Bounds for VaR Probabilities of Extreme Events

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CAS In Focus:
Elephants in the room
Chicago
October 1, 2013

Motivation

When risk management is impeded by incomplete information of variables of interest, the best way to estimate probabilities of extreme events is:

Moment Method

Find the 100% confidence interval of Value-at-Risk (VaR), i.e., semiparametric upper and lower bounds!

Value-at-Risk (VaR)

- Introduced in 1994, widely used to estimate the largest potential loss or smallest potential return that an investment may suffer during **an extreme event** with a given likelihood $\alpha \in (0,1)$.

- Incorporated in the Basel II Capital Accord in June 2004 [Kaplanski and Levy, 2007].

Value-at-Risk (VaR)

Given a confidence level α∊ (0,1)(e.g., α=1%, 5%, etc.), VaR is defined as:

For the return variable X:

$$VaR_\alpha(X) = \min\{\beta \mid \Pr(X \le \beta) \ge \alpha\}$$

$VaR_\alpha(X)$ is the smallest number β such that the probability that the return X no higher than β is at least α. **(Left tail VaR)**

For the loss variable L:

$$VaR_\alpha(L) = \min\{\beta \mid \Pr(L \ge \beta) \le \alpha\}$$

$VaR_\alpha(L)$ is the smallest number β such that the probability that the loss L exceeds β is at most α. **(Right tail VaR)**

What we do?

$$\Pr(X \leq x_0)$$

$$\Updownarrow$$

Confidence IV of $\Pr(X \leq x_0)$

$$\Updownarrow$$

100% confidence IV of $\Pr(X \leq x_0)$

$$\Updownarrow$$

Bounds on $\mathbb{E}(\mathbb{I}_{X \leq x_0})$.

General Form of Moment Problems

$$\max \ (\text{or} \ \min) \quad \mathrm{E}[\phi(X)]$$

where X is a set of random variables with specified support and moments.

These semiparametric bounds are useful in risk analysis where there is <u>only incomplete information</u> concerning the a random variable.

An Example

According to the Tchebyshev's Inequality:

$$\Pr(|x-\mu| \geq k\sigma) = \mathrm{E}(\phi(X)) \leq \frac{1}{k^2}.$$

$$\max_X \quad \mathrm{E}[\phi(X)]$$

where the support is over all X subject to

$$\mathrm{E}[X] = \mu,$$

$$\mathrm{E}[(X-\mu)^2] = \sigma^2$$

→ The solution is $\frac{1}{k^2}$.

where

$$\phi(x) = \begin{cases} 1 & \text{if } |x-\mu| \geq k\sigma, \\ 0 & \text{if } |x-\mu| < k\sigma. \end{cases}$$

Overview

We solve moment problems to find bounds on Value-at-Risk (VaR).

- *Major work*
  - Find bounds on univariate distributions given moment information with/without unimodal assumption.
    - Present a simplified maximum-entropy method to construct a representative distribution given moments.
  - Find bounds on bivariate distributions with specific forms given mean, variance, and covariance of variables.
    - Joint probability bounds
    - VaR probability bounds
    - Bounds on stop-loss payments

Overview

❑ *Numerical examples*

- ❑ Calculate value-at-risk for downside risk management.
  - ❑ Insurance loss
  - ❑ Asset return
  - ❑ Bounds on $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$
  - ❑ Bounds on $\Pr(w_1 X_1 + w_2 X_2 \leq a)$
- ❑ Define risk-neutral probability for asset pricing using maximum-entropy method.

❑ *Robustness check*

- ❑ Relationship between bounds and the number of moments considered;
- ❑ Sensitivity of the bounds to the estimated moments.

Literature Review

- ***Theories***
  - Hilbert (1888)
  - Karlin and Studden (1966)
    - Tchebycheff system and its applications in analysis and statistics.
  - Diananda (1962)
  - Second-order Cone Programming (SOCP)
    - Lobo et al. (1998), Ben-Tal and Nemirovski (2001), and Alizadeh and Goldfarb (2003)

Literature Review

- ***Univariate Moment Problems***
  - Royden (1953), Brockett and Cox (1985), Lo (1987), Heijnen (1990), and Courtois, Denuit (2006), and Schepper and Heijnen (2007)
    - Explicitly solve moment problems given two to four moments
  - Parrilo (2000), Wolkowicz, Vandenberghe and Saigal (2005), Bertsimas and Popescu (2005)
    - Numerically solve moment problems using SOS programming solvers
- ***Bivariate Moment Problems***
  - Far less complete than univariate moment problems
  - Courtois and Denuit (2008), Kaas et al. (2009), Valdez et al. (2009), Cox et al. (2010), etc.
    - Solve problems with special forms of objective functions, e.g., sum of variables, two correlated variables, etc.

Primal Problem (Univariate)

Upper bound

$$\overline{p} = \max_F \int_{\mathcal{I}} \phi(x)\,\mathrm{d}F(x)$$

$$\text{subject to} \int_{\mathcal{I}} x^i\,\mathrm{d}F(x) = \mu_i, \qquad \text{for all } i = 1, 2, \ldots, n,$$

$$\text{and } \mathcal{I} \subseteq \mathbb{R}$$

Lower bound

$$\underline{p} = \min_F \int_{\mathcal{I}} \phi(x)\,\mathrm{d}F(x)$$

subject to the same moment constraints.

Primal Problem (Univariate)

where

$$\phi(x) = \begin{cases} 1 & \text{for all } x \le t \\ 0 & \text{for all } x > t \end{cases} \quad x \in \mathcal{I}.$$

and

$$\mathcal{I} = (-\infty, a]\ ,\ \mathcal{I} = [a, b]\ ,\ \mathcal{I} = [b, +\infty)\ \text{ and } \mathcal{I} = (-\infty, +\infty)$$

Feasibility (Smith (1990))

- There are random variables with given moments if and only if M_{2n} is PSD.
  - If M_{2n} is PD (non-degenerate), one can find infinite number of distributions to match the moments.
  - If $|M_{2i}|>0$ for i=1,2,…n-1 and $|M_{2n}|=0$ (degenerate), a unique distribution matches the moments.

$$M_{2n} = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}_{(n+1)\times(n+1)}$$

In our applications, feasibility is not a problem because we use a real sample to estimate moments so the empirical distribution has those moments and the problem is always feasible.

Dual Problem (Univariate)

Upper bound

$$\overline{d} = \min_{a_0, a_1, \ldots, a_n} \quad \sum_{i=0}^{n} a_i \mu_i$$

$$\text{subject to} \quad p(x) \geq \phi(x), \quad \text{for all } x \in \mathcal{I},$$

where $p(x) = \sum_{i=0}^{n} a_i x^i$.

Lower bound

$$\underline{d} = \max_{a_0, a_1, \ldots, a_n} \quad \sum_{i=0}^{n} a_i \mu_i$$

$$\text{subject to} \quad p(x) \leq \phi(x), \quad \text{for all } x \in \mathcal{I}.$$

Strong Duality Proposition

The primal and dual problems are equivalent in term of their solutions, if one can find a solution that strictly satisfies (i.e., with > (or <)) the constraints in the dual.

As long as Φ(x) is bounded, one can find feasible solutions to make the constraint strictly hold.

Consider

$$\phi(x) = \begin{cases} 1, & \forall\, x \leq d \\ 0, & \forall\, x > d \end{cases}$$

Strong duality is satisfied as follows:

The dual solution $a_0 > 1$ *(or* $a_0 < 0$ *for the lower bound), and* $a_i = 0 \text{ for } i \neq 0$ *strictly satisfied the constraints.*

So: $\overline{p} = \overline{d}$ (or $\underline{p} = \underline{d}$)

PSD and SOS

Definition 1. *A polynomial $g(x) \in \mathbb{R}[x]$ is a sum of squares (SOS) polynomial if there exist polynomials $q_i(x) \in \mathbb{R}[x]$ so that $g(x) = \sum_i [q_i(x)]^2$.*

Definition 2. *A polynomial $g(x) \in \mathbb{R}[x]$ is positive semidefinite (PSD) on $\mathcal{I} \subseteq \mathbb{R}$ if $g(x) \geq 0$ for all $x \in \mathcal{I}$.*

Obviously, for any choice $\mathcal{I} \subseteq \mathbb{R}$, a SOS polynomial is a PSD polynomial on I. And it is a classical result that the converse is true for $\mathcal{I} = \mathbb{R}$; namely:

$g(x)$ is a (univariate) PSD polynomial on $\mathbb{R}$ $\iff$ $g(x)$ is a SOS polynomial.

SOS Program

A SOS program is an optimization program where the variables are coefficients of polynomials, the objective is a linear combination of the variable coefficients, and the constraints are given the polynomials being SOS.

$$\max_{a_0, a_1, \ldots, a_n} \ (\min) \quad a_0 + a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n$$

Subject to

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n - \phi(x) \text{ is SOS polynomial.}$$

SOS Programming

Strong duality

Dual Problems

Equivalency between PSD and SOS for univariate problems

Primal Problems

Formulate the duals as SOS programs

Solve problems with a SOS programming solvers such as SOSTOOLS , GloptiPoly or YALMIP.

Bounds for Unimodal Distribution

A continuous-type random variable X is unimodal with mode m if it satisfies one of the following two equivalent conditions:

(i) The cumulative distribution function $F(x)$ of X is convex for $x < m$ and concave for $x > m$ in its support $\mathcal{I}$.

(ii) Khintchine (1938) Representation: There are independent random variables U and Y such that $X \sim m + UY$, where U is uniformly distributed on $[0, 1]$.

We solve bounds for unimodal distribution by formulating the unimodal problems as general bounds problems after an appropriate transformation.

Unimodal Bounds Problems

Objective:

$$(1)\ \text{In this case of } t \geq m,\ \ \phi^*(y) = \begin{cases} 1 & y \leq t-m \\ \dfrac{t-m}{y} & y \geq t-m. \end{cases}$$

$$(2)\ \text{In this case of } t < m,\ \ \phi^*(y) = \begin{cases} 1 - \dfrac{t-m}{y} & y \leq t-m \\ 0 & y \geq t-m. \end{cases}$$

Moment constraints:

$$\mathrm{E}[(UY)^i] = \mathrm{E}[(X-m)^i]$$

$$\mu_i^* = \mathrm{E}[Y^i] = (i+1)\sum_{j=0}^{i} \binom{i}{j} \mu_j (-m)^{i-j}$$

Unimodal Bounds Problem

Upper bound

$$\overline{p}^* = \max \int_{\mathcal{I}^*} \phi^*(y)\,\mathrm{d}F^*(y)$$

$$\text{subject to} \int_{\mathcal{I}^*} y^i\,\mathrm{d}F^*(y) = \mu_i^*, \qquad \text{for all } i = 1, 2, \ldots, n,$$

Lower bound

$$\underline{p}^* = \min \int_{\mathcal{I}^*} \phi^*(y)\,\mathrm{d}F^*(y)$$

subject to the same moment constraints.

Maximum-Entropy Method

- A method to construct representative distribution with given moments.

$$
\begin{aligned}
\max_{f(x)} \quad & -\int_a^b f(x) \log f(x) \, \mathrm{d}x \\
\text{subject to} \quad & \int_a^b x^i f(x) \, \mathrm{d}x = \mu_i \quad \text{for all } i = 0, 1, \ldots, n \\
& f(x) \geq 0,
\end{aligned}
$$

where $\mu_0, \mu_1, \ldots, \mu_n$ is the given sequence of moments.

The optimal solution f(x) is called maximum-entropy distribution.*

Maximum-Entropy Method

- Maximum-entropy distribution can be written as:

$$f^*(x) = \exp\left(-1 - \sum_{i=0}^{n} \lambda_i x^i\right)$$

- The optimization problem can be solved using a modified Newton method (Luenberger 1984).

Bounds on Insurance Margins

The margin (M) on a line of insurance business is defined as

$$M = 1 - \mathrm{LR} - \mathrm{ER} = 1 - \mathrm{CR},$$

where

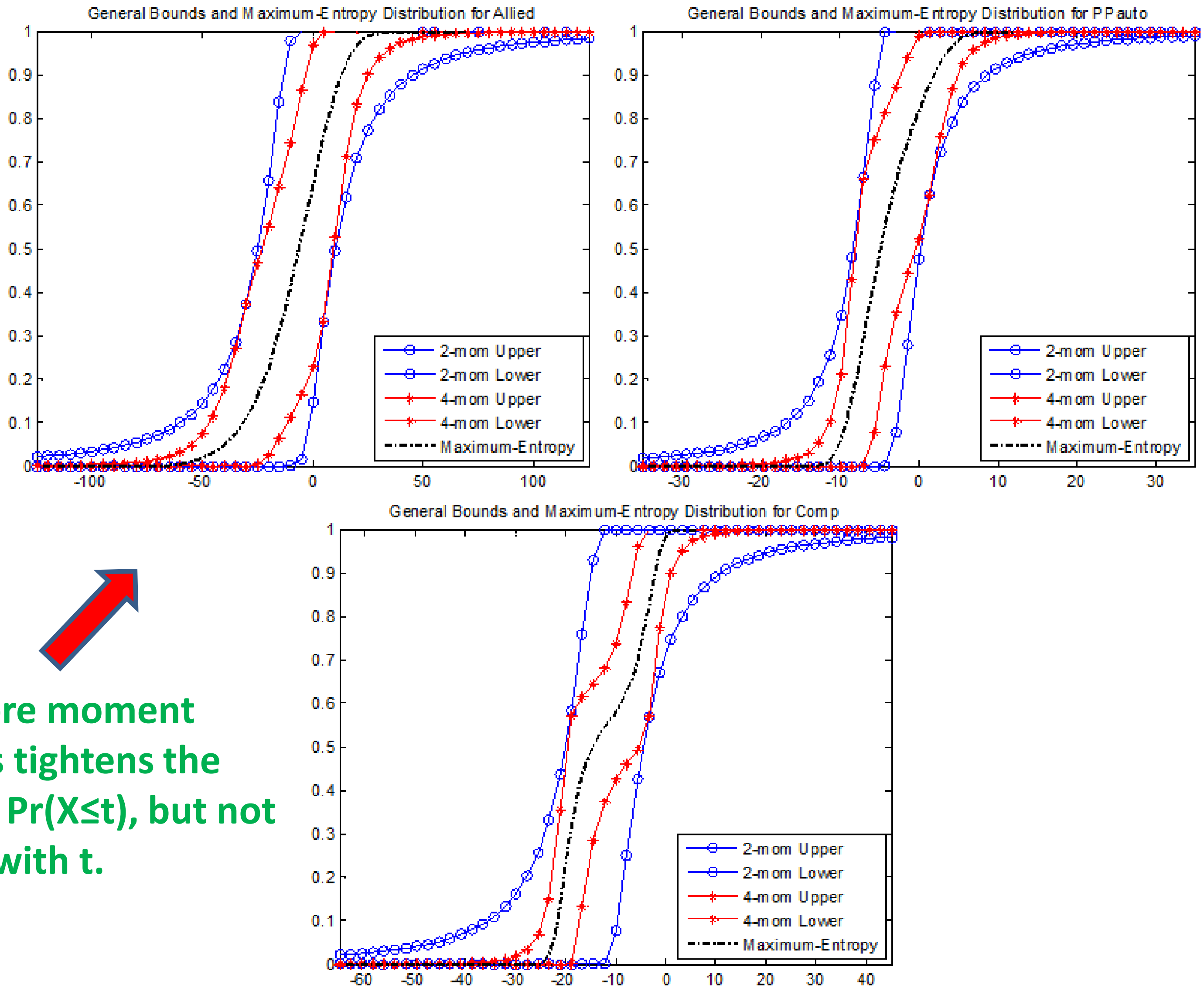
$$\mathrm{LR} = \frac{\text{Losses Incurred}}{\text{Earned premiums}}$$ is the loss ratio and

$$\mathrm{ER} = \frac{\text{Expenses}}{\text{Written premiums}}$$ is the expense ratio.

Table 2: Empirical Statistics of Margin for Allied, PPauto, and Comp During 1980 and 2005

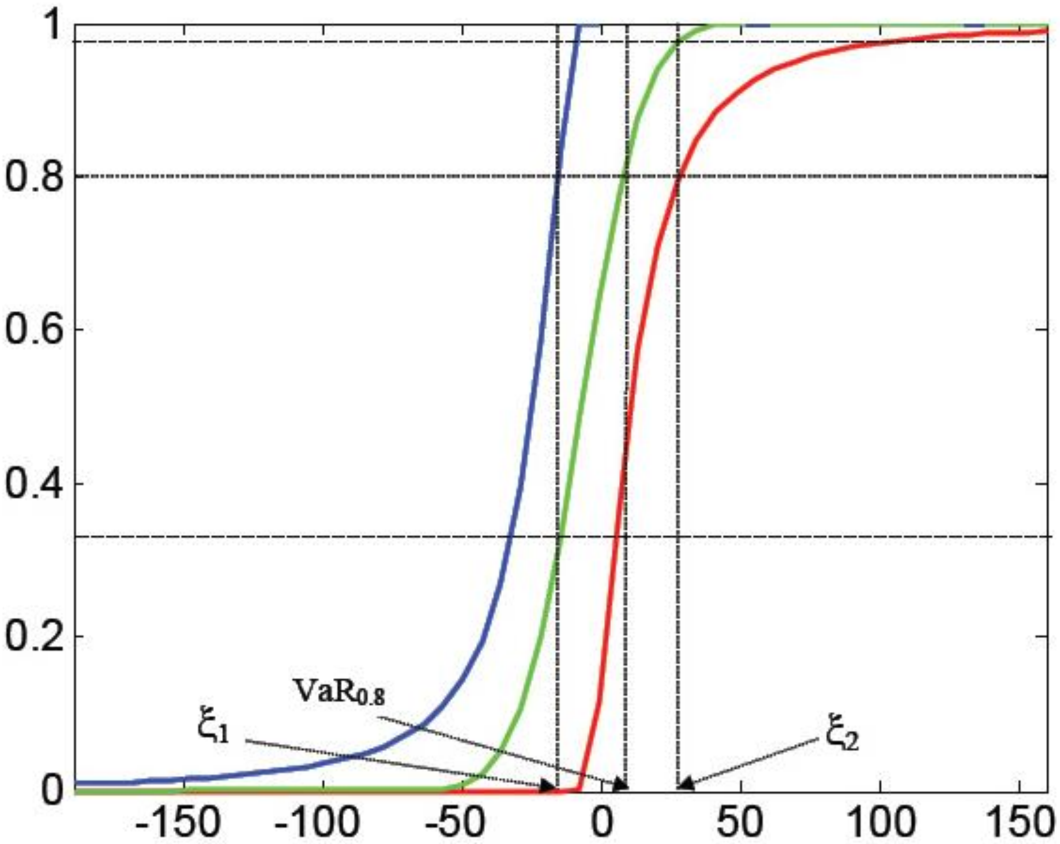
	Allied	PPauto	Comp
$\mathrm{E}(X)$	-7.32	-4.02	-12.25
$\mathrm{E}(X^2)$	360.61	33.85	209.79
$\mathrm{E}(X^3)$	-10011.00	-242.54	-3949.10
$\mathrm{E}(X^4)$	496130.00	2203.30	77445.00
Maximum	20.30	6.03	0.50
Minimum	-52.00	-11.03	-22.60
Range	72.30	17.06	23.10

General Bounds and Maximum-Entropy Distributions



Adding more moment constraints tightens the bounds on Pr(X≤t), but not uniformly with t.

Bounds on Value at Risk



The red line represents the lower bound. The blue line represents the upper bound. The green line in the middle is the true empirical distribution F(x).

$$Pr(X \leq x) = 0.8$$

$$F(\xi_1) \leq \overline{p}(\xi_1) = \alpha = \underline{p}(\xi_2) \leq F(\xi_2)$$

$$\xi_1 \leq \mathrm{VaR}_\alpha \leq \xi_2$$

Bounds on Asset Returns

We analyze stock returns by considering the following three scenarios, i.e., a period before the Great Recession, the Recession period, and the post recession period.

a: 12/2001-12/2007

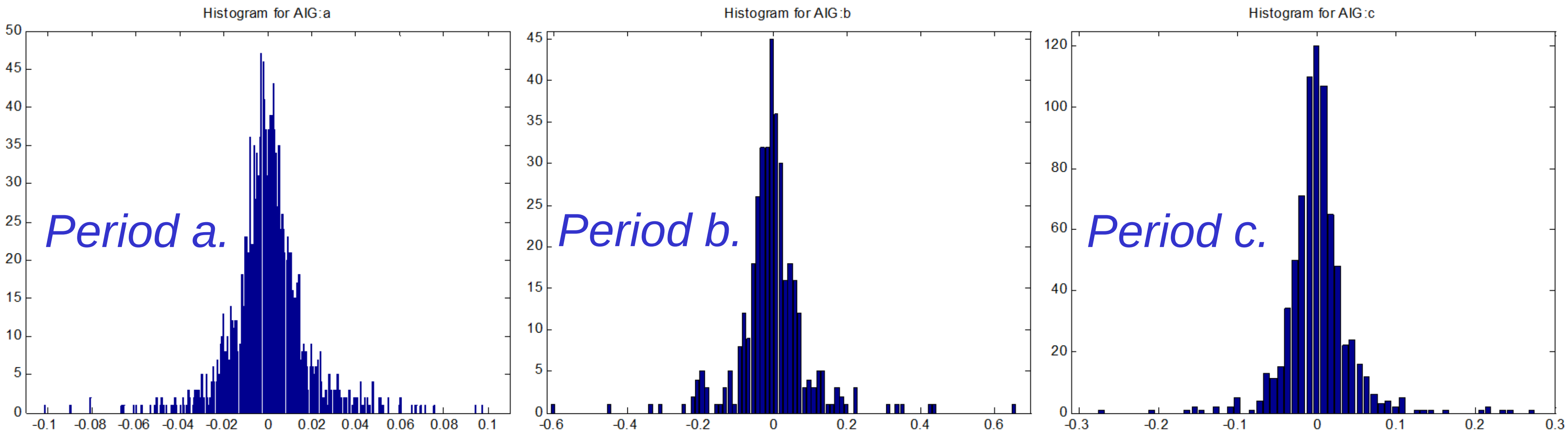
b: 01/2008-06/2009

c: 07/2009-07/2012

Statistics of AIG Stock Daily Return

	AIG:a	AIG:b	AIG:c
$E(X)$	-9.58E-05	-4.52E-03	1.63E-03
$E(X^2)$	2.81E-04	1.03E-02	2.31E-03
$E(X^3)$	5.52E-07	3.16E-04	3.75E-04
$E(X^4)$	6.61E-07	1.46E-03	2.44E-04
Mode	-2.87E-03	-5.65E-03	-7.52E-04
Variance	2.81E-04	1.03E-02	2.30E-03
Obs	1529	377	769

Histogram for AIG



The histograms show that AIG has a unimodal distribution in all three periods.

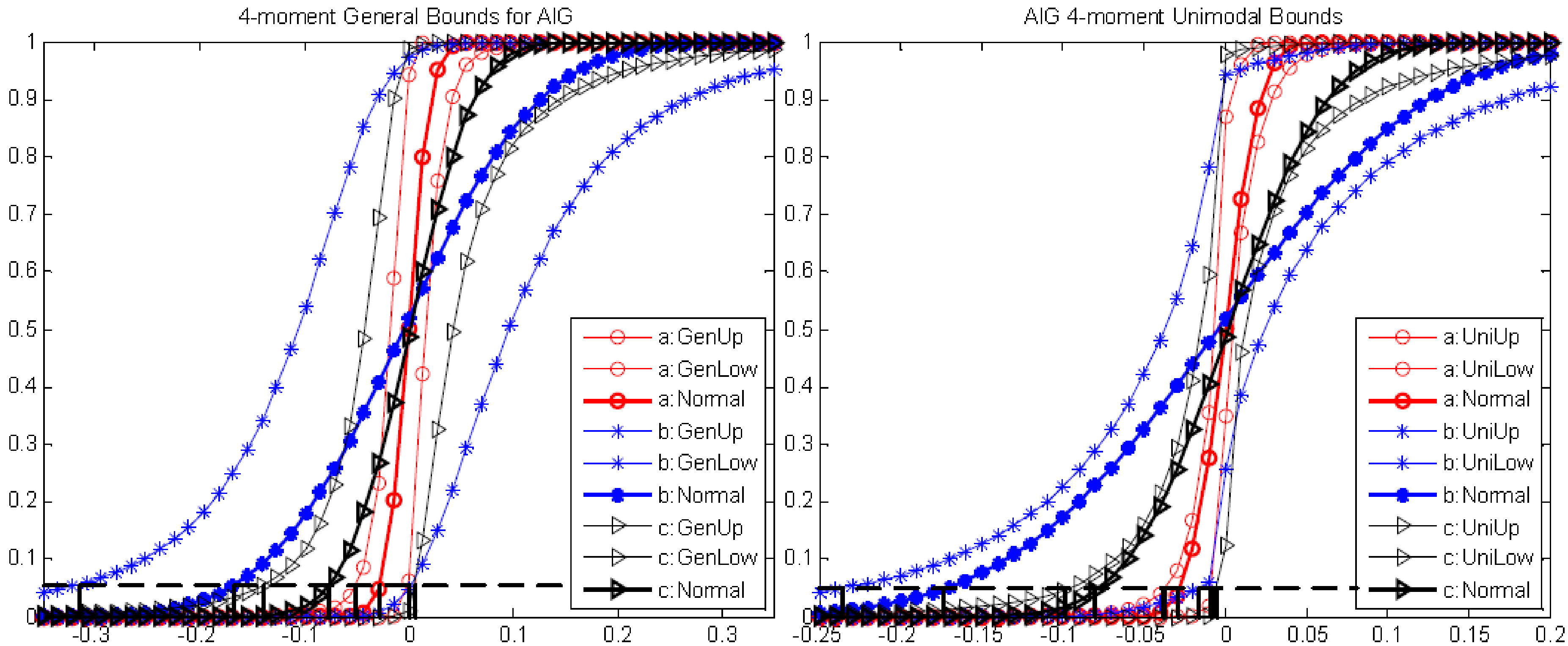
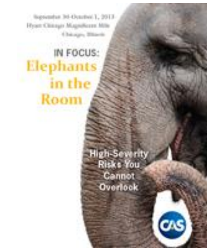


Table 4: 4-moment General and Unimodal Bounds on $VaR_{5\%}$ in Periods a, b, and c.

General Bounds	Unimodal Bounds
$-5.1\% < VaR^a_{5\%} < -0.1\%$	$-3.8\% < VaR^a_{5\%} < -1.2\%$
$-31.2\% < VaR^b_{5\%} < 0.2\%$	$-23.8\% < VaR^b_{5\%} < -1.3\%$
$-14.5\% < VaR^c_{5\%} < 0.8\%$	$-9.8\% < VaR^c_{5\%} < -0.6\%$



Bivariate Moment Problems

- **Joint Probability Bounds**: Bounds on $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$ for the non-negative variables X_1 and X_2 $(X_1, X_2 \geq 0)$, given the mean, variance and covariance, with the objective function

$$\phi(X_1, X_2) = \mathbb{I}_{\{X_1 \leq t_1 \text{ and } X_2 \leq t_2\}}$$

  where $t_1, t_2 \in \mathbb{R}^+$.

- **VaR Probability Bounds**: Bounds on $\Pr(w_1 X_1 + w_2 X_2 \leq a)$ for any $X_1, X_2 \in \mathbb{R}^2$, given the mean, variance and covariance, with or without information of expected payoff of exchange option $\mathrm{E}[(X_1 - X_2)^+] = \gamma$. The objective function is

$$\phi(X_1, X_2) = \mathbb{I}_{\{w_1 X_1 + w_2 X_2 \leq a\}}$$

  where $w_1, w_2, \gamma \in \mathbb{R}^+$ and $a \in \mathbb{R}$.

Bivariate Moment Problems

$$\phi(X_1, X_2) = \begin{cases} b & \text{if } \; X_1 + X_2 \geq a + b \\ X_1 + X_2 - a & \text{if } \; a \leq X_1 + X_2 \leq a + b \\ 0 & \text{if } \; X_1 + X_2 \leq a. \end{cases}$$

- Special cases: payoffs of call or put options
- This problem can be converted to a one variable moment problem and solved numerically.
- We also can use the explicit formulae deduced by Cox (1991) to compute the bounds.

Theorem Preparation

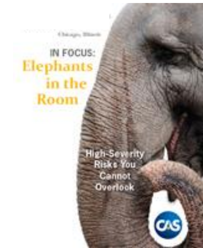
In order to numerically solve the semiparametric bounds, we reformulate the corresponding semiparametric bound problem as a sum of squares (SOS) program using the following two theories:

- Theorem (Hilbert(1888))

  Let $p(x_1, \ldots, x_n)$ be a quadratic polynomial. Then $p(x_1, \ldots, x_n) \geq 0,\ \forall\ x_1, \ldots, x_n \in \mathbb{R}$ if and only if $p(x_1, \ldots, x_n)$ is a SOS polynomial.

- Theorem (Diananda(1962))

  Let $p(x_1, \ldots, x_n)$ be a quadratic polynomial. If $n \leq 3$, then $p(x_1, \ldots, x_n) \geq 0,\ \forall\ x_1, \ldots, x_n \geq 0$ if and only if $p(x_1^2, \ldots, x_n^2)$ is a SOS polynomial.



SOS Program (Multivariate)

- A polynomial

$$p(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_n \in \mathbb{N}} y_{(i_1, \ldots, i_n)} x_i^{i_1} \cdots x_n^{i_n}$$

is said to be a SOS polynomial if

$$p(x_1, \ldots, x_n) = \sum_i q_i(x_1, \ldots, x_n)^2$$

for some polynomials $q_i(x_1, \ldots, x_n)$.

Primal and Dual Problems

- Primal problem

$$
\begin{aligned}
\overline{p}(\underline{p}) = \max(\min) \quad & \mathbb{E}_\pi(\phi(X_1, X_2)) \\
\text{such that} \quad & \mathbb{E}_\pi(1) = 1, \\
& \mathbb{E}_\pi(X_i) = \mu_i, \quad i = 1, 2, \\
& \mathbb{E}_\pi(X_i^2) = \mu_i^{(2)}, \quad i = 1, 2, \\
& \mathbb{E}_\pi(X_1 X_2) = \mu_{12}, \\
& \pi \text{ a probability distribution in } \mathcal{D}
\end{aligned}
\tag{1}
$$

- Dual problem [Popescu (2005)]

$$
\begin{aligned}
& \overline{d}(\underline{d}) = \min(\max) y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12} \\
& \text{such that} \\
& p(x_1, x_2) \geq \ (\text{or } \leq) \ \phi(x_1, x_2), \forall \ (x_1, x_2) \in \mathcal{D},
\end{aligned}
\tag{2}
$$

where the quadratic polynomial

$$
p(x_1, x_2) = y_{00} + y_{10}x_1 + y_{01}x_2 + y_{20}x_1^2 + y_{02}x_2^2 + y_{11}x_1x_2.
$$

Feasibility and Strong Duality

- **Feasibility**: The dual problem (2) is feasible if and only if Σ is a positive semidefinite matrix (i.e., all eigenvalues are greater than or equal to zero), where Σ is the moment matrix. If $\mathcal{D} = {\mathbb{R}^+}^2$, all elements of Σ are required to be non-negative.

$$\Sigma = \begin{bmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_1^{(2)} & \mu_{12} \\ \mu_2 & \mu_{12} & \mu_2^{(2)} \end{bmatrix}.$$

- **Strong Duality** The solution to the dual is equivalent to the primal in the sense that the numerical values of the dual is equal to that of the primary, if and only if one can find a special case in which the constraints in the dual is strictly satisfied.

Joint Probability Bounds on $\Pr(X_1 \leq t_1 \text{ and } X_2 \leq t_2)$

- **Upper Bounds**

$$\overline{d} = \min \; y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \geq 1, \forall \; 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2$$

$$p(x_1, x_2) \geq 0, \forall \; x_1, x_2 \geq 0.$$

- **Lower Bounds**

$$\underline{d} = \max \; y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \leq 1, \forall \; 0 \leq x_1 \leq t_1, 0 \leq x_2 \leq t_2$$

$$p(x_1, x_2) \leq 0, \forall \; x_1, x_2 \geq 0.$$

Numerical Example of Probability Bounds

- Consider the American International Group (AIG). We find bounds on the probability of joint extreme events, *i.e.*, unanticipated poor asset returns and unexpectedly high claims, $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$.
- X_1: **Weighted Average Net Return.** Suppose AIG's portfolio contains 6 assets. For asset i, $r_i^G = \mathbf{r}_i + 1 = \frac{P_{i,t}}{P_{i,t-1}}$. $\mathbf{r}$ is the weighted average return:

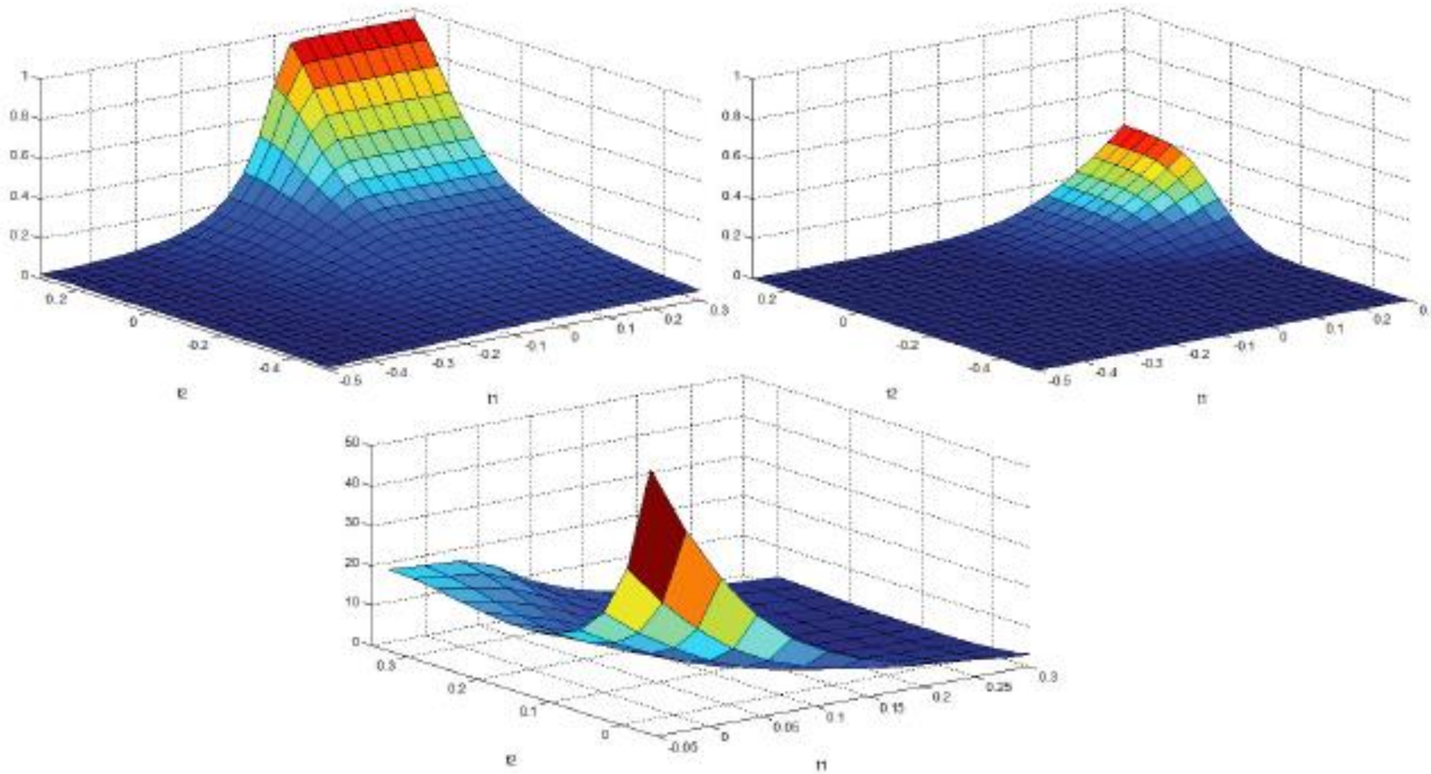
$$\mathbf{r} = \sum_{i=1}^{6} w_i r_i^G - 1.$$

- X_2: **Margin.** The margin on insurance business $\mathbf{m}$ is defined as

$$\mathbf{m} = 1 - \mathbf{LR},$$

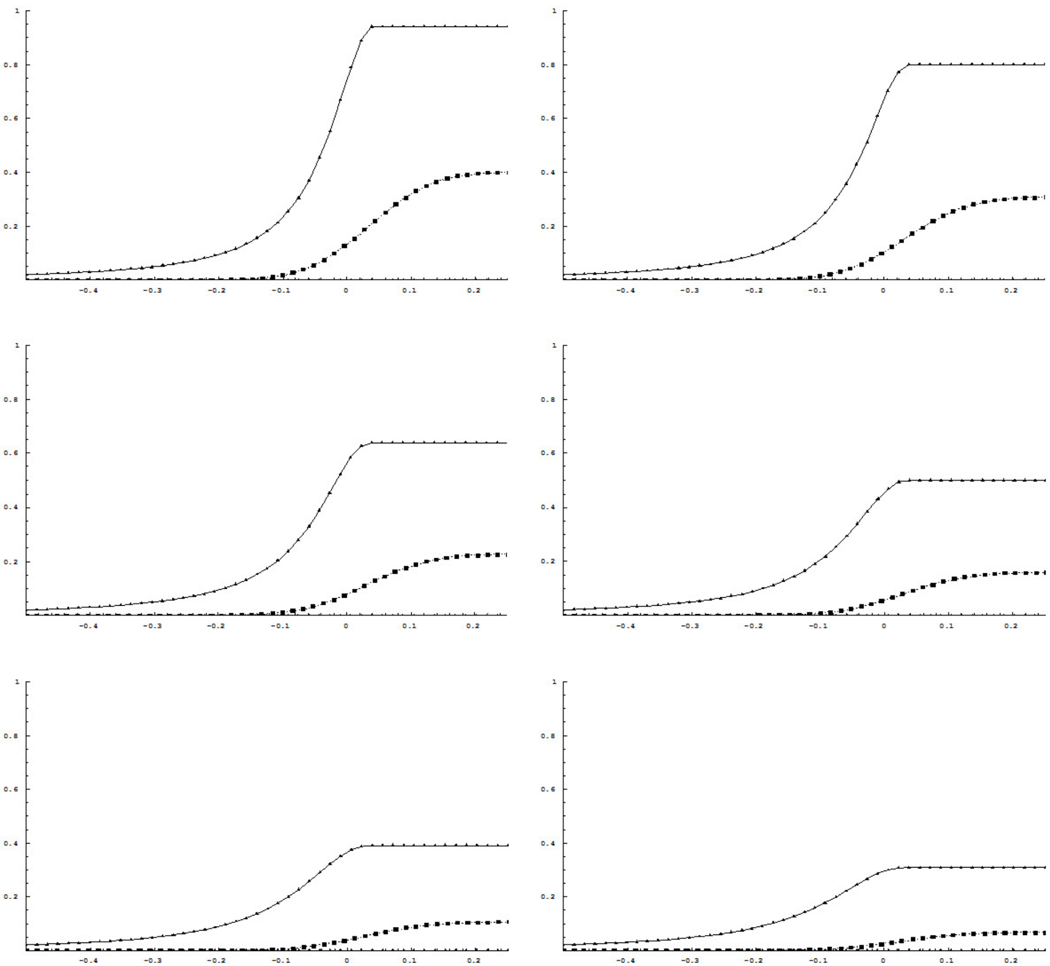
where $\mathbf{LR}$ is the economic loss ratio.

AIG Example



The upper left plot shows the upper bound of $\Pr(\mathbf{r} \le t_1, \mathbf{m} \le t_2)$. The upper right one is the CDF of bivariate normal with the same moments as AIG. The ratio of the upper bound to the bivariate normal CDF is shown in the third graph. The vertical axis of the graphs is probability. It is the ratio in the third graph.

AIG Example (Continued)



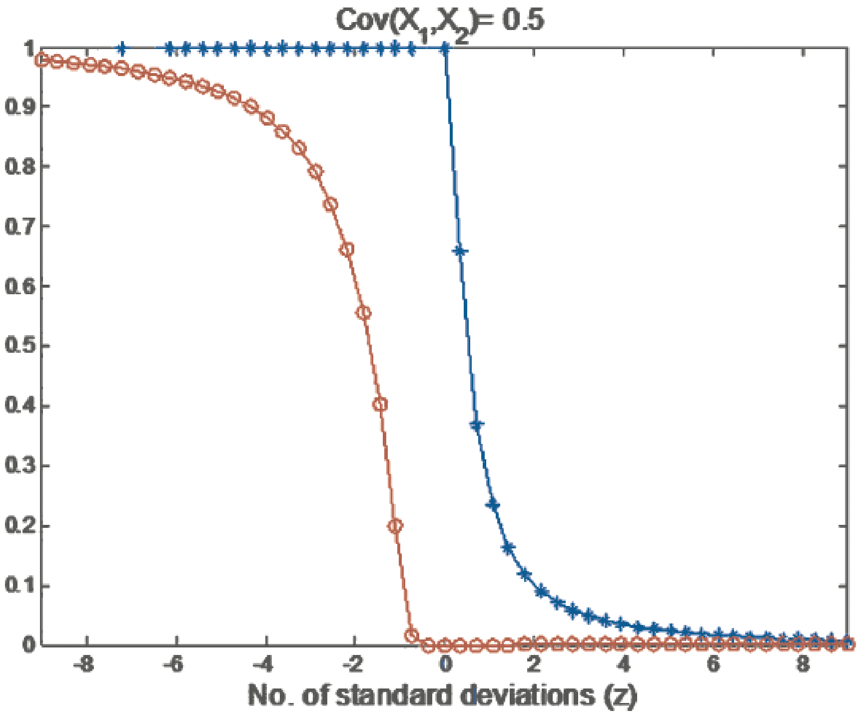
The higher curve is the upper bound on $\Pr(\mathbf{r} \leq t_1, \mathbf{m} \leq t_2)$. The lower curve is the CDF of bivariate normal with the same moments as AIG.

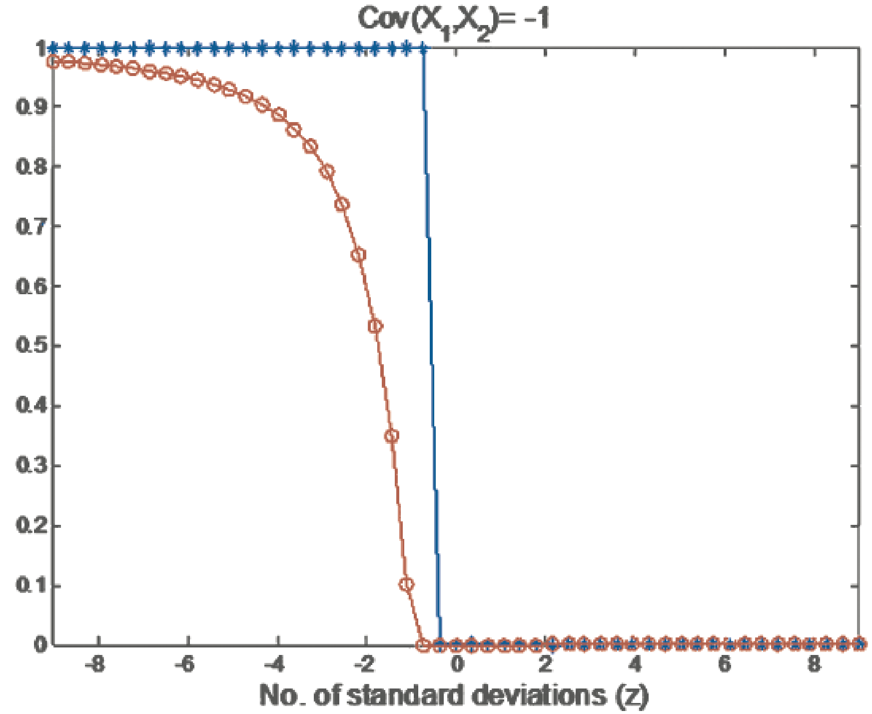
x-axis stands for t_1. And t_2 is fixed at $\mathrm{E}(\mathbf{m}) - k\sigma(\mathbf{m})$ where k = 0.25, 0.5, . . . , 1.5, with k = 0.25 on the upper left and running to the right and then down.

Bounds on Joint Right-Tail Events

Bounds on $\Pr(X_1 \geq t_1,\ X_2 \geq t_2)$. The left and right graphs show bounds with covariance of X_1 and X_2 equals 0.5 and -1, respectively. The vertical axis stands for probability, and the horizontal axis is the number of standard deviations from the mean, z. That is:

$$t_1 = \mu_1 + z\sigma_1 \text{ and } t_2 = \mu_2 + z\sigma_2.$$





VaR Bounds on $\Pr(w_1X_1 + w_2X_2 \leq a)$

- **Upper Bounds**

$$\overline{d}_{\mathrm{VaR}} = \min \; y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \geq 1, \forall \; x_1, x_2 \text{ s.t. } w_1x_1 + w_2x_2 \leq a$$

$$p(x_1, x_2) \geq 0, \forall \; x_1, x_2 \in \mathbb{R}.$$

- **Lower Bounds**

$$\underline{d}_{\mathrm{VaR}} = \max \; y_{00} + y_{10}\mu_1 + y_{01}\mu_2 + y_{20}\mu_1^{(2)} + y_{02}\mu_2^{(2)} + y_{11}\mu_{12}$$

such that

$$p(x_1, x_2) \leq 1, \forall \; x_1, x_2 \text{ s.t. } w_1x_1 + w_2x_2 \leq a$$

$$p(x_1, x_2) \leq 0, \forall \; x_1, x_2 \in \mathbb{R}.$$

Numerical Example of VaR Bounds

We analyze the tail joint probability of total return of a portfolio investing in the S&P 500 Index and the Dow Jones U.S. Small-Cap Index.

Let X_1 and X_2 be the log-return of the S&P 500 Index and Dow Jones U.S. Small-Cap Index in percentage per day.

For day t: $X_i(t) = 100 \log(S_i(t+1) = S_i(t))$

Their moments are as follows:

$$\mathrm{E}(X_1) = 0.0059, \qquad \mathrm{E}(X_1^2) = 1.2158$$

$$\mathrm{E}(X_2) = -0.2117, \qquad \mathrm{E}(X_2^2) = 112.8609$$

$$\mathrm{E}(X_1X_2) = 1.4161, \qquad \mathrm{Cov}(X_1, X_2) = 1.41736$$

$$\mathrm{Var}(X_1) = 1.2158, \qquad \mathrm{Var}(X_2) = 112.8160$$

$$\rho = 0.1210, \qquad \mathrm{E}((X_1 - X_2)^+) = 0.4464$$

Numerical Example of VaR Bounds

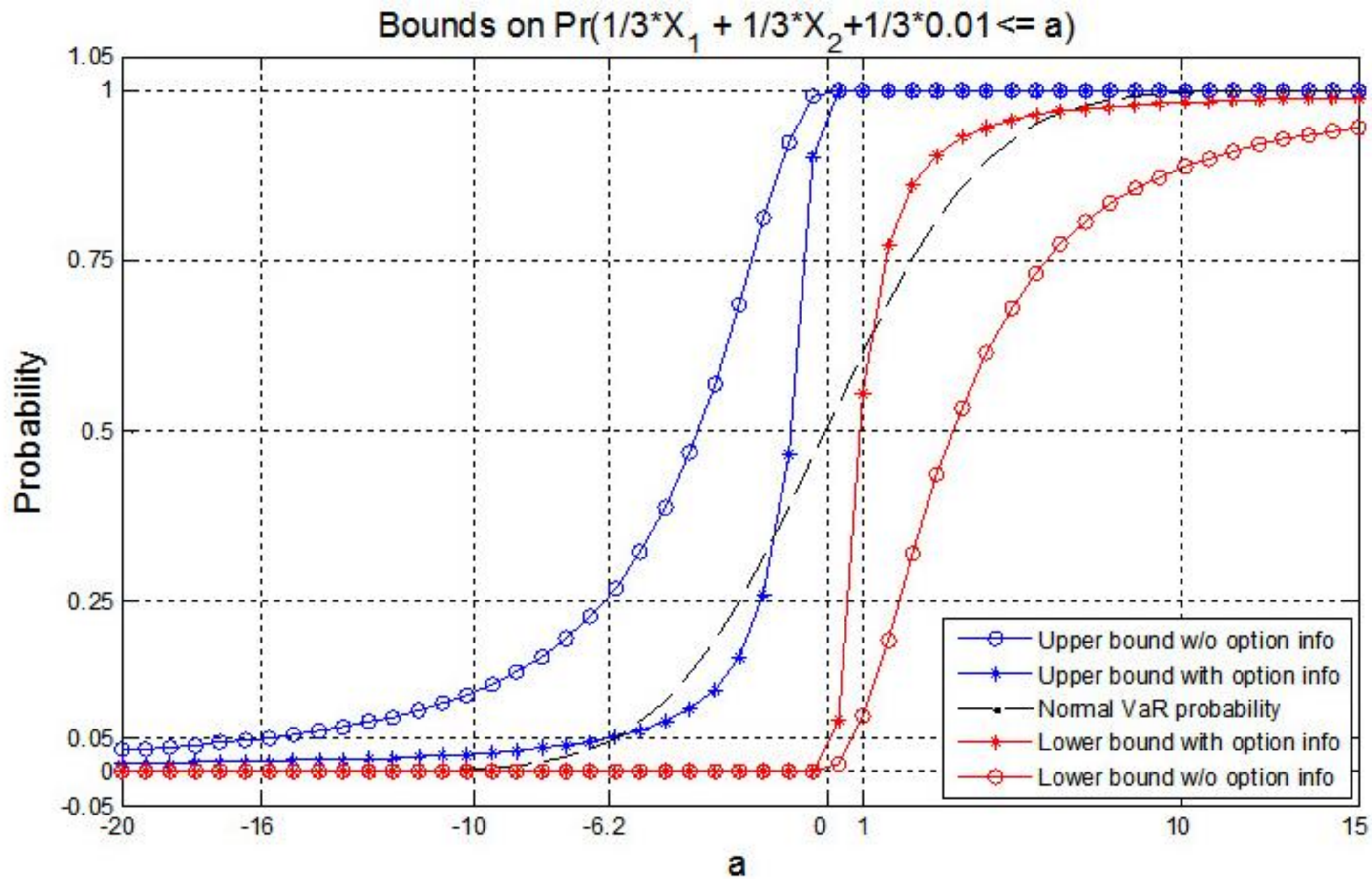
Suppose we invest 1/3 of our assets in the S&P 500 Index, 1/3 in the Dow Jones U.S. Small-Cap Index, and 1/3 in a risk-free fund paying a flat 0.01 percent per day. Thus, our portfolio daily return is:

$$\frac{1}{3}X_1 + \frac{1}{3}X_2 + \frac{1}{3}0.01$$

We now calculate the upper and lower bounds for the probability when the portfolio return falls

$$\Pr((1/3)X_1 + (1/3)X_2 + (1/3)0.01 \leq a)$$

Numerical Example of VaR Bounds



The lines with —o— and —*— represent the upper and lower bounds on the VaR probability, without and with using the exchange option information.

Numerical Example of VaR Bounds

The figure on the previous slide gives us an idea of how likely the return of this portfolio will be lower than a in 1 day under different conditions.

Without the exchange option information (—o— curves):

$$-16\% < \mathrm{VaR}_{0.05} < 1\%$$

With the exchange option information (—*— curves):

$$-6.2\% < \mathrm{VaR}_{0.05} < 0\%$$

Maximum-Entropy Risk Neutral Probability

Li (2010) suggests to using the maximum-entropy approach to calculate risk neutral probabilities for asset pricing.

Consider a market with n assets, whose prices at time 0 are:

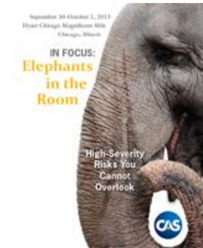
$$\mu_1, \mu_2, \ldots, \mu_n.$$

Assume there are m states of nature and discounted by the risk-free rate, the payoff of asset i in state j is

$$x_i(s_j), i = 1, \ldots, n \text{ and } j = 1, \ldots, m.$$

Let $\mathbb{P}$ be the physical probability measure and $\mathbb{Q}$ be an equivalent risk neutral probability measures. The martingale measure $\mathbb{Q}$ can be chosen by minimizing the Kullback-Leibler (Kullback and Leibler, 1951) information criterion

$$\mathrm{E}^{\mathbb{P}}\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right]$$



Maximum-Entropy Risk Neutral Probability

With this setup, the discrete risk-neutral probabilities $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$ for the states of nature $s = [s_1, s_2, \ldots, s_m]$ can be determined by solving the following maximum-entropy problem:

$$\max_{\pi=[\pi_1,\pi_2,\ldots,\pi_m]} \quad -\sum_{j=1}^{m} \pi_j \log \pi_j$$

$$\text{subject to} \quad \sum_{j=1}^{m} \pi_j = 1$$

$$\sum_{j=1}^{m} x_i(s_j) \cdot \pi_j = \mu_i \qquad \text{for all } i = 1, \ldots, n$$

$$\pi_j \geq 0 \qquad \text{for all } j = 1, \ldots, m.$$

Pricing An European Call Option

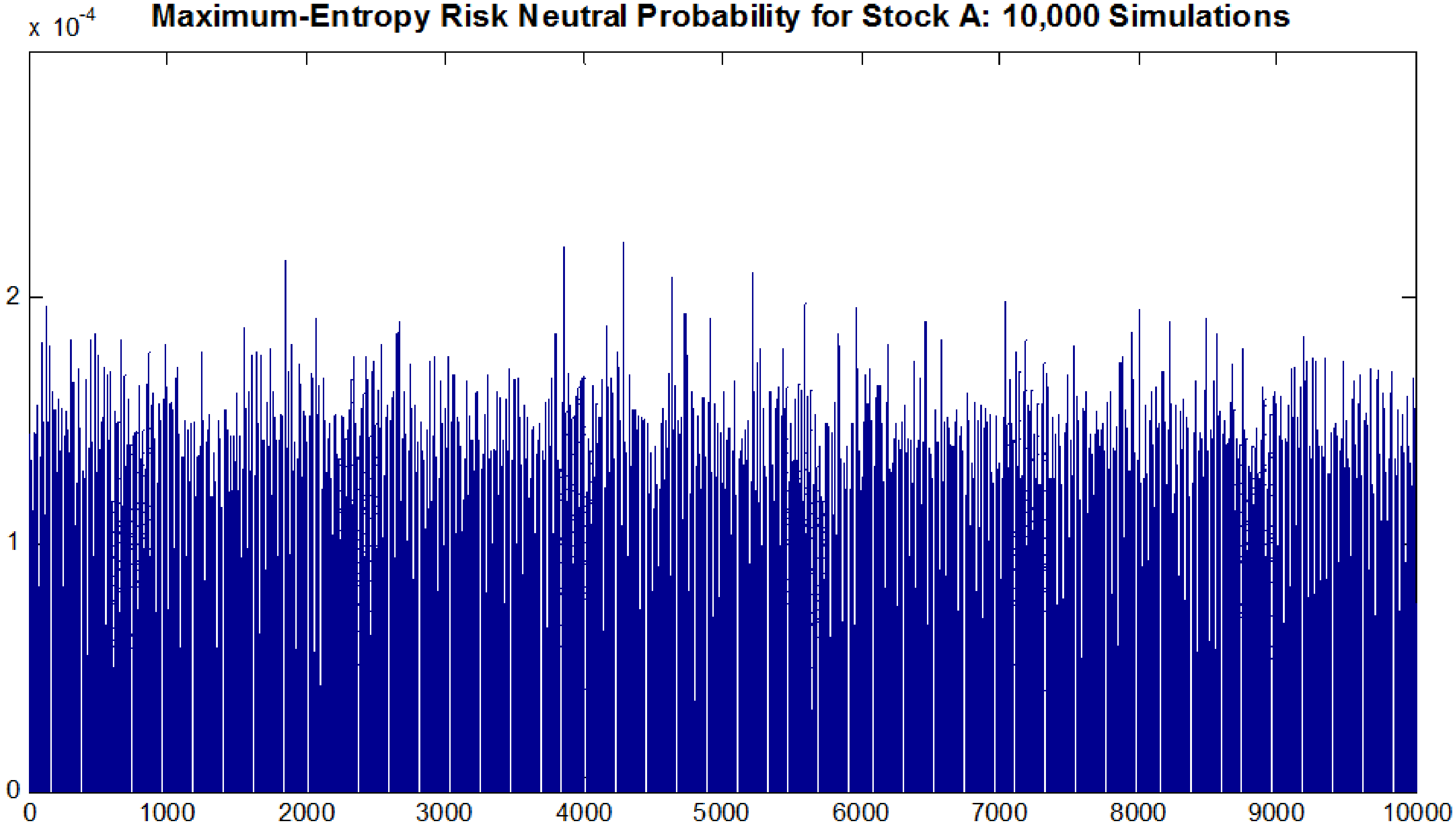
Consider a non-dividend paying stock (Stock A) whose price follows a geometric Brownian motion with spot price $S_0 = 20$, volatility $\sigma = 0.15$, and rate of return $\mu = 0.08$. We further assume that the risk-free rate is $r_f = 0.04$.

We first simulate the stock price at T:

$$S_T = S_0 e^{(\mu - \sigma^2/2)T + \sigma W_T},$$

where W_T is a Wiener process under $\mathbb{P}$.

We then solve the Maximum-Entropy risk neutral problem with one price constraint where $\mu_1 = S_0 = 20$.


Maximum-Entropy Risk Neutral Probability for Stock A: 10,000 Simulations
x 10^-4
2
1
0
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000


IN FOCUS:
Elephants in the Room
High-Severity Risks You Cannot Overlook

Pricing An European Call Option

Consider an European call option written on stock A with strike price K = 18 and exercise time T = 1. Based on the risk neutral measure and the present value of the call option

$$e^{-r_f T}\max(S_T - K, 0)$$

we calculate the value of the call option as:

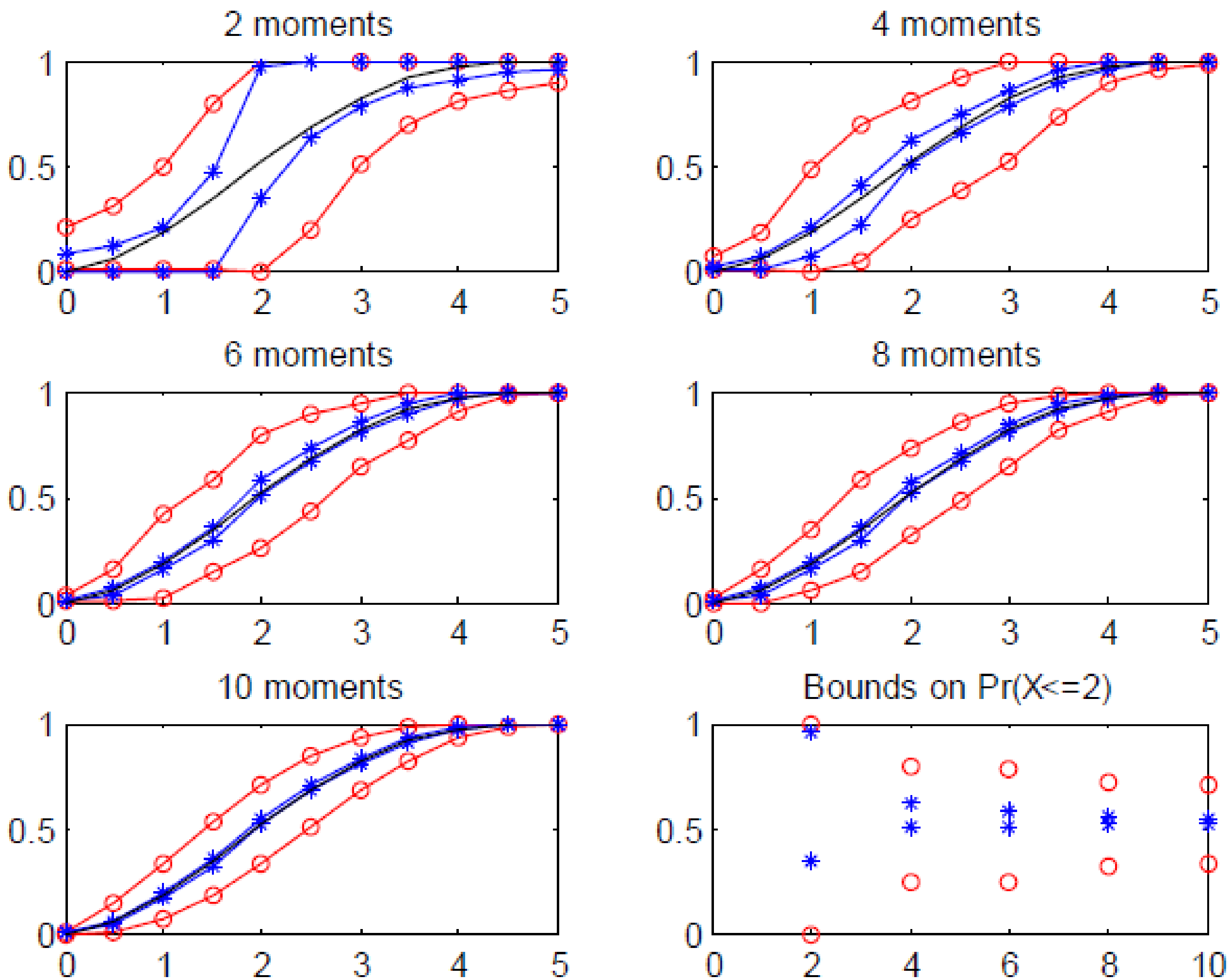
$$C = e^{-r_f T}\sum_{j=1}^{m}\max(S_T^j - K, 0)\cdot\pi_j$$

where m = 10, 000 for our case. Our estimation of the value of the call option from the maximum-entropy distribution is <u>2.94</u>. While following the Black-Scholes model, the value of the call is <u>2.95</u>.

Robustness Check

- Does it matter how many moments are considered to calculate bounds?
  - Check the bounds accuracy with known distribution
- Stability experiments
  - Examine the sensitivity of bounds with respect to moment estimates by altering the data sample sizes.
    - Long-tail distribution – Pareto with $\alpha=5$ (or $\alpha=1$), $\theta=10$.
    - Sample size n=25, 100, 500, and 1000.

General and unimodal Bounds on Beta Distribution (a = 2, b = 3 and θ = 5)



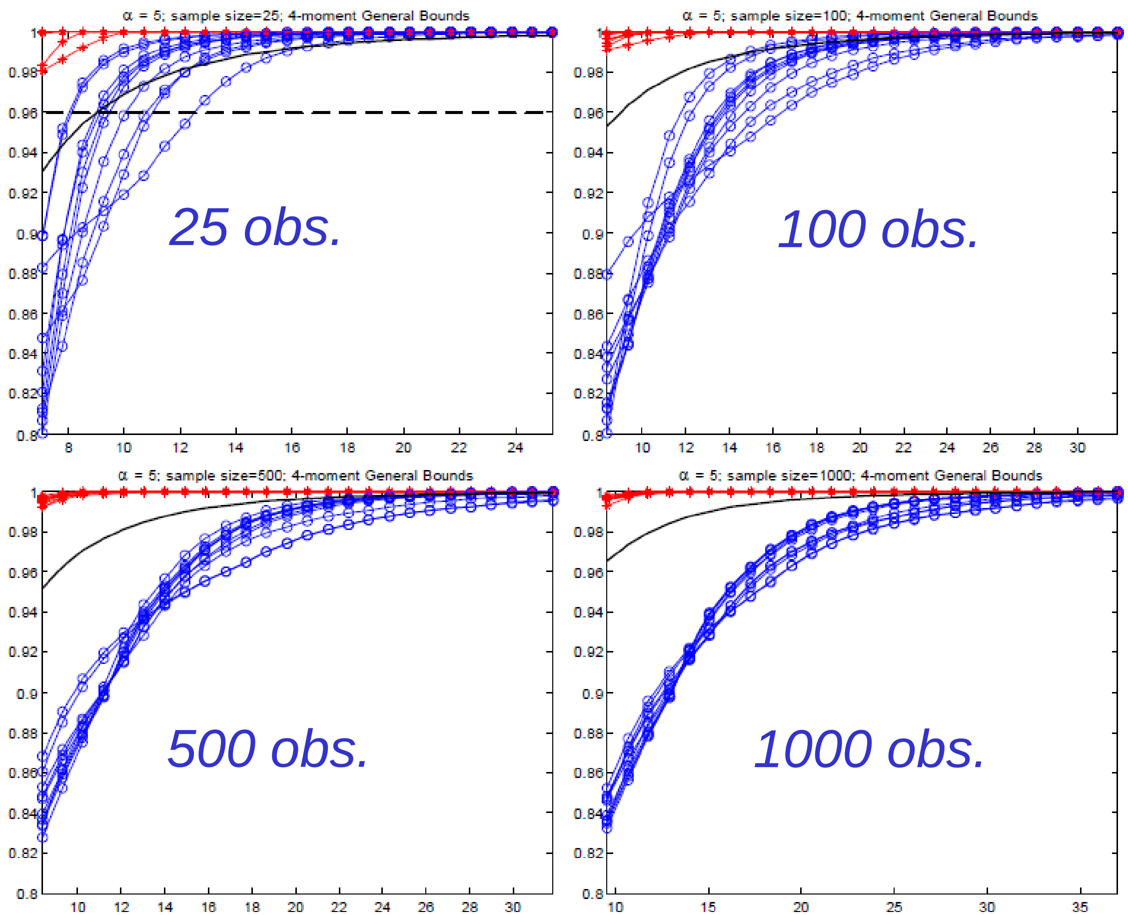
The improvement of bounds wears out after 4 or 6 moments.

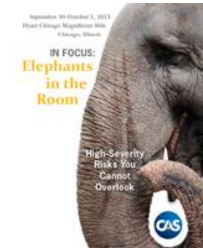
Stability Analysis

General bounds on

$$\underline{p}(t) \le \Pr(X \le t) \le \overline{p}(t), \quad \text{for } t \ge E(X) + 2\sqrt{\text{Var}(X)}$$

for Pareto distribution with α= 5 and θ= 10, given 4 moments.





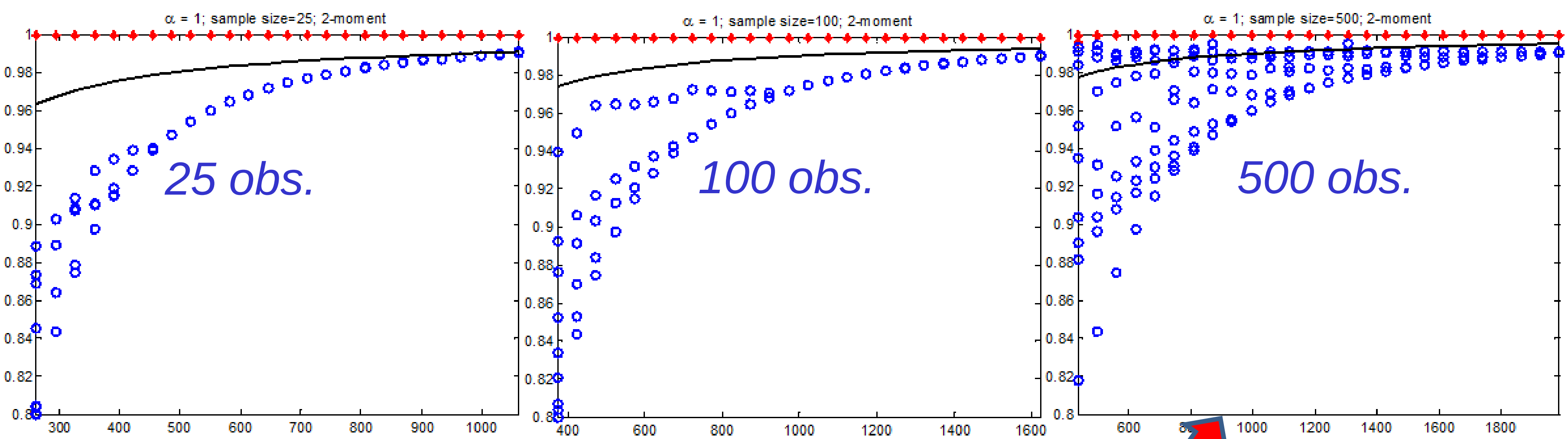
Stability Analysis (Continued)

General bounds on

$$\underline{p}(t) \le \Pr(X \le t) \le \overline{p}(t), \quad \text{for } t \ge \mathrm{E}(X) + 2\sqrt{\mathrm{Var}(X)}$$

for Pareto distribution with α= 1 and θ= 10, given 2 moments.

This distribution has no finite raw moments. We calculate the upper and lower bounds on the distribution based on the "non-existent" moments approximated from samples.



<u>A big proportion of the lower bound dots fall above the true distribution.</u>

Stability Analysis (Continued)

- Larger samples make the bounds estimation more reliable and accurate.
- Reliable bounds could be provided when samples with more than 500 observations are analyzed.
- Moment method can be used to detect the existence of higher moments.

Conclusion

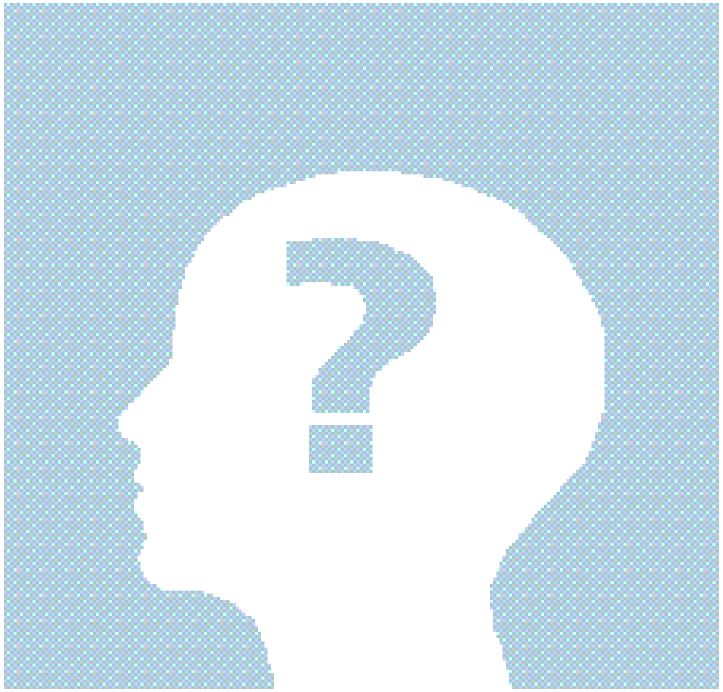
We solve univariate and bivariate moment problems numerically with SOS programming solver.

- *Calculate semiparametric upper and lower general bounds by reformulating problems as SOS programs.*
  - *Provide the best bounds on Value-at-risk given moment information.*
- *Compute the improved bounds on a distribution when the unimodal assumption is added.*
  - *Effectively narrow the general bounds.*
- *Construct Maximum-Entropy distribution with given moments.*
  - *Provide a representative distribution which has given moments, but otherwise uses as little information as possible.*
  - *Price assets with Maximum-Entropy risk neutral probabilities.*

Conclusion (Continued)

- *Analyze two bound problems on bivariate distributions*
  - *"100% confidence intervals" on extreme events given moments and support*
  - *Bounds on the sum of two variables given moments and support.*
- ***Other possible applications of our approach***
  - *Default probabilities*
  - *Prices of different fix-income securities*
  - *Inventory and supply chain management*
  - *Sensitivity analysis of the joint probabilities and VaR estimates to model misspecification*

Questions?



Thank you for your attention!