TOWERS PERRIN

CANE 2008 Meeting

What Color is Your Copula: The Language of Uncertainty

Terminology Surrounding Loss Reserve Variability/Ranges

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Why analyze the variability of claim liabilities?

ASOP 43 effective Sep. 1, 2007	■ "The actuary should consider the uncertainty associated with the unpaid claim estimate."
	  ■ Actuary not required to, nor prohibited from, measuring uncertainty
	■ "The actuary should consider the types and sources of uncertainty."
	  ■ "May include model risk, parameter risk, and process risk."

NAIC	■ Actuarial opinions are produced on a "reasonableness" standard ■ Variation from the "best estimate" is the issue ■ Actuarial Opinion Summary (AOS) includes focus on ranges
SEC	■ Require public companies to discuss reserve uncertainty in 10-K filings ■ Increasing pressure…hand-waving rationale will soon be inadequate
Rating Agencies	■ Capital adequacy analyses usually assume reserve shortfalls ■ Management is expected to consider more than just the best estimate
Solvency II	■ **Technical Provisions** – amounts set aside to fulfil obligations towards policyholders and other beneficiaries; includes a risk margin ■ **Solvency Capital Requirement (SCR)** –capital that enables absorption on significant unforeseen losses and gives reasonable assurance to policyholders (0.5% probability of ruin over a one year timeframe)

ASOP 43 includes various definitions of "estimate"

- ***Unpaid Claim Estimate*** – "The actuary's estimate of the obligation for future payment resulting from claims due to past events"
- ***Scope of the Unpaid Claim Estimate*** should identify its *intended measure*, examples of which include
  - Mean, median, mode, or specific percentile
  - High estimate, low estimate
  - **Actuarial Central Estimate** – "An estimate that represents an expected value of the range of reasonably possible outcomes."
    - — May not include all conceivable outcomes, e.g., "extreme events where the contribution of such events to an expected value is not reliably estimable."
    - — May or may not be the result of a probabilistic/statistical analysis
- ASOP 43 deems the terms *best estimate* and *actuarial estimate* as insufficient identifiers of the unpaid claim estimate's intended measure

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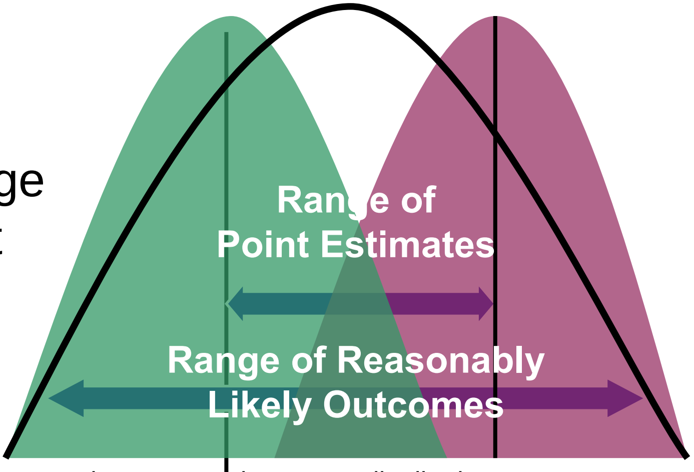
Booked reserves are estimates

- A probabilistic *point estimate* of the ultimate value of future claim obligations is a prediction of the mean (or another "central tendency": median, mode) of that random variable from a given algorithm
- The *prediction error* of that point estimate can arise from three basic sources
  - Your estimate will differ from the average value of all the potential estimates that your algorithm might produce (**Parameter Risk**)
  - The average value of all the potential estimates from your algorithm might not coincide with the "true mean" of the random variable being estimated (**Model Risk** or **Bias**)
  - Ultimate future obligations will undoubtedly differ from their own true mean (**Process risk**)



The green and magenta distributions represent different actuarial reserving methods

A “real life” analogy

- Let’s say we want to predict the age of the next person, Brian, to walk through the door
- If we knew the actual mean age of all CANE attendees, that would be our estimate
  - That estimate will be “off the mark” because Brian’s age will be different from the actual/true mean (**Process Risk**)
  - Nevertheless, the mean, if known, would be our best estimate
- Since we don’t know the true mean age, let’s ask everyone in the room what their age is and take the average
  - That estimate will also be “off the mark” because, in addition to the above, the average age in this room will be different from the true CANE mean (**Parameter Risk**)
  - Furthermore, asking people in this room to divulge their ages may give us a biased – downward(!) – result (**Model Risk**)

Several distinct types of risks are inherent in the estimation of claim liabilities

Total Risk

<u>Estimation error</u>
Difference between estimated mean and true mean

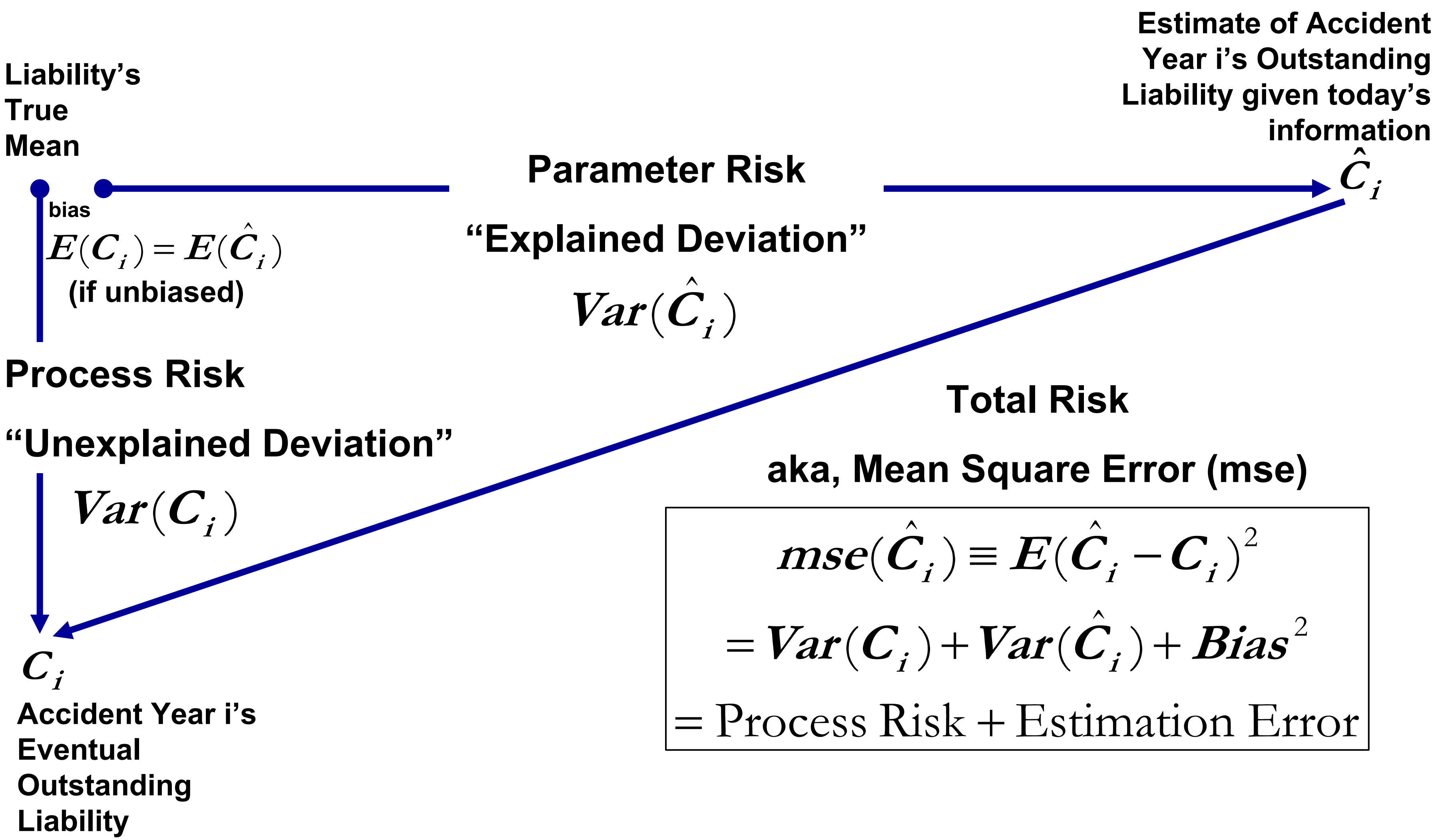
<u>Parameter Risk</u> Variability due to fact that our model's parameters are estimates	**<u>Model Risk</u>** Variability due to fact that our model imperfectly represents reality	**<u>Process Risk</u>** Difference between actual costs and true mean
Our room's average age could be different from the CANE mean	This room's population may not be representative of the entire meeting population (or entirely forthcoming!)	The age of the next entrant will differ from the true CANE mean due to random variation

Estimate of Expected Outcome

True Mean Outcome

Eventual Outcome

Total Risk, aka Mean Square Error, is the statistical equivalent of the Pythagorean Theorem



$$mse(\hat{C}_i) \equiv E(\hat{C}_i - C_i)^2$$

$$= Var(C_i) + Var(\hat{C}_i) + Bias^2$$

$$= \text{Process Risk} + \text{Estimation Error}$$

A Risk by any other name ...

- The word "risk" can be ambiguous and confusing; for example
  - "Variance" or "standard deviation"
  - Value at Risk (VaR) which is a quantile (e.g., the 99.5^{th} percentile)
  - Tail Value at Risk (TVar) which is the expected value of tail losses
- Coefficient of Variation, or CV, is a popular measure of relative risk
  - $CV(x) = \frac{StdDev(x)}{Mean(x)}$
    - — StdDev(Ultimate Loss) = StdDev(Outstanding + Paid)
      = StdDev(Outstanding)
      because paid loss is a scalar
    - — So the numerators of the CV(Ultimate Loss) and CV(Outstanding) are the same
  - Most of the popular stochastic methods estimate the risk of ultimate loss first, then back into the risk of outstanding loss
    - — Follow the deterministic method on which they are based

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- Popular Stochastic Methods

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Mack Method: Overview

- Mack Method derives formulas for the standard error of the liability projected by the chain ladder method
- Towers Perrin recommends using the recursive formulas from Murphy’s 1994 paper “Unbiased Loss Development Factors”
- The formulas provide for process and parameter risk, separately and in total
- The method can be extended to incorporate age-to-age factors other than the volume weighted average
- Mack recommends fitting a normal or lognormal distribution to the mean and variance (or CV) of the liability to yield a distribution of liabilities
- The variability of the tail beyond the triangle can be incorporated in various ways

But first, Mack's model and formula

- These three assumptions comprise "the model" which forms the basis of Mack's formulas (see his 1993 paper)

  (CL1) $E(C_{i,k+1} \mid \text{the triangle}) = C_{ik} f_k$

  (CL2) $\boldsymbol{Var}(\boldsymbol{C}_{i,k+1} \mid \text{the triangle}) = \boldsymbol{C}_{ik} \sigma_k^2$ for unknown parameters σ_k^2

  (CL3) accident years are independent

- From those assumptions, Mack derives that

$$m\hat{s}e(\hat{C}_{iI}) = \hat{C}_{iI}^2 \sum_{k=I+1-i}^{I-1} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2} \left(\frac{1}{\hat{C}_{ik}} + \frac{1}{\sum_{j=1}^{I-k} C_{jk}} \right)$$

where

$$\hat{\sigma}_k^2 = \frac{1}{I-k-1} \sum_{i=1}^{I-k} C_{ik} \left(\frac{C_{i,k+1}}{C_{ik}} - \hat{f}_k \right)^2$$

the "f-hats" are the weighted average link ratios, and the "C-hats" are the chain ladder estimates of future loss for accident yr i.

Mack Example

XYZ ABC Insurance Company
Paid Losses

AY/DY	1	2	3	4	5	6	7	8	9
1999	10,238	24,654	38,025	46,550	52,842	58,722	65,227	67,604	69,559
2000	5,508	16,235	25,586	32,863	38,111	42,315	45,171	47,666	49,045
2001	7,374	20,620	34,220	43,438	50,898	55,475	58,367	60,943	62,706
2002	6,153	19,182	31,005	40,424	46,949	50,942	54,931	57,354	59,014
2003	7,253	25,066	40,134	51,063	58,376	64,144	69,166	72,218	74,307
2004	10,855	38,520	62,348	82,710	95,382	104,806	113,011	117,998	121,411
2005	10,313	34,341	51,110	65,632	75,688	83,166	89,677	93,634	96,343
2006	16,411	42,228	66,770	85,743	98,879	108,649	117,155	122,324	125,863
2007	21,234	63,281	100,059	128,491	148,177	162,818	175,564	183,311	188,614
sum below diagonal	0	63,281	166,829	279,866	418,125	523,583	619,504	707,782	777,301
								Total Ult=	846,861
LDFs	2.980	1.581	1.284	1.153	1.099	1.078	1.044	1.029	1.000
CDFs	8.883	2.981	1.885	1.468	1.273	1.158	1.074	1.029	1.000

Mack Example

XYZ ABC Insurance Company
Paid Losses
Total Variance of Chain Ladder Projection, Each Accident Year

Accident Year	12	24	36	48	60	72	84	96	108
1999									
2000									137,728
2001								841,057	1,086,801
2002							3,540,395	4,635,215	5,086,951
2003						665,924	5,515,915	7,073,998	7,742,360
2004					1,125,201	2,609,513	12,209,351	15,466,710	16,923,627
2005				3,419,436	5,389,037	7,430,518	15,317,689	18,231,588	19,680,224
2006			4,769,700	12,580,548	17,909,193	22,936,094	36,320,849	41,873,885	44,914,250
2007		50,150,681	133,218,960	227,546,190	304,630,553	370,111,398	447,776,706	492,460,032	522,519,845
Sum		50,150,681	140,722,079	255,259,323	349,048,948	436,068,529	647,762,497	751,833,825	811,554,968

- Process variance and parameter variance are calculated separately and recursively using the Murphy formulas
- Total variance is the sum of the those two

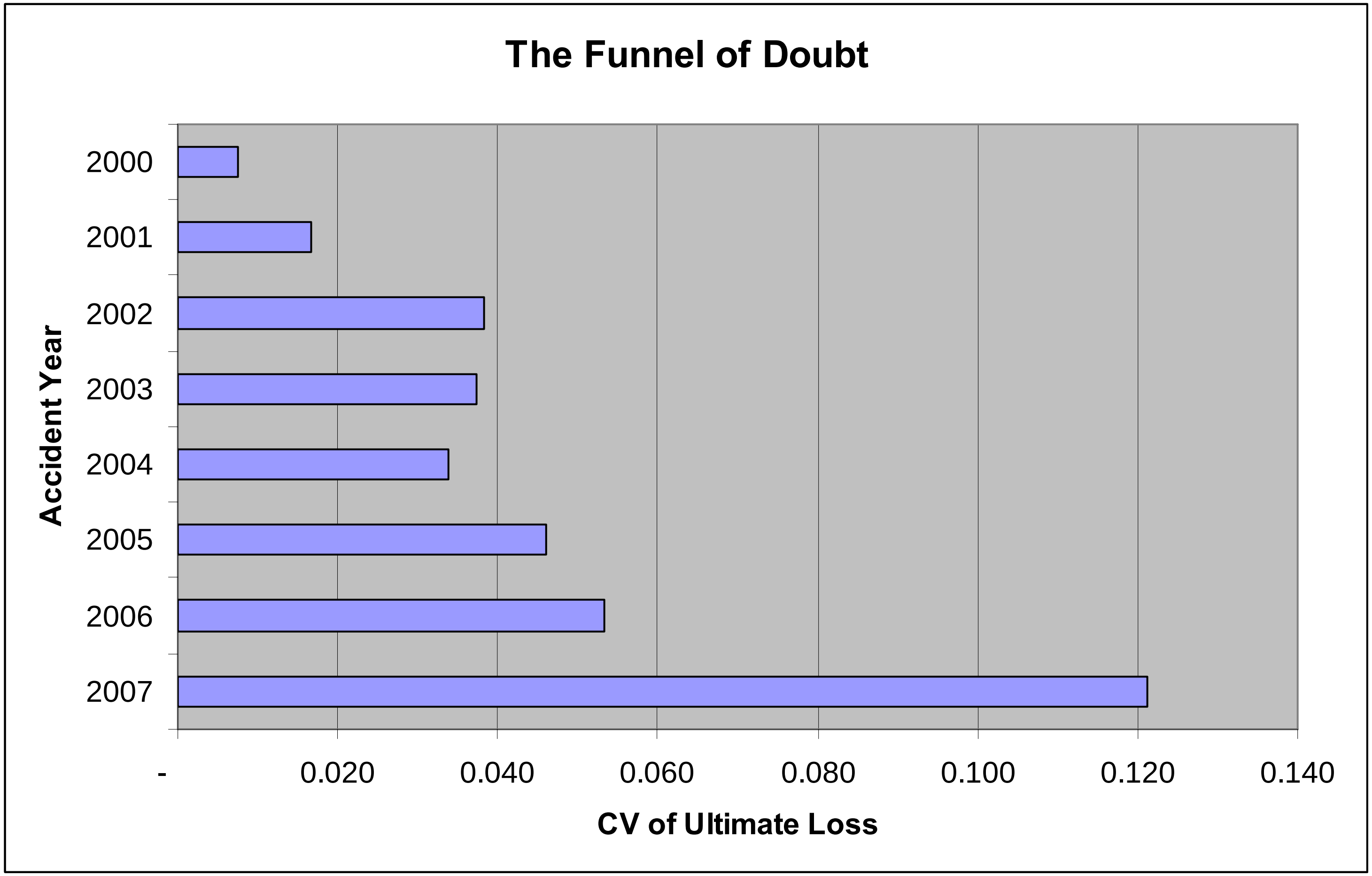
Mack Output

XYZ ABC Insurance Company
Paid Losses

	Ci,10	Ri	Mack Variance Formulation			CV of OS Loss			
AY	Ultimate	O/S	Process risk	Parameter risk	Total risk	Process	Parameter	Total	CV of Ultimate
1999	69,559	-	-	-	-				
2000	49,045	1,379	284	239	371	0.206	0.173	0.269	0.008
2001	62,706	4,338	828	633	1,042	0.191	0.146	0.240	0.017
2002	59,014	8,071	1,931	1,166	2,255	0.239	0.144	0.279	0.038
2003	74,307	15,931	2,319	1,538	2,783	0.146	0.097	0.175	0.037
2004	121,411	38,701	3,178	2,612	4,114	0.082	0.067	0.106	0.034
2005	96,343	45,233	3,748	2,373	4,436	0.083	0.052	0.098	0.046
2006	125,863	83,635	5,662	3,585	6,702	0.068	0.043	0.080	0.053
2007	188,614	167,380	19,858	11,321	22,859	0.119	0.068	0.137	0.121
Total:	846,861	364,669	21,458	18,738	28,488	0.059	0.051	0.078	0.034

- CVs of outstanding loss tend to “smile”
- CVs of ultimate loss tend to “blow up”

Mack Output



- The farther away an accident year is from ultimate resolution, the more uncertain the estimate of its ultimate value.

Mack VaR Estimates

XYZ ABC Insurance Company
Paid Losses

AY	Ci,10 Ultimate	Ri O/S	Mack Variance Formulation: Process risk	Parameter risk	Total risk
Total:	846,861	364,669	21,458	18,738	28,488

- Percentiles of the estimated outstanding liability can be estimated using an assumed probability distribution
  - 99.5%
    - Norminv(.995,364669,28488) = $438,048
    - Loginv(.995,12.804,0.078) = $444,462, ~1.5% higher
  - 99.95%
    - Norminv(.9995,364669,28488) = $438,048
    - Loginv(.9995,12.804,0.078) = $444,462, ~2.5% higher
- Mack Method has been criticized for understating tail risk (GIRO working party, July 2007)

Mack: Summary

Advantages	Disadvantages

■ Widely regarded in the industry
■ Founded in statistical theory
■ Works with chain-ladder eligible triangles
■ Can reflect tail variability
	■ Data outliers can have a leveraged effect on the results
	■ May over-parameterize the risk
	  ■ A 10x10 triangle will estimate 9 link ratios from 36 observations
	■ Tail risk may be understated, even when its assumptions are fully satisfied
	  ■ It is fundamentally a regression method

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Bootstrap Method: Overview

- Bootstrapping is a simulation technique that generates empirical probability distributions of complex functions
- It can be useful in situations in which the variability of an estimated parameter (e.g., ultimate loss) can be difficult to determine analytically
- It is based on the idea that a "might-have-been" historical dataset can be recast from the original dataset by sampling from the original dataset with replacement
  - The parameter of interest is estimated from each historical recast
  - The process is repeated many times to get a full distribution of the parameter
- Sometimes it does not make sense to resample from the original data
  - An alternative bootstrap approach is to fit a model to the data and resample from the residuals (difference between your model of the data an the actual data value)
  - The residuals are considered to hold all the random noise information
  - The historical dataset is recast by adding noises to the fitted values that are randomly sampled from the residuals

Bootstrap Method
Loss Development

XYZ ABC Insurance company
Paid Losses

Step 1: Observed cumulative historical data

Accident Year	12	24	36	48	60	72	84	96	108
1999	10,238	24,654	38,025	46,550	52,842	58,722	65,227	67,604	69,559
2000	5,508	16,235	25,586	32,863	38,111	42,315	45,171	47,666	
2001	7,374	20,620	34,220	43,438	50,898	55,475	58,367		
2002	6,153	19,182	31,005	40,424	46,949	50,942			
2003	7,253	25,066	40,134	51,063	58,376				
2004	10,855	38,520	62,348	82,710					
2005	10,313	34,341	51,110						
2006	16,411	42,228							
2007	21,234								
ATA factors	2.980	1.581	1.284	1.153	1.099	1.078	1.044	1.029	1.000
CDFs	8.883	2.981	1.885	1.468	1.273	1.158	1.074	1.029	1.000

Step 2: Recast cumulative values

Accident Year	12	24	36	48	60	72	84	96	108
1999	7,831	23,337	36,901	47,386	54,646	60,046	64,747	67,604	69,559
2000	5,521	16,455	26,018	33,411	38,530	42,337	45,651	47,666	
2001	7,059	21,038	33,265	42,717	49,262	54,130	58,367		
2002	6,644	19,799	31,306	40,202	46,362	50,942			
2003	8,365	24,930	39,420	50,621	58,376				
2004	13,668	40,734	64,408	82,710					
2005	10,846	32,324	51,110						
2006	14,170	42,228							
2007	21,234								

Bootstrap Method
Loss Development

XYZ ABC Insurance company
Paid Losses

Step 3: Incremental historical data

Accident Year	12	24	36	48	60	72	84	96	108
1999	10,238	14,416	13,371	8,525	6,292	5,880	6,505	2,377	1,956
2000	5,508	10,727	9,351	7,277	5,248	4,204	2,856	2,495	
2001	7,374	13,246	13,600	9,218	7,460	4,577	2,892		
2002	6,153	13,029	11,823	9,419	6,525	3,993			
2003	7,253	17,813	15,068	10,929	7,313				
2004	10,855	27,665	23,828	20,362					
2005	10,313	24,028	16,769						
2006	16,411	25,817							
2007	21,234								

Step 4: Incremental recast data

Accident Year	12	24	36	48	60	72	84	96	108
1999	7,831	15,507	13,563	10,485	7,260	5,399	4,701	2,857	1,956
2000	5,521	10,933	9,563	7,393	5,119	3,807	3,314	2,014	
2001	7,059	13,979	12,227	9,452	6,545	4,867	4,238		
2002	6,644	13,156	11,507	8,896	6,159	4,581			
2003	8,365	16,565	14,489	11,201	7,755				
2004	13,668	27,066	23,674	18,302					
2005	10,846	21,477	18,786						
2006	14,170	28,058							
2007	21,234								

Bootstraping Loss Triangles

- A triangle of cumulative fitted values for the past triangle is obtained by backwards recursion on the most recent diagonal using chain ladder link ratios
- A set of Pearson residuals is calculated from the fitted and actual data
  - The Pearson residuals attempt to normalize residuals across the columns
- Each simulated sampling scenario produces a new "realization" of triangular data that has the same statistical characteristics as the actual data
- Since each realization yields new ultimates based on new estimates of the LDF parameters, the Bootstrap Method without additional enhancements calculates only parameter risk
  - Our model calculates both parameter and total risk
- Options
  - Our bootstrapping implementation can calculate tail volatility by employing curve fitting to each realization of average loss development factors
  - Bornhuetter-Ferguson option
  - Outlier observations can be restricted
  - The sampling of residuals can be restricted for the first development period

Bootstrapping: Summary

Advantages	Disadvantages
■ Easy to understand and explain ■ Commonly used in industry ■ Accommodates BF method ■ Facilitates the calculation of tail volatility	■ Data outliers can have a leveraged effect on the results ■ Method does not work well with negative loss development (due to underlying theoretical model) ■ Heteroskedasticity can yield wild results ■ Bootstrap can also understate tail risk, even when its assumptions are satisfied

Three stochastic methods in popular use today

Mack Method

Bootstrapping

Practical Method

Practical Method: Overview

- The Practical Method – first published in 2002 by a working party of the UK Institute of Actuaries – takes a Monte Carlo approach to stochastic reserving
- The Practical Method uses Monte Carlo simulation to estimate liability distributions based on the three most popular deterministic methods – Chain Ladder, Loss Ratio, and Bornhuetter-Ferguson
- This method simulates age-to-age (ATA) factors and loss ratios as normal or lognormal random variables
  - Means and variances of those distributions are selected inputs
  - For BF method, LDFs can be "fixed" based on the ATA means, or "variable" based on the ATA simulations
- Explicitly reflects process risk only
  - Parameter can be incorporated with some additional analysis

Practical Method
Loss Development

XYZ ABC Insurance company
Paid Losses

Complete the triangle using random draws from parameters' assumed distributions

Accident Year	12	24	36	48	60	72	84	96	108	Ult	O/S
1999	10,238	24,654	38,025	46,550	52,842	58,722	65,227	67,604	69,559	69,559	0
2000	5,508	16,235	25,586	32,863	38,111	42,315	45,171	47,666	48,707	48,707	1,042
2001	7,374	20,620	34,220	43,438	50,898	55,475	58,367	60,744	61,954	61,954	3,586
2002	6,153	19,182	31,005	40,424	46,949	50,942	55,184	57,698	59,862	59,862	8,920
2003	7,253	25,066	40,134	51,063	58,376	63,269	66,890	68,761	70,478	70,478	12,102
2004	10,855	38,520	62,348	82,710	93,915	100,708	113,350	119,007	122,500	122,500	39,790
2005	10,313	34,341	51,110	67,588	77,486	83,657	96,560	100,820	102,121	102,121	51,011
2006	16,411	42,228	67,637	89,391	102,672	110,104	114,974	120,212	126,063	126,063	83,835
2007	21,234	66,040	104,952	*133,027*	151,933	167,329	177,918	188,734	194,857	194,857	173,623
ATA factors											
Mean	2.980	1.581	1.284	1.153	1.099	1.078	1.044	1.029	1.000		373,908
Std Dev	0.471	0.061	0.040	0.016	0.014	0.032	0.013	0.009			

1.26750 =norminv(rand(),mean,stdev)

Practical: Summary

Advantages	Disadvantages
■ Easy to understand and explain ■ Accommodates the three most popular actuarial deterministic methods ■ Can incorporate tail variability	■ Not as well known in the actuarial community ■ Does not explicitly measure parameter risk

The flexibility of a simulation model can be both an advantage and a disadvantage

Three stochastic methods in popular use today

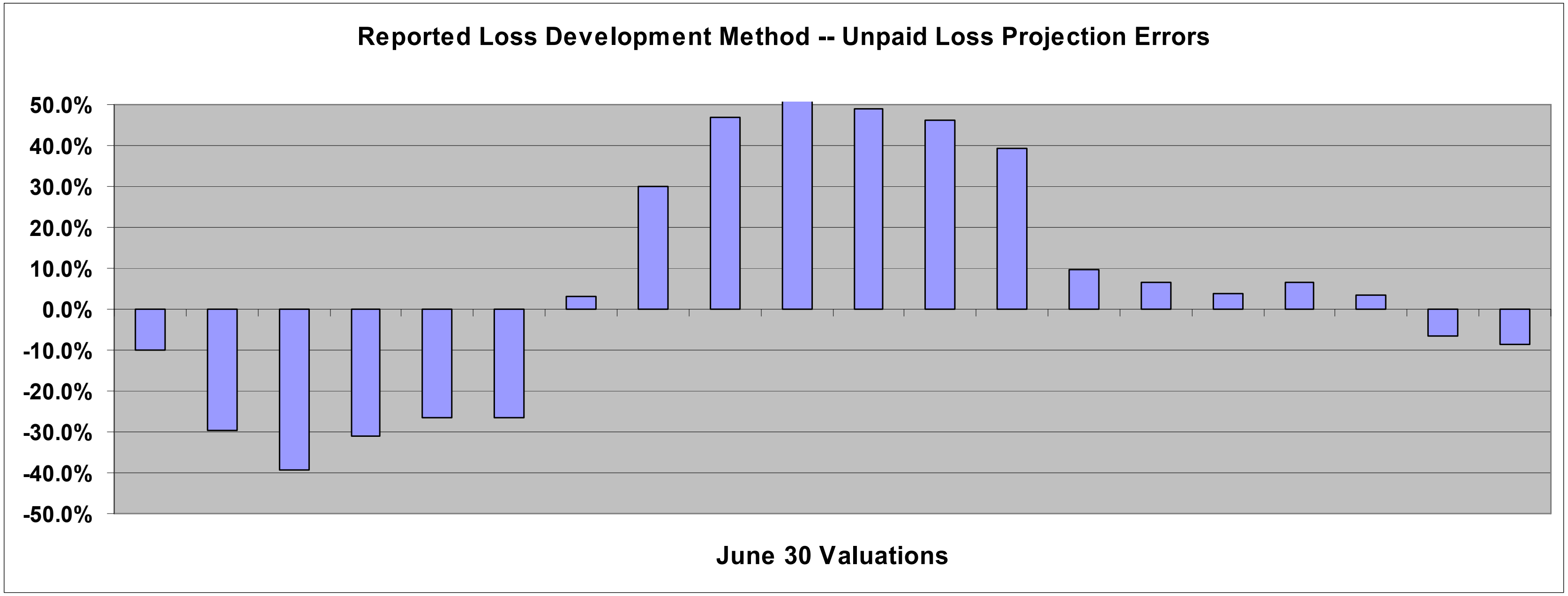
Mack Method

Hindsight Method

Bootstrapping

Practical Method

Hindsight Method: Overview



- Consists of testing the performance of past estimates of ultimate losses by comparing them to actual emergence with the benefit of hindsight
- Uses actuarial central estimates from actual past reserve reviews; for older periods it is usually necessary to imitate current reserving methods to obtain past best estimates
- Method is non-parametric; captures all sources of risk

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- Tail Volatility

Tail Volatility

- Many of the popular stochastic methods only measure risk to the edge of the triangle
- Variability for development beyond the triangle – so called "tail volatility" – must be measured and incorporated separately
- Mack
  - Heuristic approach to tail variability in his 1999 paper
- Bootstrap
  - England and Verrall (1998) only measure risk to the edge of the triangle
- Practical
  - Assume you incorporate a tail in your deterministic analysis
  - For a stochastic simulation you will need to have some idea of the variability of that tail factor

Example: Tail variability can be reflected with the Mack Method using the heuristic in his 1999 paper …

XYZ ABC Insurance Company
Paid Losses

AY/DY	1	2	3	4	5	6	7	8	9	Ult
1999	10,238	24,654	38,025	46,550	52,842	58,722	65,227	67,604	69,559	73,037
2000	5,508	16,235	25,586	32,863	38,111	42,315	45,171	47,666	49,045	51,497
2001	7,374	20,620	34,220	43,438	50,898	55,475	58,367	60,943	62,706	65,841
2002	6,153	19,182	31,005	40,424	46,949	50,942	54,931	57,354	59,014	61,964
2003	7,253	25,066	40,134	51,063	58,376	64,144	69,166	72,218	74,307	78,022
2004	10,855	38,520	62,348	82,710	95,382	104,806	113,011	117,998	121,411	127,482
2005	10,313	34,341	51,110	65,632	75,688	83,166	89,677	93,634	96,343	101,160
2006	16,411	42,228	66,770	85,743	98,879	108,649	117,155	122,324	125,863	132,156
2007	21,234	63,281	100,059	128,491	148,177	162,818	175,564	183,311	188,614	198,045
LDFs	2.980	1.581	1.284	1.153	1.099	**1.078**	**1.044**	1.029		
CDFs	9.327	3.130	1.979	1.541	1.337	1.216	1.128	1.080	**1.050**	
σ_k^2	1,835.79	91	54.80	9.82	8.71	**52.43**	**9.43**	1.69	**16.823**	
σ_β^2	0.02477	0.00051	0.00024	0.00005	0.00005	**0.00034**	**0.00009**	0.00003	**0.00013**	

Practical Method
Loss Development with Tail

XYZ ABC Insurance company
Paid Losses

Complete the triangle using random draws from parameters' assumed distributions

Accident Year	12	24	36	48	60	72	84	96	108	Ult	O/S
1999	10,238	24,654	38,025	46,550	52,842	58,722	65,227	67,604	69,559	74,329	4,770
2000	5,508	16,235	25,586	32,863	38,111	42,315	45,171	47,666	48,456	50,572	2,906
2001	7,374	20,620	34,220	43,438	50,898	55,475	58,367	60,917	62,749	67,388	9,021
2002	6,153	19,182	31,005	40,424	46,949	50,942	53,745	57,740	60,126	62,296	11,353
2003	7,253	25,066	40,134	51,063	58,376	64,868	68,005	70,317	71,718	75,171	16,795
2004	10,855	38,520	62,348	82,710	93,351	103,425	110,016	113,893	118,202	123,615	40,905
2005	10,313	34,341	51,110	64,884	76,010	83,628	89,582	93,485	95,820	100,663	49,553
2006	16,411	42,228	66,138	86,723	100,295	110,679	112,473	116,807	120,615	124,426	82,198
2007	21,234	65,143	101,274	*134,630*	156,817	172,525	184,264	194,022	198,081	205,524	184,290
ATA factors											
Mean	2.980	1.581	1.284	1.153	1.099	1.078	1.044	1.029	**1.050**		401,792
Std Dev	0.471	0.061	0.040	0.016	0.014	0.032	0.013	0.009	**0.016**		

1.32936 =norminv(rand(),mean,stdev)

- The only thing new is that the tail factor is now simulated too

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- Aggregation

Aggregation: combining lines

- Means aggregate without much fuss:
  - E(X+Y) = E(X) + E(Y)
  - I.e., to get the aggregate mean, just aggregate the marginals
- Variances aggregate without much fuss when the lines are independent (more precisely, uncorrelated)
  - Var(X+Y) = Var(X) + Var(Y)
  - I.e., to get aggregate variance, just aggregate the marginals, but only when the lines are uncorrelated
- When the lines are correlated, there is an extra covariance term
  - Var(X+Y) = Var(X) + 2Cov(X,Y) + Var(Y) (1)
    - Covariance is to the formula for the variance of the sum of two random variables as the cross product term is to the square of a binomial
- Entire distributions aggregate without much fuss when the random variable pairs are joint normally distributed
  - Otherwise, more advanced techniques are required

Aggregation continued: correlation

- Correlation scales the covariance of two lines by dividing by their standard deviations

$$\text{Correlation Coefficient}: \quad \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

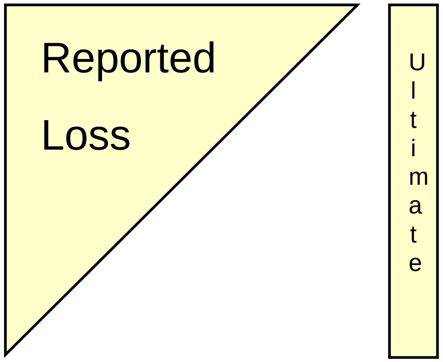
  - Correlation is “standardized” covariance
  - Allows comparison of two lines of difference sizes

- Such relationships between N lines of business are encapsulated in the covariance matrix and the correlation matrix

$$\Sigma = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_N) \\ Cov(X_2,X_1) & Var(X_2) & \cdots & Cov(X_2,X_N) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(X_N,X_1) & Cov(X_N,X_2) & \cdots & Var(X_N) \end{bmatrix} \quad Corr = \begin{bmatrix} 1 & corr(X_1,X_2) & \cdots & corr(X_1,X_N) \\ corr(X_2,X_1) & 1 & \cdots & corr(X_2,X_N) \\ \vdots & \vdots & \ddots & \vdots \\ corr(X_N,X_1) & corr(X_N,X_2) & \cdots & 1 \end{bmatrix}$$

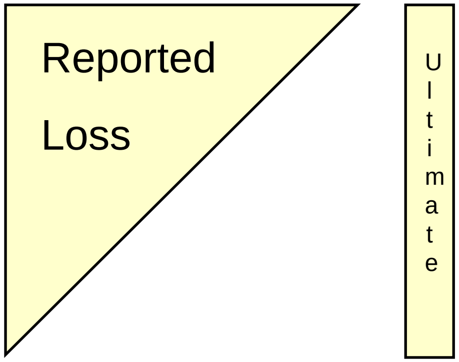
  - If you don’t make a mistake in building these matrices, they are always positive-semidefinite (you can take their “square root,” as standard deviation is the square root of variance)
  - Can be inverted only if positive-definite (cannot be “zero”)

Correlation between two quantities measures the degree to which deviations from the mean move – or don't move – in conjunction with each other



Auto BI

$$C_{ABI}(i,Ult)-\overline{C}_{ABI}(i,Ult)$$



GL

$$C_{GL}(i,Ult)-\overline{C}_{GL}(i,Ult)$$

- Given pairwise estimates of ultimates from two lines for I accident years, the strength to which the estimated ultimates "co-vary" can be measured by the sample correlation coefficient

$$\frac{1}{I}\sum_{i=1}^{I}\left(C_{ABI}(i,Ult)-\overline{C}_{ABI}(i,Ult)\right)\left(C_{GL}(i,Ult)-\overline{C}_{GL}(i,Ult)\right)\Big/ s_{ABI}s_{GL}$$

Simple Example

	Outstanding Loss			
	Mean	**CV**	**StDev**	**Var**
Line A	364,669	0.078	28,488	811,554,968
Line B	182,334	0.078	14,244	202,888,742
Correlation	20%		Cov=	81,155,497
A+B	547,003	0.063	34,304	1,176,754,703

- Line A is XYZ ABC Insurance Company Paid Loss
  - CV from Mack calculation without tail
- Line B is a similar LOB in another state
  - CVs are similar
  - Correlation was previously calculated on the side
- Covariance of the sum uses formula (1)
  - Then the standard deviation and CV are calculated
- The smaller CV of the sum demonstrates the "diversification benefit"
  - Minimum CV = 0.058 when correlation = 0

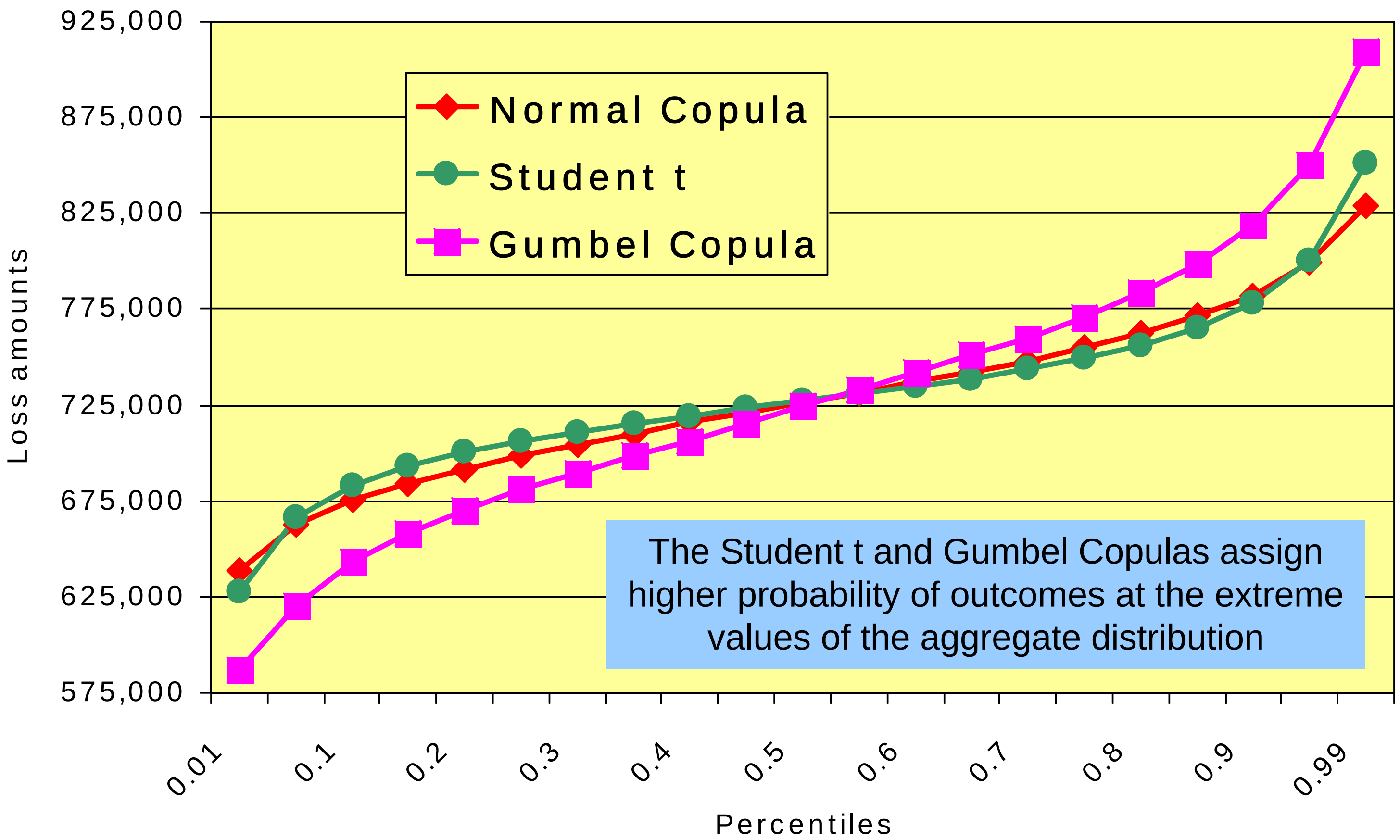
Aggregation continued: combining entire distributions

- Correlation measures the <u>average</u> strength of the relationship between lines over the entire distribution
- When is the correlation coefficient not enough?
  - When the strength of the relationship between two lines changes in different parts of their distributions
  - Example: Correlation between property lines might be higher in the tails of their distributions, which could be important to an actuary parameterizing a CAT contract
- Ideally, a company writing N lines of business one would like the complete joint distribution of all N lines
- It turns out that every joint distribution of N lines of business can be decomposed into N marginal distributions by virtue of an amalgamating function called a "copula"
- Vice versa, given the marginal distributions of N lines of business, the joint distribution can be calculated with the help of an appropriate copula

Copulas provide a convenient way to express the aggregate distribution of several lines

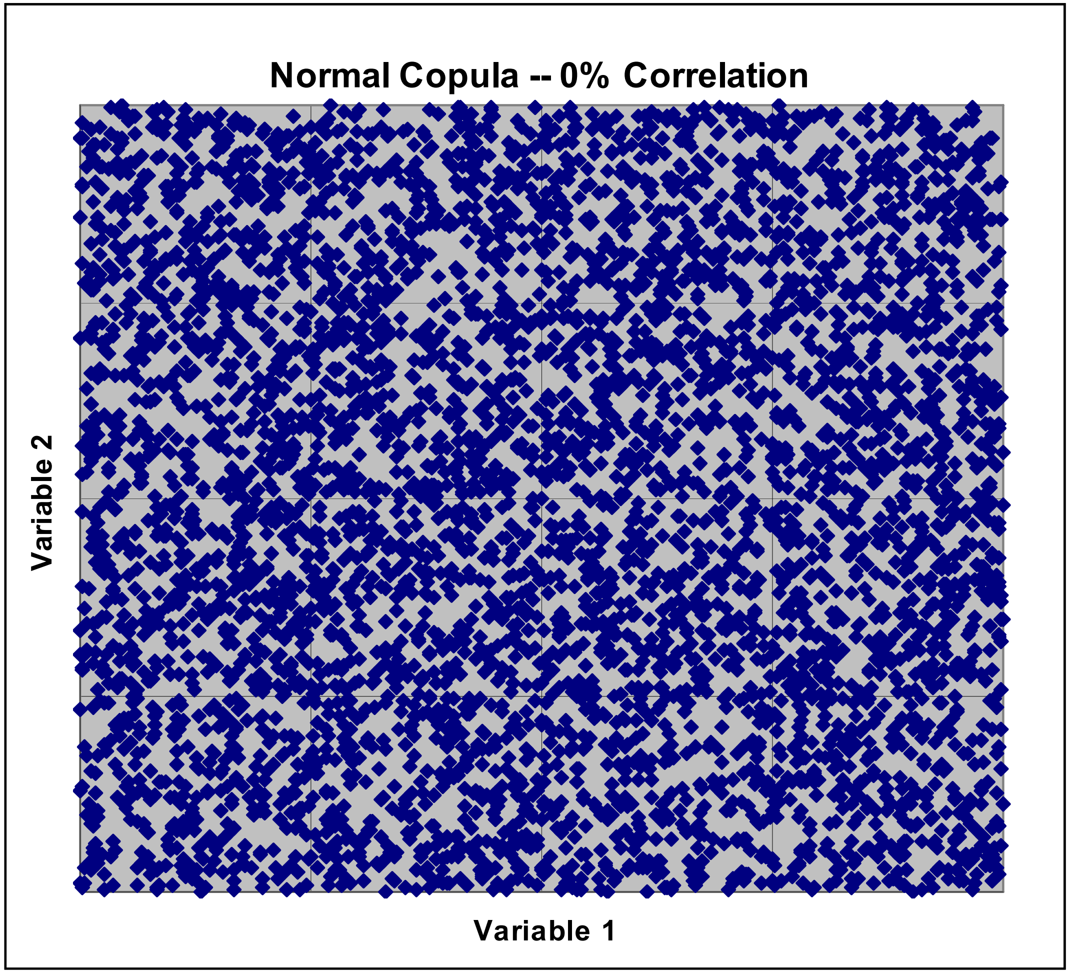
- Three popular copulas in actuarial use today are
  - The Normal copula
  - The Student-t copula
  - The Gumbel copula
- Copula required components (with the exception of Gumbel):
  - The marginal distributions of the individual lines
  - Correlations among these lines
- The Gumbel copula is different from the Normal and Student-t
  - It does not need a complete correlation matrix
  - Association is expressed by a single parameter applying to all lines
  - Upper tail dependence is strong while lower tail dependence always equals 0

The choice of the appropriate copula is a matter of judgment

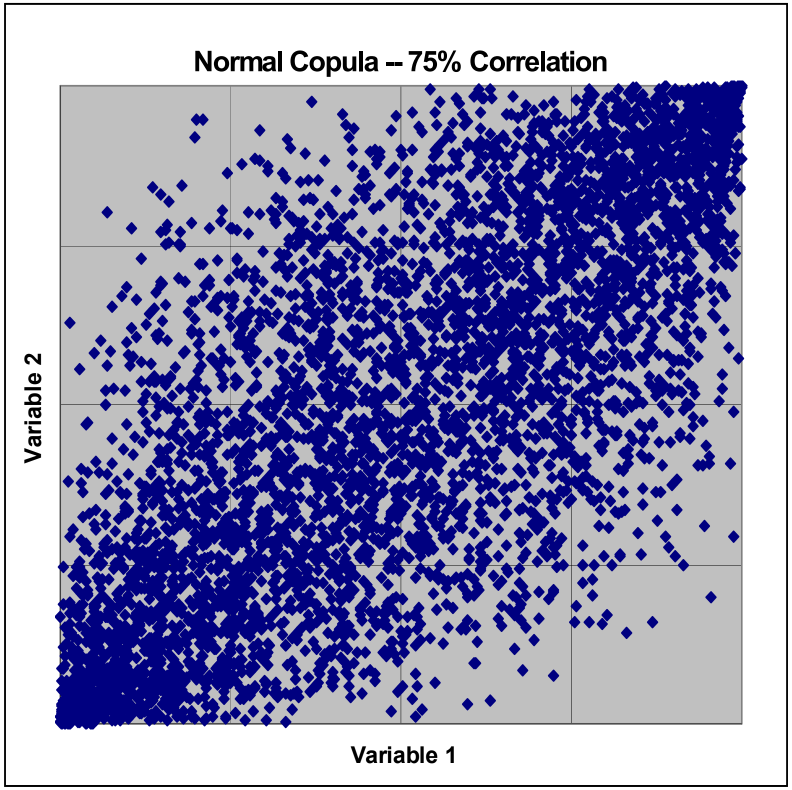


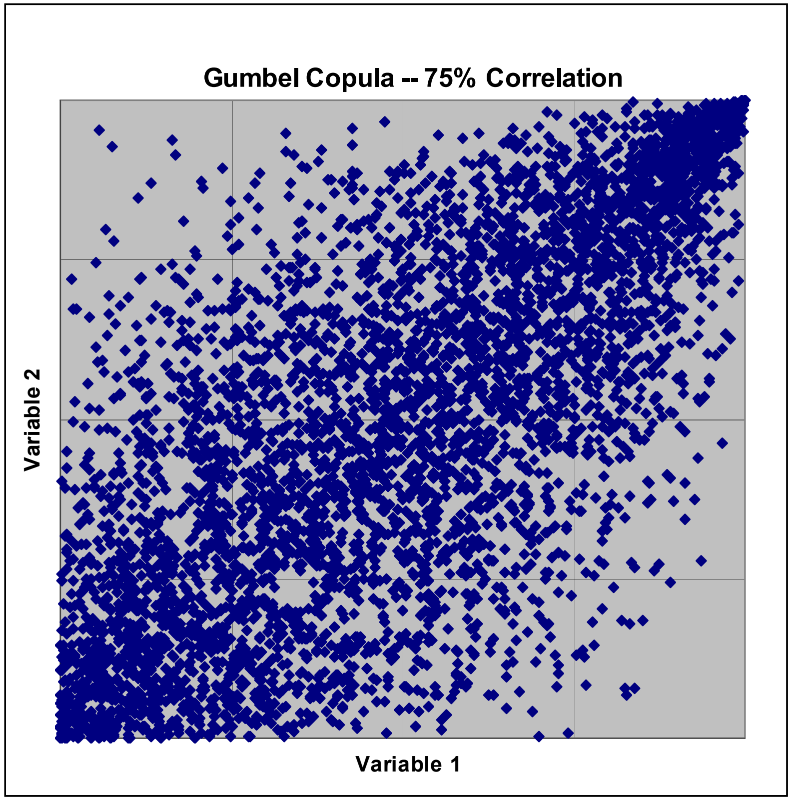
- The portfolio of liabilities can be stress-tested under varying copula assumptions

With independent variables results are not correlated

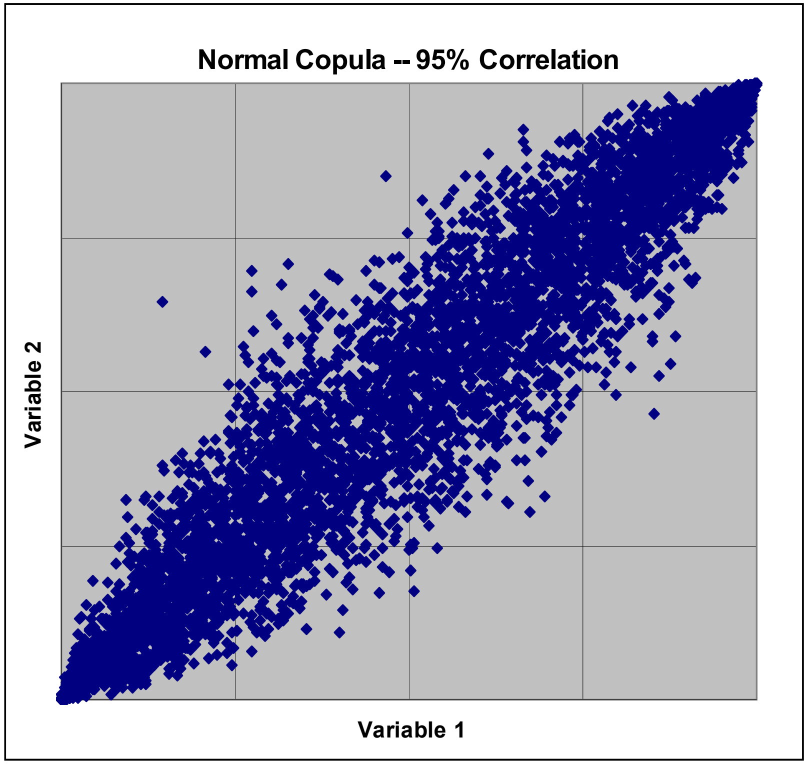


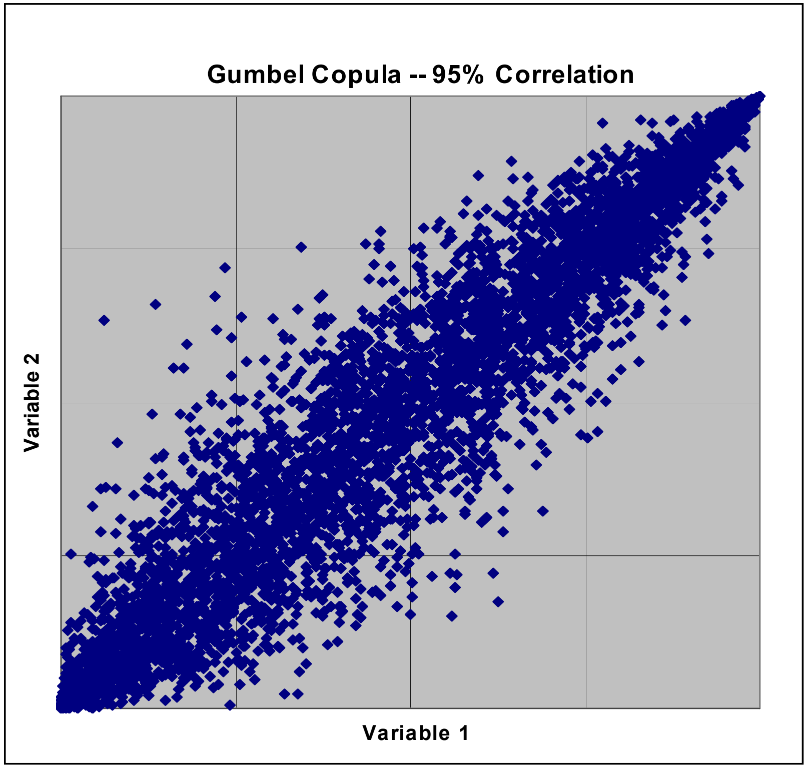
75% correlation: bad results in one line make it more likely to have bad results in the second line





The relationship is even more pronounced with 95% correlation





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- Risk Horizon: Runoff versus One Year

Runoff Risk vs. One-Year Risk

- Suppose you use the paid loss development method as your deterministic method for estimating ultimate loss
- The result of the chain ladder calculations is a number, based on many other numbers that are fixed, historical values.
- Isn't that number a fixed scalar value?
- In what sense does that number have "variance"?
- Answer: the variance of the estimate is the variance of the difference between the true ultimate value and the estimate today
- The risk is that the estimate can change – up or down – as we runoff the claims to ultimate
- Similarly, the **one-year risk** is that the estimate can change between now and 12 months from now

Converting the established stochastic methods from runoff to one-year horizons

- Define the **change in the estimate** as

$$\Delta \hat{C}_{iJ}^{(1)} \equiv \hat{C}_{iJ}^{(1)} - \hat{C}_{iJ}^{(0)}$$

- Define the **change in the noise** as

$$\Delta \varepsilon_{iJ}^{(1)} \equiv \varepsilon_{iJ}^{(1)} - \varepsilon_{iJ}^{(0)}$$

- Define the **calendar year mean square error** (CYMSE) as the sum of two components: the variance arising from the change in noise plus the variance arising from the change in the estimate

$$CYMSE_{ij}^{(1)} = Var(\Delta \varepsilon_{ij}^{(1)}) + Var(\Delta \hat{C}_{ij}^{(1)})$$

- The paper *Uncertainty of the Claims Development Result in the Chain Ladder Method* (2007) by Wuthrich, Merz, and Lysenko analyzes the MSE of this change based on Mack's approach, but with a slightly stronger variance assumption

Questions?

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