Measuring risk

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Risk

- A simplified conceptual framework:

  risk = uncertainty = a random variable X

  - we picture X as the grand total of all cash flows, positive and negative,
  - we disregard the temporal component - all uncertainty is resolved at a unique fixed future time T,
  - our models are perfect.

- We also assume that
  - we have a pretty good idea about what we like - our preferences.
  - If given a choice between any two risks (gambles, options, etc.) we are able to choose the one we like better: $X \preceq Y$.

- The problem of risk measurement is the following:

  explain the process of choosing between X and Y.

Risk II

- operationally, we would like to come up with a procedure which
  - takes a random variable X as input, and
  - and assigns to it a number $\phi(X)$ so that the comparison between X and Y is reduced to a comparison between $\phi(X)$ and $\phi(Y)$, i.e.,

$$X \preceq Y \text{ if (and only if) } \phi(X) \leq \phi(Y).$$

- Typically, one compares X to 0 and calls X **acceptable** if $\phi(X) \geq 0$:

Should I take the risk in X or do nothing?

- A naïve example:

$$\phi(X) = \mathbb{E}[X].$$

3 gambles

- Consider the following offer. Would you take it?

$$\begin{cases} \text{You get } \$\,5 & \text{with probability 0.5, and} \\ \text{You lose } ¢\,50 & \text{with probability 0.5.} \end{cases}$$

- How about this one?

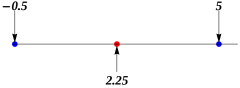
$$\begin{cases} \text{You get } \$\,50 & \text{with probability 0.5, and} \\ \text{You lose } \$\,5 & \text{with probability 0.5.} \end{cases}$$

- Would you take this one?

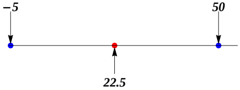
$$\begin{cases} \text{You get } \$\,500,000 & \text{with probability 0.5, and} \\ \text{You lose } \$\,50,000 & \text{with probability 0.5.} \end{cases}$$

Risk aversion

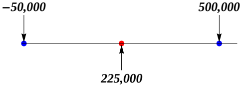
- From the point of naïve risk measurement, the offers get more and more attractive:



  - ¢5 vs. $5
    $\phi(X) = \mathbb{E}[X] = 0.5 \times \$5 + 0.5 \times (-\$0.5) = \$2.25$



  - $5 vs. $50
    $\phi(X) = \mathbb{E}[X] = 0.5 \times \$50 + 0.5 \times (-\$5) = \$22.5$



  - $\$50,000$ vs. $\$500,000$
    $\phi(X) = \mathbb{E}[X] =$
    $0.5 \times \$500,000 + 0.5 \times (-\$50,000) = \$225,000$

and yet, we are less and less inclined to take them.

Risk aversion II

- Humans are “risk averse”: we prefer certainty to uncertainty, we prefer less risk to more risk and losses hurt more than gains of the same magnitude make us happy.
- From the evolutionary point of view, risk aversion is hard-wired into our brains.

Bernoulli's paradox

- In 1738 Daniel Bernoulli proposed the following problem:

  How much would you pay for the right to play the following game?
  You toss a fair coin until you get heads.

  - *If heads show on the first toss, you get a dollar.*
  - *If the pattern is tails and then heads, you get two dollars.*
  - *Tails, tails and then heads get you four dollars.*
  - *. . .*
  - *In general, if it takes n tails to get the first head, you get 2^n dollars.*

  (English translation of the original article: Bernoulli, Daniel: 1738, "Exposition of a New Theory on the Measurement of Risk", Econometrica 22 (1954), 23-36)

- Hacking says: *"few of us would pay even* \$25 *to enter such a game."*

  Hacking, Ian: 1980, "Strange Expectations", Philosophy of Science 47, 562-567.

Bernoulli II

- The expected payoff B of one round of Bernoulli's game is

$$\begin{aligned}\mathbb{E}[\text{payoff}] &= 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{2^3} + \cdots + 2^n \times \frac{1}{2^{n+1}} \\ &= \infty,\end{aligned}$$

- Therefore, if you choose to pay $\$x$ for the right to play, the risk you face is $X = B - x$ and $\phi(X) = \mathbb{E}[B] - x = +\infty$.
(You got yourself a great deal, no matter what price you pay.)
- Bernoulli himself argues that the payoff should be computed in terms of satisfaction (utility) and not in monetary terms.
- If we assume that the utility value of $\$x$ is $\log_2(x)$, the expected utility gained is finite:

$$\mathbb{E}[\log(B)] = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{2^3} + \cdots + n \times \frac{1}{2^{n+1}} + \cdots = 2$$

Bernoulli III

- In other words, Bernoulli suggests $\phi(X) = \mathbb{E}[\log(X)]$ as the measure of risk. In this case

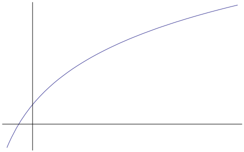
$$\phi(B) = 2 = \log_2(4) = \phi(4),$$

so that the certain pay-off of \$4 dollars is equivalent to a run of Bernoulli's game.

- Believe it or not, the idea of using anything but $\phi(X) = \mathbb{E}[X]$ was considered highly controversial in Bernoulli's time. In fact, it took 200 years for von Neumann and Morgenstern (1944) to pick up where Bernoulli left off.

Utility functions

- A utility function U is
  - increasing
  - concave
  - has diminishing marginal utility, i.e. $U'(x) \to 0$, as $x \to \infty$.



- a **utility-based** risk measure is any ϕ of the form

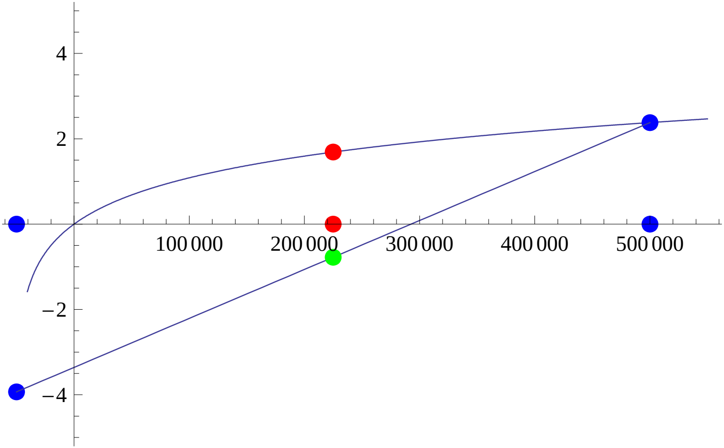
$$\phi(X) = \mathbb{E}[U(X)],$$

  where U is a utility function.
- A related measure is the so-called **mean-variance** measure

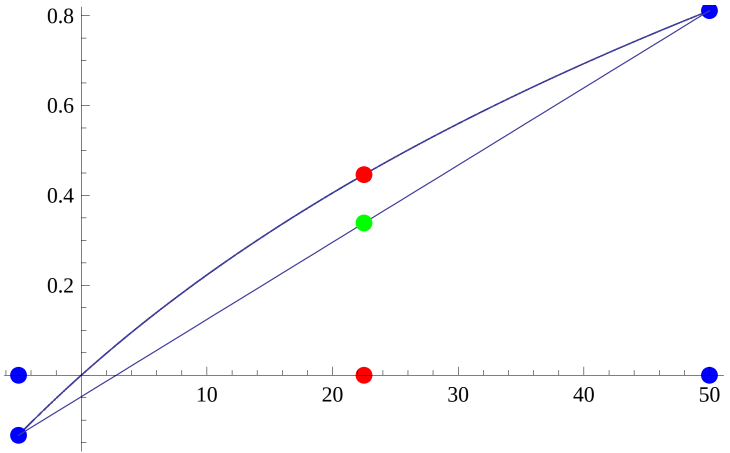
$$\phi(X) = \mathbb{E}[X] - \gamma\sqrt{\mathrm{Var}[X]}.$$

3 gambles via utility

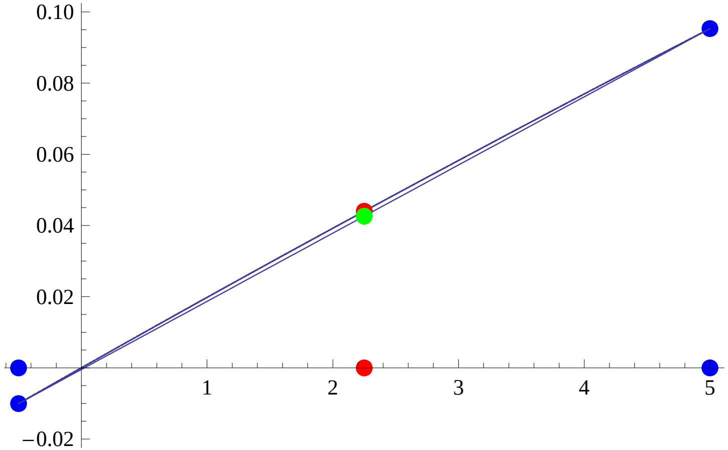
Let us applly a utility risk measurement on the 3 gambles:



3 gambles via utility II



3 gambles via utility III



Value at Risk

A popular measure of risk is the **value at risk.**

- Pick a **confidence level** $p \in (0,1)$ and define

$$\mathrm{V@R}_p = \min\{l \in \mathbb{R} : \mathbb{P}[X < l] \geq p\}$$

- If you know that $\mathrm{V@R}_{0.05}(X) = -\$50,000$, you are 95% sure that you will not bear a loss of more than $\$50,000$.
- Typically, you would combine $\mathrm{V@R}$ with $\mathbb{E}[X]$ to get a more useful risk measure.

A critique of $\mathrm{V@R}$

- Insensitivity to extreme events:

$$X_1 = \begin{cases} -\$10,000, & \text{with probability } 5\% \\ +\$10,000, & \text{with probability } 99\% \end{cases}$$

$$X_2 = \begin{cases} -\$10,000,000 & \text{with probability } 1\% \\ -\$10,000 & \text{with probability } 4\% \\ +\$10,000, & \text{with probability } 99\% \end{cases}$$

have the same $\mathrm{V@R}$ at $p = 0.05$.

A critique of $\mathrm{V@R}$ II

- Lack of convexity: there are risks X_1 and X_2 such that

$$\mathrm{V@R}(X_1) > 0, \mathrm{V@R}(X_2) > 0 \text{ but } \mathrm{V@R}(X_1 + X_2) < 0.$$

  Diversification *increases* risk.
- Statistical problems - very hard to estimate without abundant data.
- A remedy:

$$\mathrm{AV@R}_p = \mathbb{E}[X | X \leq \mathrm{V@R}_p(X)].$$

Convex risk measures

- Proposed by Artzner, Delbaen, Eber, Heath (and others) in late 1990s. Axiomatize!
- A risk measure ϕ is called an **ADEH-risk measure** if
    - $\phi(X) \geq 0$ is $X \geq 0$
    - $\phi(\alpha X + (1-\alpha)Y) \geq \alpha\phi(X) + (1-\alpha)\phi(Y)$, $\alpha \in (0,1)$,
    - $\phi(X + c) = c + \phi(X)$
- $\mathrm{AV@R}$ is a typical example.

Conclusions

- Risk measurement is a non-trivial matter, often neglected.
- Main issue is the translation from psychology to mathematics.
- Widely used concepts are less than adequate (V@R)
- How about an axiomatic approach?