Reserve Variability

2017 CLRS

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Session Outline

1. Convolution Equations in Loss Reserving

2. Properties of Convolutions

3. Reserve Risk = Inflation Risk + Insurance Risk

4. Duality and Transforms

Convolution

Statistical Meaning: The sum of two independent random variables

Algebraic Meaning: A linear transformation to or from diagonals

Sum of Two Dice

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Joint Probability of Two Dice

	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Convolution of Two Dice

$$\text{Prob}(7) = \sum \text{Prob}(A)\ \text{Prob}(7\text{-}A)$$

	1	2	3	4	5	6
1						1/36
2					1/36	
3				1/36		
4			1/36			
5		1/36				
6	1/36					

Convolution

The sum of two independent random variables:

T = X + Y

P(T = t) = $\int_{-\infty}^{\infty} f(x)g(t-x)dx$

Paid Loss Development

1. $\text{Ultimate(AY)} = \text{CumPaid(AY)} \times \text{LDF(age)}$

2. $\text{Unpaid} = \sum \text{Ultimate(AY)} - \text{CumPaid(AY)}$

Substitute age = CY-AY in 1, then combine 1 & 2:

$$\text{Unpaid(CY)} = \sum \text{CumPaid(AY)}\ (\text{LDF(CY-AY)} - 1)$$

Traditional Methods as Convolutions

- Paid Loss Method:

  $\text{Unpaid(CY)} = \int \text{CumPaid(AY) (LDF(CY-AY) -1) dAY}$

- Incurred Loss Method:

  $\text{IBNR(CY)} = \int \text{CumInc(AY)(LDF(CY-AY) -1) dAY}$

- Bornheutter-Ferguson:

  $\text{IBNR(CY)} = \int \text{ExpLoss(AY) (1-1/LDF(CY-AY)) dAY}$

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Notation

We will use the symbol * to denote convolution,

i.e. h = f * g means

$$h(y) = \int_{-\infty}^{\infty} f(x) g(x-y)\, dx$$

Linearity

- Commutative

  $f * g = g * f$

- Associative

  $f * (g * h) = (f * g) * h$

- Distributive over addition

  $f * (\alpha g + \beta h) = \alpha f * g + \beta f * h$

Operator Algebra

Operator	Distribution
Identity	Point mass at 0, or δ
Shift by t	Point mass at t, or δ(t)
Definite Integral	1
Indefinite Integral	1^- or 1^+ (right or left limit)
Derivative	dδ/dt, denote Δ

Derivative Rule

$(f * g)' = f' * g = f * g'$

$(f * g) * \Delta = (f * \Delta) * g = f * (g * \Delta)$

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Three Factor Model

Incremental:

IncrLoss(AY, age) = X(AY) Y(CY) Z(age)

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$$Ultimate(AY) = \int_{age=0}^{\infty} X(AY)Y(CY)Z(age)dage$$

Three Factor Model

Incremental:

IncrLoss(AY, age) = X(AY) Y(CY) Z(age)

$$Ultimate(AY) = \int_{age=0}^{\infty} X(AY)Y(CY)Z(age)dage$$

$$Ultimate(AY) = \int_{CY=AY}^{\infty} X(AY)Y(CY)Z(CY-AY)dCY$$

Rearrange Terms

$$\text{Ultimate(AY)} = \int X(AY)\ Y(CY)\ Z(CY\text{-}AY)\ dCY$$

$$\text{Ultimate(AY)} = \int Y(CY)\ [X(AY)Z(CY\text{-}AY)]dCY$$

Ultimate(AY) = Inflation * Insurance Risk

Unpaid(AY,t) = Inflation * Insurance Risk * $1^{+}(t)$

Expanded Version

Reserve Risk(AY) = Inflation Risk * Insurance Risk

Reserve Risk = Inflation Risk *

Frequency Risk *

Claim Intensity Risk

Risk Theory

Traditional

Frequency – Systemic

Severity - Diversifiable

Convolution

Frequency – Systemic

Intensity – Diversifiable

Inflation - Systemic

Risk Theory

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Frequency – Systemic

Severity - Diversifiable

Convolution

Frequency – Systemic

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Transform

An operation on distributions that changes convolution to multiplication

$T(f * g) = T(f)\,T(g)$

Drawback:

Actuaries don't like "magic formulas"

General Form

$$T(f(x)) = \int_a^b f(x)g(s)\,ds$$

Familiar Transforms

Transform	**Object Space**
	Limit points of sequences of smooth functions with...
MGF	... Bounded Support
Fourier Transform	... Bounded
LaPlace Transform	... Exponentially Bounded

LaPlace vs. Fourier Transforms

Theorem:

$\mathcal{L}(f(x)) = \mathcal{F}(e^{-\alpha x} f(x))$

The LaPlace Transform of a distribution equals the Fourier Transform of the "de-trended" distribution

Count – Time Duality

Duality:

An alternate formulation of a problem that preserves basic properties

Equivalence between a discrete distribution and a continuous distribution

Poisson – Exponential Duality

Poisson (Discrete)

Let N_t denote the number of occurrences in [0,t]

Exponential (Continuous)

The time between occurrences (T_1) is Exponential

Let T_n denote the time of the n^{th} occurrence

$P[N_t \geq n] = P[T_n \leq t]$

Geometric Distribution

The number of failures until the first success

The Geometric distribution can be used as a discrete approximation to the Exponential

For a short time interval t, define failure as no claims, The Geometric distribution will give the discrete time until a claim occurs.

P(no claims) = q = $1-e^{-\lambda t}$

Negative Binomial

Traditional Formulation

Poisson mixed by Gamma

Alternate Formulation

N-fold convolution of Geometric Distributions

LaPlace Transform

$\mathcal{L}(x^n/n!) = (1/s)^n$

The Laplace Transform maps from Poisson to Geometric

Collective Risk Model

Traditional Formulation

A combinatoric mix of convolutions

Alternative Formulation

A convolution of convolutions

Collective Risk Model

Aggregate Loss = $\sum_{n=0}^{\infty} P(n) f(x)^{*n}$

Aggregate Loss = Normal * $g(x)^{*m}$

Collective Risk Model

Aggregate Loss = $\sum_{n=0}^{\infty} P(n) f(x)^{*n}$

Aggregate Loss = Normal * $g(x)^{*m}$

Aggregate Loss = $P(N)\ n\ \mu_X * f(x-\mu_X)^{*\mu_N}$