_4$	$\alpha_1 + \beta_5$
$\alpha_2 + \beta_1$	α_2	$\alpha_2 + \beta_3$	$\alpha_2 + \beta_4$	
$\alpha_3 + \beta_1$	α_3	$\alpha_3 + \beta_3$		
$\alpha_4 + \beta_1$	α_4			
$\alpha_5 + \beta_1$				

Start with fitting a GLM to the triangle to get the parameters, …

Part III – Reserve Projections

Reserve projection with 2d model.

$$\eta_{ij} = \alpha_i + \beta_j,$$

where $\beta_2 = 0$.

$\alpha_1 + \beta_1$	α_1	$\alpha_1 + \beta_3$	$\alpha_1 + \beta_4$	$\alpha_1 + \beta_5$
$\alpha_2 + \beta_1$	α_2	$\alpha_2 + \beta_3$	$\alpha_2 + \beta_4$	$\alpha_2 + \beta_5$
$\alpha_3 + \beta_1$	α_3	$\alpha_3 + \beta_3$	$\alpha_3 + \beta_4$	$\alpha_3 + \beta_5$
$\alpha_4 + \beta_1$	α_4	$\alpha_4 + \beta_3$	$\alpha_4 + \beta_4$	$\alpha_4 + \beta_5$
$\alpha_5 + \beta_1$	α_5	$\alpha_5 + \beta_3$	$\alpha_5 + \beta_4$	$\alpha_5 + \beta_5$

… and extend the triangle using the fitted parameters. The reserve is the sum of all expected values in the bottom half of the "squared" triangle.

Snag with 2d+1 model.

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_1 = \beta_1 = \gamma_3 = 0.$

		β_3	$\beta_4 + \gamma_4$	$\beta_5 + \gamma_5$
	$\alpha_2 + \beta_2$	$\alpha_2 + \beta_3 + \gamma_4$	$\alpha_2 + \beta_4 + \gamma_5$	$\alpha_2 + \beta_5 + \gamma_6$
α_3	$\alpha_3 + \beta_2 + \gamma_4$	$\alpha_3 + \beta_3 + \gamma_5$	$\alpha_3 + \beta_4 + \gamma_6$	$\alpha_3 + \beta_5 + \gamma_7$
$\alpha_4 + \gamma_4$	$\alpha_4 + \beta_2 + \gamma_5$	$\alpha_4 + \beta_3 + \gamma_6$	$\alpha_4 + \beta_4 + \gamma_7$	$\alpha_4 + \beta_5 + \gamma_8$
$\alpha_5 + \gamma_5$	$\alpha_5 + \beta_2 + \gamma_6$	$\alpha_5 + \beta_3 + \gamma_7$	$\alpha_5 + \beta_4 + \gamma_8$	$\alpha_5 + \beta_5 + \gamma_9$

Want to extend triangle using fitted parameters as for 2d model …

Part III – Reserve Projections

Snag with 2d+1 model.

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_1 = \beta_1 = \gamma_3 = 0.$

		β_3	$\beta_4 + \gamma_4$	$\beta_5 + \gamma_5$
	$\alpha_2 + \beta_2$	$\alpha_2 + \beta_3 + \gamma_4$	$\alpha_2 + \beta_4 + \gamma_5$	$\alpha_2 + \beta_5 + \cancel{\gamma_6}$
α_3	$\alpha_3 + \beta_2 + \gamma_4$	$\alpha_3 + \beta_3 + \gamma_5$	$\alpha_3 + \beta_4 + \cancel{\gamma_6}$	$\alpha_3 + \beta_5 + \cancel{\gamma_7}$
$\alpha_4 + \gamma_4$	$\alpha_4 + \beta_2 + \gamma_5$	$\alpha_4 + \beta_3 + \cancel{\gamma_6}$	$\alpha_4 + \beta_4 + \cancel{\gamma_7}$	$\alpha_4 + \beta_5 + \cancel{\gamma_8}$
$\alpha_5 + \gamma_5$	$\alpha_5 + \beta_2 + \cancel{\gamma_6}$	$\alpha_5 + \beta_3 + \cancel{\gamma_7}$	$\alpha_5 + \beta_4 + \cancel{\gamma_8}$	$\alpha_5 + \beta_5 + \cancel{\gamma_9}$

Want to extend triangle using fitted parameters as for 2d model …
… but we don't know the values of γ_6, γ_7, γ_8, and γ_9!

Part III – Reserve Projections

Extrapolating trend parameters.

We can fix the "snag" using linear extrapolation with weights that add up to one:

$$\eta_{ij} = \alpha_i + \beta_j + \gamma_{i+j-1},$$

where $\alpha_1 = \beta_1 = \gamma_3 = 0.$

		β_3	$\beta_4 + \gamma_4$	$\beta_5 + \gamma_5$
	$\alpha_2 + \beta_2$	$\alpha_2 + \beta_3 + \gamma_4$	$\alpha_2 + \beta_4 + \gamma_5$	$\alpha_2 + \beta_5 + \gamma_6$
α_3	$\alpha_3 + \beta_2 + \gamma_4$	$\alpha_3 + \beta_3 + \gamma_5$	$\alpha_3 + \beta_4 + \gamma_6$	$\alpha_3 + \beta_5 + \gamma_7$
$\alpha_4 + \gamma_4$	$\alpha_4 + \beta_2 + \gamma_5$	$\alpha_4 + \beta_3 + \gamma_6$	$\alpha_4 + \beta_4 + \gamma_7$	$\alpha_4 + \beta_5 + \gamma_8$
$\alpha_5 + \gamma_5$	$\alpha_5 + \beta_2 + \gamma_6$	$\alpha_5 + \beta_3 + \gamma_7$	$\alpha_5 + \beta_4 + \gamma_8$	$\alpha_5 + \beta_5 + \gamma_9$

$$\gamma_k = \sum_{\ell=3}^{k-1} \ddot{\gamma}_\ell$$

fitted

where $\ddot{\gamma}_i = \gamma_{i+1} - \gamma_i$ **for** $i = 1, \ldots, 4$, **and**

$$\ddot{\gamma}_i = \kappa_i + \sum_{\ell=1}^{4} \omega_{i\ell} \cdot \ddot{\gamma}_\ell, \text{ for } i = 5, \ldots, 8, \text{ and } \sum_{\ell=1}^{4} \omega_{i\ell} = 1.$$

extrapolated

This is just the mechanics – the practical implications will be explored with hands-on examples during the concurrent session.

Part IV – Stochastic Reserve Ranges

Part IV - Stochastic Reserve Ranges

Learning Objectives.

Readers should (re-)familiarize themselves with the following concepts:

- **Stochastic payment process** (reserve outcome = estimated - actual)
- **Two components of variability** (parameter & process)
- **Model risk is not considered** (all results are conditional on model structure)
- **Simulation of parameter variability** (bootstrap & MV-normal approximation)
- **Simulation of process variability** (template uses resampling)

This part is of interest to readers who want to understand the different bootstrap options offered by the VBA template used for this session. If you are less familiar with stochastic reserving you can skip this part.

Part IV - Stochastic Reserve Ranges

Stochastic payment process.

Expected Incremental Amounts

250	400	200	100	50
275	440	220	110	55
300	480	240	120	60
325	520	260	130	65
375	600	300	150	75

Expected Unpaid = 1,815

Random Draw **Reserve Outcome = -39**

237	439	140	134	48
296	446	200	59	53
310	472	172	188	64
244	494	183	107	39
411	582	312	135	93

Estimated Unpaid = 1,716

Actual Paid = 1,755

Random Draw **Reserve Outcome = -444**

221	440	164	97	30
314	526	284	63	66
315	441	189	191	23
328	508	283	126	69
336	599	304	138	89

Estimated Unpaid = 1,443

Actual Paid = 1,887

Random Draw **Reserve Outcome = 244**

200	399	203	99	26
265	427	190	122	51
327	484	345	119	68
373	493	283	85	42
421	543	326	141	74

Estimated Unpaid = 1,977

Actual Paid = 1,733

Two components of variability.

In our example on the previous slide we know all the parameters of the underlying stochastic process. Therefore we can precisely calculated the true reserve (i.e. expected unpaid amounts).

In practice we need to estimate the model parameters from the random amounts observed in the triangle, …

… which leads to a complicated multivariate sampling distribution for the model parameters.

Parameter Risk

Part IV - Stochastic Reserve Ranges

Two components of variability.

In our example on the previous slide we know all the parameters of the underlying stochastic process. Therefore we can precisely calculated the true reserve (i.e. expected unpaid amounts).

When it comes to actual future payments, however, we still have to recon with the random nature of the stochastic process, …

… and we do not know what amount will be.

Process Risk

Part IV - Stochastic Reserve Ranges

Model risk is not considered.

Most stochastic reserving models, including the methods implemented in the template used for this session, are conditional on the assumed model structure being correct.

While omitting model risk may be considered “state of the art,” practitioners should be aware of the limitation.

Part IV - Stochastic Reserve Ranges

Simulation of parameter variability.

Most stochastic reserving models rely on Monte Carlo simulations to generate information about the distribution of reserve outcomes.

Bootstrapping is a resampling technique that uses observed residuals to approximate the error structure of the underlying stochastic process. Pseudo data is repeatedly generated, and the parameters are re-estimated from the pseudo data. This results in a Monte Carlo sample of reserve estimates.

There is no universally accepted way of resampling residuals. A commonly implemented approach, namely linear rescaling of Pearson residuals, often breaks down for real data, and the template used for this session offers two resampling methods that are more widely applicable.

Bootstrapping

Simulation of parameter variability.

Both bootstrapping methods use a parameter to define a minimum value (% of expected) for resampled pseudo data.

Split-linear rescaling is the same as linear rescaling of Pearson residuals, if none of the rescaled residuals result in a resampling value below the minimum. If a resampling value drops below the minimum, the Pearson residuals are split into two sets with separate scaling factors in such a way that the resulting resampling distribution retains the assumed mean-variance relationship.

Limited Pareto sampling is a computationally efficient parametric resampling method (not based on residuals) that ensures that the resampled values have the assumed mean-variance relationship.

Bootstrapping

Simulation of parameter variability.

GLM's use maximum likelihood estimation, and there are asymptotic results for the parameter estimates. In particular, there is a generalization of the Mean Value Theorem which states, that for large data sets, the sampling distribution of the parameters approaches a multivariate normal distribution, with a known covariance structure.

This motivates a way of sidestepping the bootstrapping scheme altogether, and directly sampling parameters from the asymptotic multivariate normal distribution. Once the parameters have been sampled, the resulting reserve estimate can easily be calculated. Again we end up with a Monte Carlo sample of reserve estimates.

This method is computationally efficient, but the asymptotic distribution can differ from the actual sampling distribution of the parameters.

Asymptotic Approximation

Part IV - Stochastic Reserve Ranges

Simulation of process variability.

All models supported by the template rely on the same semi-parametric assumptions about the structure of the underlying stochastic process: namely that we are dealing with a GLM where the mean-variance relationship is defined by the choice of variance function.

Consistent with these assumptions, the uncertainty of future payments is simulated using the expected values resulting from the original model fit, and the expected variances implied by the expected values.

For each of the bootstrapping methods, the same resampling or sampling technique is both for past and future payment amounts. For the asymptotic approximation, Limited Pareto sampling is employed for future payment amounts.

Thank you

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