Mack Model

Mack Model

Mack Model

- Mack developed a distribution-free method for computing the Variances of chain ladder age-to-age and age-to-ultimate factors
- The model is applied to weighted average development factors
- While no distribution assumptions are required to estimate factor Variances, they are needed to compute percentiles of ultimate loss and unpaid distributions

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Chain Ladder Assumptions

[1] E[c(w,d+1)|c(w,1),…,c(w,d)] = c(w,d)F(d)

[2] c(i,1), …, c(i,n)} & {c(j,1), …, c(j,n)} are independent for $i \neq j$

[3] Var[c(w,d+1)|c(w,1),…,c(w,d)] = $\alpha_d^2 \sigma(w,d)^2$

– OR –

Var[c(w,d+1)/c(w,d)|Data] = α_d^2/c(w,d)

for proportionality constants α

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Mack Model

Mack Mean

- Under these assumptions, the best estimate of the age-to-age factor is a weighted average

$$E[F(d)] = \sum_w \frac{c(w,d)}{\sum_w c(w,d)} \times \frac{c(w,d+1)}{c(w,d)} = \frac{\sum_w c(w,d+1)}{\sum_w c(w,d)}$$

- The Ultimate estimate is:

$$E[c(w,n)|D] = c(w,d) \times F(d) \times F(d+1) \times \ldots \times F(n-1)$$

where D is known data

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Mack Variance

- Since the mean is weighted, the variance is also weighted.
- Variance associated with one age-to-age factor or column of losses, σ_d^2:

$$\sigma_d^2 = \frac{1}{N-d-1} \sum_{j=1}^{N-d} c(j,d) \left(\frac{c(j,d+1)}{c(j,d)} - F(d) \right)^2$$

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Exercises using Mack Data

- Compute the weighted average age-to-average factors for each column
- Compute the weighted variances for age 1 for the factors in the exercise triangle
- Bonus: Calculate weighted averages and variances for age 1 in the complete Mack data triangle

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Mack Model

Answer (Variance of Column 1)

Computation of σ_1^2 for Development Age 1

Accident Year (w)	$c(w,1)$	$F(w,1)$	$F(w,1) - F(1)]$	Weighted Deviation
(1)	(2)	(3)	(4)	(5)
			$[(3) - 2.334]^2$	(2) x (4)
1981	5,012	1.650	0.47	2,345.0
1982	106	40.425	1,450.90	153,795.4
1983	3,410	2.637	0.09	313.3
1984	5,655	2.043	0.08	477.3
Weighted Average [*F(1)*]		2.334	Σ (5) =	156,930.9
			$\sigma_1^2 = \Sigma(5) / (N-d-1) =$	52,310.3

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Variance of Ultimates

- We want variance of future payments or future incurred loss changes
- MSE[c(w,n)] = E[{c(w,n) – E[c(w,n)]}^{2} |D] where D is data
- Iterative rule of expectations
- MSE[c(w,n)] = Var[c(w,n)|D]+ {E[c(w,n)|D] – E[c(w,n)]}^{2}
- Mean squared error = process variance of Ultimate + Parameter variance of estimate of ultimate
- Does not take into account changes in underlying model in the future.

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Variance of Ultimates

- Iterative computation to get variance of ultimate

$$Var[c(w,n)] = E[c(w,n-1)]\sigma_{n-1}^2 + E[c(w,n-1)]^2 F(n-1)^2 =$$
$$c(w,n-k+1)F(n-k+1)...F(n-2)\sigma_{n-1}^2 +$$
$$\{E[c(w,n-2)]^2 F(n-2)^2 F(n-1)^2 + E[c(w,n-2)]F(n-1)^2 \sigma_{n-2}^2\}$$

- Variance of unpaid = variance of ultimate

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Mack Model

Mack Variance of a Row (AY)

- The Mack formula for the variance of the reserve estimate for accident year *w* is:

$$Var[R(w,d)] = U(w)^2 \sum_{d=n+1-w}^{n-1} \frac{\sigma_d^2}{F(d)^2}\left(\frac{1}{E[c(w,d)]} + \frac{1}{\sum_{j=1}^{N-d} c(j,d)}\right)$$

Process Variance (variance of the column of observed development factors) → $\frac{1}{E[c(w,d)]}$

Parameter Variance (variance of the calculated weighted average development factor) → $\frac{1}{\sum_{j=1}^{N-d} c(j,d)}$

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Example

Computation of *Standard Error* of Ultimate for AY 1985

d	$F(d)$	$Sigma_d^2$	$\frac{Sigma_d^2}{F(d)^2}$	$E[c(1,d)]$	$SUM[c(j,d)]$	$\frac{1}{E[c(1,d)]} + \frac{1}{SUM}$	$x\ \frac{Sigma_d^2}{F(d)^2}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			(3) / (2)^2			1/(5) + 1/(6)	(4) x (7)
4	1.147	161.2	122.6	4,920	11,805	0.0002880	0.04
3	1.378	2,887.3	1,519.8	3,569	16,303	0.0003415	0.52
2	1.401	161.2	82.2	2,549	21,546	0.0004388	0.04
1	2.334	52,310.3	9,603.8	1,092	14,183	0.0009863	9.47
			$U(w)$ =	5,642		Sum =	10.06
						$Var[R(w,d)^2] = U(w)^2$ x Sum =	320,349,084
						SE =	17,898

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Exercise

- Compute the standard error for 1984
  - Compute σ_d^2 for ages 2 through 4 for the exercise data (use minimum of 2 & 3 for 4)
  - Assume age 5 is the ultimate valuation.
- Bonus: Compute the standard error for 1990 in the complete Mack data triangle

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Answer

Computation of *Standard Error* of Ultimate for AY 1984

d	$F(d)$	$Sigma_d^2$	$\frac{Sigma_d^2}{F(d)^2}$	$E[c(1,d)]$	$SUM[c(j,d)]$	$\frac{1}{E[c(1,d)]} + \frac{1}{SUM}$	$\frac{Sigma_d^2}{F(d)^2}$ x
(1)	(2)	(3)	(4) (3) / (2)^2	(5)	(6)	(7) 1/(5) + 1/(6)	(8) (4) x (7)
4	1.147	161.2	122.6	22,306	11,805	0.0001295	0.02
3	1.378	2,887.3	1,519.8	16,183	16,303	0.0001231	0.19
2	1.401	161.2	82.2	11,555	21,546	0.0001330	0.01
			$U(w)$ =	25,582		Sum =	0.21
						$Var[R(w,d)^2] = U(w)^2$ x Sum =	140,015,175
						SE =	11,833

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Mack Total Variance

- The total unpaid is the sum of the unpaid random variable for each accident year
- $R_{tot} = R_1 + R_2 + \ldots + R_N$
- The Variance of a sum equals the sum of the Variances plus twice the co-variances

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Mack Total Variance

- The Mack formula for the Variance of the total unpaid estimate is:

$$SE(R_{tot})^2 = \sum_{w=2}^{N} \left\{ SE(R_w)^2 + U(w) \underbrace{\left(\sum_{i=w+1}^{N} c(i,n) \right) \sum_{d=n+1-w}^{n-1} \left(\frac{2\sigma_d^2 / F(d)^2}{\sum_{j=1}^{N-d} c(j,d)} \right)}_{\text{Covariance term}} \right\}$$

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Mack Model

Calculation Pointers

- It is easiest to set up a set of triangles to perform the calculations
  - First create a row of column sums of cumulative losses x the last observation
  - Create a triangle of weighted squared deviations of development factors from their mean
  - Create a projected runoff triangle that computes each estimate of cumulative losses, $E[c(w,d)]$, for all future periods
  - Create a triangle of inverses of projected runoff plus inverse of sum of cumulative losses
  - A spreadsheet showing the calculation for the Mack data is provided

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Using Mack Parameters

- We have a mean and a variance for unpaid (or IBNR) amounts. Now what?
- To get confidence intervals or probability distribution, assumptions must be made
- Assume unpaid (or IBNR) amounts follow a probability distribution, say the Gamma
- Use mean and variance of unpaid (or IBNR) amounts to derive parameters for distribution
- Use this distribution to estimate percentiles and other statistics for unpaid (or IBNR) amounts

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Group Exercise

- Compute the variance of the total reserve amount using the Mack data
- Assume total reserve amount follows a lognormal (or Gamma) distribution and compute the parameters μ & σ. Compute the 75th percentile of the reserve (IBNR) amount.

Refer to Mack Model workbook for results

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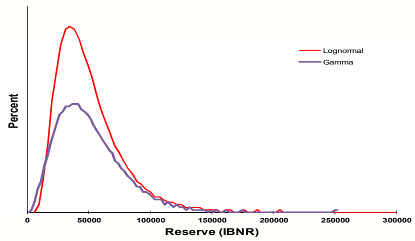
Results



Mean Reserve (IBNR)	52,135
Mean Squared Error	26,909
75th Percentile (logNormal)	64,299
75th Percentile (Gamma)	66,934

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Open issues

- Covariance term for the oldest year
  - Multiply ultimate for year by
    1. Sum of ultimates for all subsequent years
    2. Times the factor variance (σ_d^2) for last age-to-age factor
    3. Divide by square of last age-to-age factor
  - For Other years, need a sum of the ratio computed in 2 and 3
- Tail Factors? Recursion formula is useful
- Assumption testing

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Testing Assumption 1

[1] $E[c(w,d+1)|c(w,1),\ldots,c(w,d)] = c(w,d)F(d)$

- Graph two consecutive cumulative development periods.
  - Does it look like a linear relationship?
  - Does it look like the intercept is zero?
- Fit a regression of the form

  $c(w,d+1) = \alpha + \beta c(w,d)$
- Test α for statistical significance

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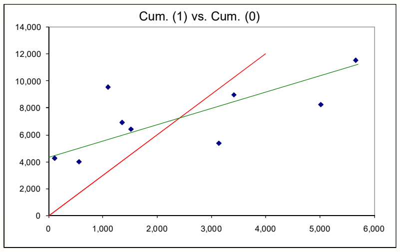
Mack Model

Testing Assumption 1



P-Value = 0.115 Int. P-Value = 0.626

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Testing Assumption 2

[2] {c(i,1), …, c(i,n)} & {c(j,1), …, c(j,n)} are independent for i≠j

- Mack suggests that the independence assumption can be tested by searching for calendar year effects
- His test looks for the probability you would observe "s" large or small development factors (relative to the median) along a diagonal out of j factors

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Testing Assumption 2

Development Factors

	1	2	3	4	5	6	7	8	9
1	1.650	1.319	1.082	1.147	1.195	1.113	1.033	1.003	1.009
2	40.425	1.259	1.977	1.292	1.132	0.993	1.043	1.033	
3	2.637	1.543	1.163	1.161	1.186	1.029	1.026		
4	2.043	1.364	1.349	1.102	1.113	1.038			
5	8.759	1.656	1.400	1.171	1.009				
6	4.260	1.816	1.105	1.226					
7	7.217	2.723	1.125						
8	5.142	1.887							
9	1.722								
Median	4.260	1.599	1.163	1.166	1.132	1.033	1.033	1.018	1.009

Above Median Below Median

Conclusion?

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Mack Model

Testing Assumption 3

[3] $\mathrm{Var}[c(w,d+1) | c(w,1),\ldots,c(w,d)] = \alpha_w^2\sigma^2$
for unknown proportionality constants α

- Graph weighted residuals v. cumulative losses
  $[c(w,d+1) - c(w,d)F(d)]/c(w,d)^{1/2}$ v. $c(w,d)$

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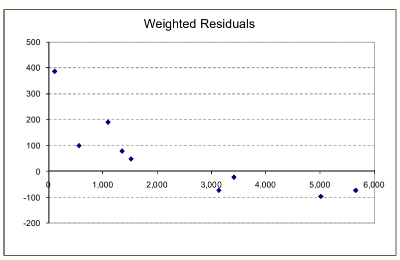
Testing Assumption 3



Conclusion?

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Some Tail Factor Methods

- Industry tail factor
- Judgmental selections
- Generalized Bondy
- Simple Inverse Power $F(d) = 1+(b/d)^a$
  – Can use regression to fit

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Mack Model

Mack: Tail Variances

- Look at pattern of standard deviations or variances from triangle
- Judgmentally select SD (or Var) for development ages with no data
- If you use industry factors to get tail, variance should be larger than indicated by triangle, because entity specific tail may be different from industry (an additional uncertainty)

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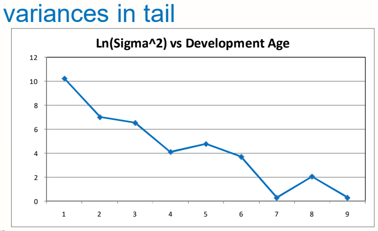
Tail Variances

- Plot ln(*Sigma*^{2}) vs. development age
- If relationship looks linear, fit regression to get variances in tail



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Exercises

- Use judgment to select tail variances for the Mack data.
- Fit a regression to ln(*Sigma*^{2}) vs development age for the Mack data and use it to estimate the ages 9 and 10 variances.
- What is the variance of the next (age 10-11) factor?

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