Rehabilitating Traditional Reserving Methods

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Traditional Methods

- Arrange data in two-dimensional arrays
- Use patterns observed in the data to estimate unpaid losses

Strengths of Traditional Methods

- Visual
- Intuitive, easy to learn
- Relatively simple to set up and apply
- Easy to interpret

Weaknesses of Traditional Methods

- Traditional methods are deterministic
- Methods seem “ad hoc”, lack a clear mathematical basis
- Methods do not respond to changing inflation projections

Joint Probability of Two Dice

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Sum of Two Dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Sum of Two Dice

Prob(7) = $\sum$ Prob(A) Prob(7-A)

	1	2	3	4	5	6
1						1/36
2					1/36	
3				1/36		
4			1/36			
5		1/36				
6	1/36					

Convolution

Statistical Meaning:

The sum of two independent random variables

Paid Loss Development

1. Ultimate(AY) = CumPaid(AY) x LDF(age)

2. Unpaid = $\sum$ Ultimate(AY) – CumPaid(AY)

Substitute age = CY-AY in 1, then combine 1 & 2:

Unpaid(CY) = $\sum$ CumPaid(AY) (LDF(CY-AY) -1)

Traditional Methods as Convolutions

- Paid Loss Method:

  $$\text{Unpaid(CY)} = \int \text{CumPaid(AY) (LDF(CY-AY) -1) dAY}$$

- Incurred Loss Method:

  $$\text{IBNR(CY)} = \int \text{CumInc(AY)(LDF(CY-AY) -1) dAY}$$

- Bornheutter-Ferguson:

  $$\text{IBNR(CY)} = \int \text{ExpLoss(AY) (1-1/LDF(CY-AY)) dAY}$$

Question

How does it make sense that a multiplicative method (LDF) can be the sum of two distributions?

If we consider a distribution to be composed of a pattern and an error term, the patterns are multiplied while the error terms add.

Notation

We will use the symbol * to denote convolution,

i.e. h = f * g means

$$h(y) = \int_{-\infty}^{\infty} f(x)g(x-y)\,dx$$

Convolution of Error Terms

The linearity of convolution says that

$$f(x)(1+\varepsilon_f)*g(y)(1+\varepsilon_g) = f(x)g(y)(1+(\varepsilon_f*\varepsilon_g))$$

$$(1+\varepsilon_f)*(1+\varepsilon_g) = \int 1+\varepsilon_f+\varepsilon_g+\varepsilon_f\varepsilon_g$$

but $\int \varepsilon_f\varepsilon_g = 0$

Question

What is the error term for the LDF method?

Consider the emerged loss as a point mass

$\varepsilon_{emerged} = \delta(0)$

$\varepsilon_{emerged} * \varepsilon_{LDF} = \varepsilon_{LDF}$

$\delta(0)$ is the identity element for convolution

Question

What is the error term for the B-F method?

$\varepsilon_{ELR} * \varepsilon_{IBNR\%}$

Observation

As convolutions, traditional methods are stochastic

We can compute the error distribution for each method

Weaknesses of Traditional Methods

☑Traditional methods are ~~deterministic~~ stochastic

☐Methods seem “ad hoc”, lack a clear mathematical basis

☐Methods do not respond to changing inflation projections

Linearity

- Commutative

  $f * g = g * f$

- Associative

  $f * (g * h) = (f * g) * h$

- Distributive over addition

  $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$

More Algebraic Properties

- One-to-one and Onto
  - One-to-one implies uniqueness of solutions
  - Onto implies existence of solutions

- Invertible
  - Identity is the δ function (point mass)

Modern Algebra

Algebra	3 operations, 2 vector spaces, linear in each
Field	All of the properties of the real numbers
Ring	2 operations, linearity
Group	1 operation, inverses

Weaknesses of Traditional Methods

☑Traditional methods are ~~deterministic~~ stochastic

☑Traditional Methods ~~seem "ad hoc~~" are convolution equations

☐Methods do not respond to changing inflation projections

Convolution

Statistical Meaning:

The sum of two independent random variables

Algebraic Meaning:

A linear transformation to or from diagonals

Sum of Two Dice

Prob(7) = $\sum$ Prob(A) Prob(7-A)

	1	2	3	4	5	6
1						1/36
2					1/36	
3				1/36		
4			1/36			
5		1/36				
6	1/36					

Convolution from CY to AY

		CY Pattern					
		12	24	36	48	60	72
Inflation Rate	Cost Index	0.28	0.25	0.14	0.09	0.05	0.04
6%	1.338						0.054
6%	1.263					0.063	
6%	1.191				0.107		
6%	1.124			0.157			
6%	1.060		0.265				
	1.000	0.280					

AY Incremental Partial Severities

AY	12	24	36	48	60	72	84	96	108
1998									61
1999								121	47
2000							136	86	81
2001						231	109	87	148
2002					302	173	78	195	144
2003				541	269	159	168	206	128
2004			791	473	243	202	253	192	12
2005		1,351	808	437	284	300	238	35	87
2006	1,625	1,484	869	487	295	397	90	150	
2007	1,758	1,580	863	607	543	213	225		
2008	1,995	1,596	1,052	831	375	322			
2009	1,936	1,895	1,233	585	495				
2010	2,025	1,957	1,059	753					
2011	2,128	1,928	1,173						
2012	2,140	1,910							
2013	2,071								

Age

CY Incremental Partial Severities

CY	Age 12	24	36	48	60	72	84	96
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Averages in Two Directions

Age

CY	12	24	36	48	60	72	84	96	Average	Inflation Index	Y-O-Y Change
2006	1,625	1,351	791	541	302	231	136	121	637	0.83	
2007	1,758	1,484	808	473	269	173	109	86	645	0.84	1%
2008	1,995	1,580	869	437	243	159	78	87	681	0.89	6%
2009	1,936	1,596	863	487	284	202	168	195	716	0.93	5%
2010	2,025	1,895	1,052	607	295	300	253	206	829	1.08	16%
2011	2,128	1,957	1,233	831	543	397	238	192	940	1.22	13%
2012	2,140	1,928	1,059	585	375	213	90	35	803	1.05	-15%
2013	2,071	1,910	1,173	753	495	322	225	150	887	1.16	10%
									767		
Average	1,960	1,713	981	589	351	250	162	134	6,139		
CY Pattern	0.32	0.28	0.16	0.10	0.06	0.04	0.03	0.02			

Convolution from CY to AY

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6%	1.338						0.054
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6%	1.124			0.157			
6%	1.060		0.265				
	1.000	0.280					

Incremental Loss Model

$$\text{Incremental Paid}(AY, age) = X(AY)\ Y(CY)\ Z(age)$$

Cumulative AY Losses

$$\text{Cum. Paid(AY,t)} = \int_{age=o}^{t} X(AY)Y(CY)Z(age)\,dage$$

Cumulative AY Losses

Cum. Paid(AY,t) = $\displaystyle\int_{age=o}^{t} X(AY)Y(CY)Z(age)\,dage$

Cum. Paid(AY,t) = $\displaystyle\int_{CY=AY}^{AY+t} X(AY)Y(CY)Z(CY-AY)\,dCY$

Pull X(AY) Outside the Integral

Cum. Paid(AY,t) = $$X(AY)\int_{CY=AY}^{AY+t} Y(CY)Z(CY-AY)\,dCY$$

Question

How do we interpret an exponential function (inflation) as a distribution?

The distribution that we are working with is the error term around the exponential pattern

In Transform Analysis this is a theorem:

$\mathcal{L}(f(x)) = \mathcal{F}(e^{-\alpha x} f(x))$

Conclusions

Traditional methods are distributional if we view them as solving convolution equations. We can derive error terms.

The AY emergence pattern is the convolution of an inflation index and a constant cost CY emergence pattern.

Extensions

- The convolution approach works with incurred losses as well as paid. No log transforms were used.
- "Reserving cycles" reflect the image under convolution of variations in the inflation rate.