PL-2
The Matrix Inverted: A Primer in GLM Theory



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2005 CAS Seminar on Ratemaking

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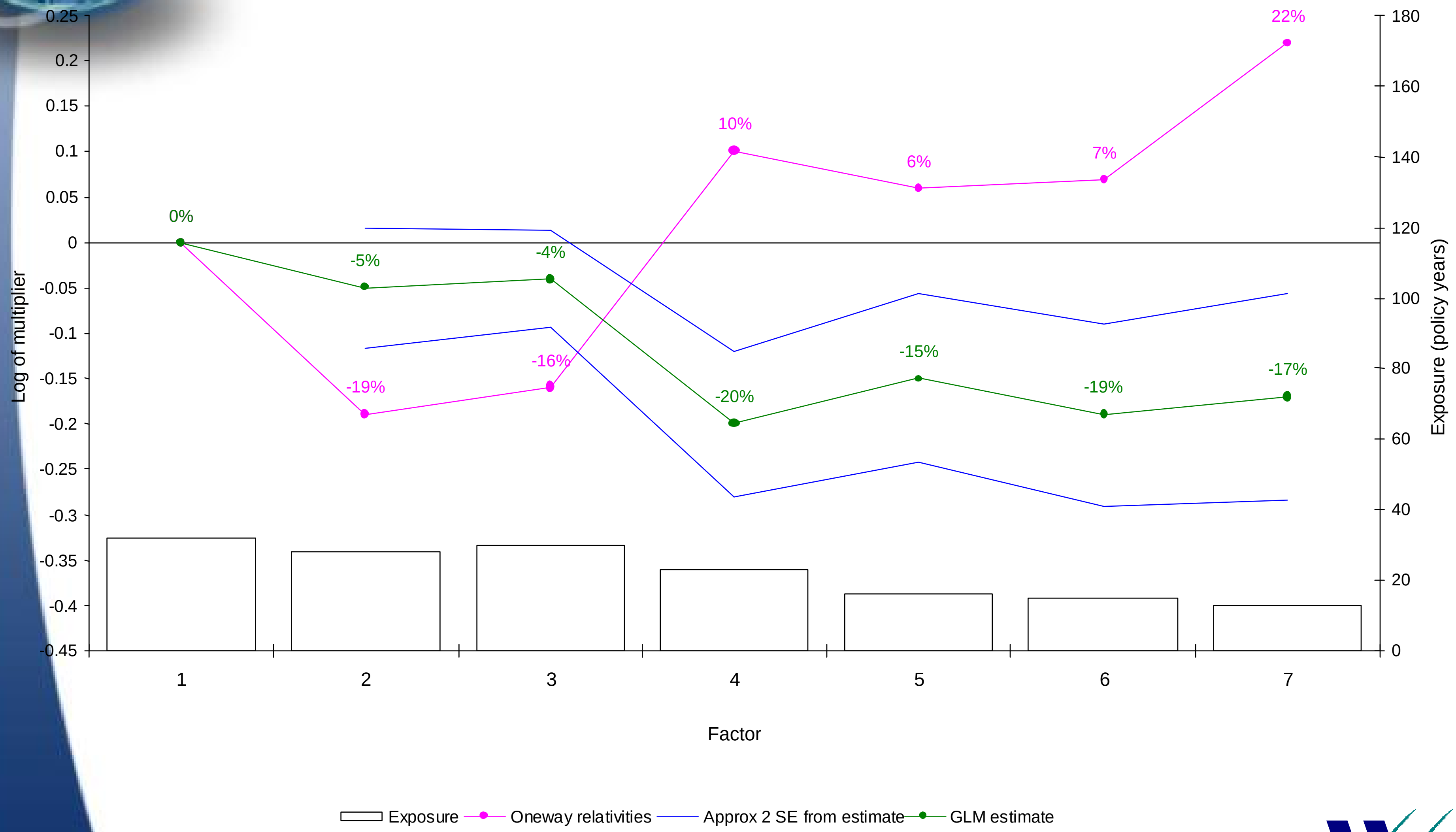
Generalized linear model benefits

- Consider all factors simultaneously
- Allow for nature of random process
- Provide diagnostics
- Robust and transparent

Example of GLM output (real UK data)



Agenda

- Formularization of GLMs
  - linear predictor, link function, offset
  - error term, scale parameter, prior weights
  - typical model forms
- Model testing
  - use only variables which are predictive
  - make sure model is reasonable
- Aliasing

Agenda

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Linear models

- Linear model $Y_i = \mu_i$ + error
- μ_i based on linear combination of measured factors
- Which factors, and how they are best combined is to be derived

$$\mu_i = \alpha + \beta.\text{age}_i + \gamma.\text{age}_i^2 + \delta.\text{height}_i.\text{age}_i \quad ✓$$

$$\mu_i = \alpha + \beta.\text{age}_i + \gamma.(\text{sex}_i\text{=female}) \quad ✓$$

$$\mu_i = (\alpha + \beta.\text{age}_i) * \exp(\delta.\text{height}_i.\text{age}_i) \quad ✗$$

Linear models - formularization

$$E[Y_i] = \mu_i = \Sigma X_{ij}\beta_j$$

$$Var[Y_i] = \sigma^2$$

$$Y_i \sim N(\mu_i, \sigma^2)$$

What is $\Sigma X_{ij}\beta_j$?

- **X** defines the explanatory variables to be included in the model
  - could be continuous variables - "variates"
  - could be categorical variables - "factors"
- $\underline{\beta}$ contains the parameter estimates which relate to the factors / variates defined by the structure of **X**
  - "the answer"

What is $X.\underline{\beta}$?

- Write $\sum X_{ij}\beta_j$ as $\mathbf{X}.\underline{\beta}$
- Consider 3 rating factors

  - age of driver ("age")

  - sex of driver ("sex")

  - age of vehicle ("car")
- Represent $\underline{\beta}$ by α, β, γ, δ, ...

What is $X.\underline{\beta}$?

- Suppose we wanted a model of the form:

$$\underline{\mu} = \alpha + \beta.\underline{age} + \gamma.\underline{age}^2 + \delta.\underline{car}^{27}.\underline{age}^{52½}$$

- **X**.$\underline{\beta}$ would need to be defined as:

$$\begin{pmatrix} 1 & age_1 & age_1^2 & car_1^{27}.age_1^{52½} \\ 1 & age_2 & age_2^2 & car_2^{27}.age_2^{52½} \\ 1 & age_3 & age_3^2 & car_3^{27}.age_3^{52½} \\ 1 & age_4 & age_4^2 & car_4^{27}.age_4^{52½} \\ 1 & age_5 & age_5^2 & car_5^{27}.age_5^{52½} \\ & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \end{pmatrix} . \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

What is $X.\underline{\beta}$?

- Suppose we wanted a model of the form:

$$\underline{\mu} = \alpha + \beta_1 \text{ if } \underline{\text{age}} < 30$$

$$+ \beta_2 \text{ if } \underline{\text{age}} \text{ } 30 - 40$$

$$+ \beta_3 \text{ if } \underline{\text{age}} > 40$$

$$+ \gamma_1 \text{ if } \underline{\text{sex}} \text{ male}$$

$$+ \gamma_2 \text{ if } \underline{\text{sex}} \text{ female}$$

What is X.β ?

$$
\begin{array}{c}
\\
\\
1\\
2\\
3\\
4\\
5\\
\\
\\
\end{array}
\left(
\begin{array}{c c c c c c}
 & \multicolumn{3}{c}{\text{Age}} & \multicolumn{2}{c}{\text{Sex}} \\
 & <30 & 30\text{-}40 & >40 & M & F \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\right)
\cdot
\left(
\begin{array}{c}
\alpha \\
\beta_1 \\
\beta_2 \\
\beta_3 \\
\gamma_1 \\
\gamma_2
\end{array}
\right)
$$

What is $X.\underline{\beta}$?

- Suppose we wanted a model of the form:

$$\underline{\mu} = \alpha + \beta_1 \text{ if } \underline{\text{age}} < 30$$

$$+ \beta_2 \text{ if } \underline{\text{age}} \text{ 30 - 40}$$

$$+ \beta_3 \text{ if } \underline{\text{age}} > 40$$

$$+ \gamma_1 \text{ if } \underline{\text{sex}} \text{ male}$$

$$+ \gamma_2 \text{ if } \underline{\text{sex}} \text{ female}$$

What is $X.\underline{\beta}$?

- Suppose we wanted a model of the form:

$$\underline{\mu} = \alpha + \beta_1 \text{ if } \underline{\text{age}} < 30$$

~~$+ \beta_2$ if age 30 - 40~~

"Base levels"

$$+ \beta_3 \text{ if } \underline{\text{age}} > 40$$

~~$+ \gamma_1$ if sex male~~

$$+ \gamma_2 \text{ if } \underline{\text{sex}} \text{ female}$$

X.β having adjusted for base levels

		Age			Sex	
		<30	30-40	>40	M	F
1	1	0	1	0	1	0
2	1	1	0	0	1	0
3	1	1	0	0	0	1
4	1	0	0	1	1	0
5	1	0	1	0	0	1
	....	....	....	....	....	....
	....	....	....	....	....	....

$$\cdot \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_1 \\ \gamma_2 \end{pmatrix}$$

Linear models - formularization

$$E[Y_i] = \mu_i = \Sigma X_{ij}\beta_j$$

$$Var[Y_i] = \sigma^2$$

$$Y_i \sim N(\mu_i, \sigma^2)$$

Generalized linear models

$$\mu_i = f(\ \alpha + \beta.age_i + \gamma.age_i^2 + \delta.height_i.age_i\ )$$

$$\mu_i = f(\ \alpha + \beta.age_i + \gamma.(sex_i{=}female)\ )$$

Generalized linear models

$$\mu_i = g^{-1}(\alpha + \beta.age_i + \gamma.age_i^2 + \delta.height_i.age_i)$$

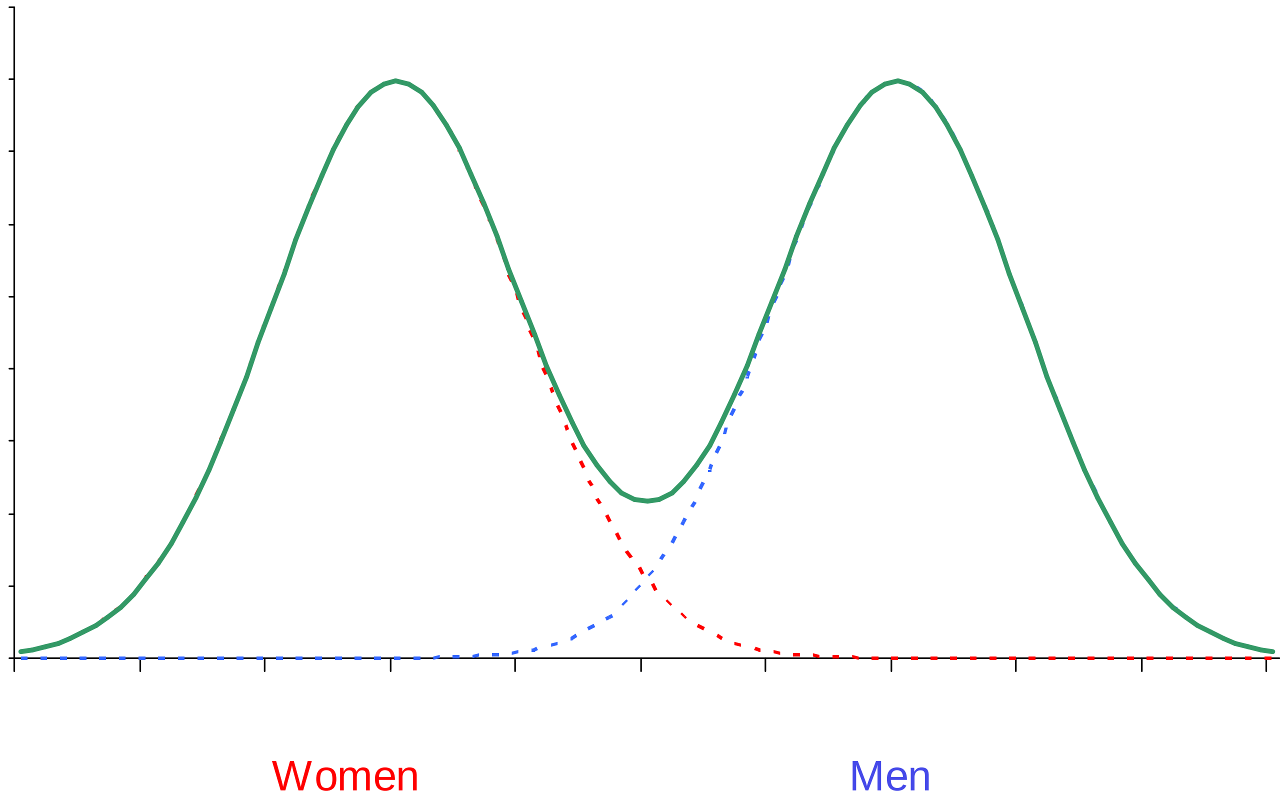
$$\mu_i = g^{-1}(\alpha + \beta.age_i + \gamma.(sex_i\text{=female}))$$

Generalized linear models

Linear Models	Generalized Linear Models
$E[Y_i] = \mu_i = \Sigma X_{ij}\beta_j$	$E[Y_i] = \mu_i = g^{-1}(\Sigma X_{ij}\beta_j + \xi_i)$
$Var[Y_i] = \sigma^2$	$Var[Y_i] = \phi V(\mu_i)/\omega_i$
Y from Normal distribution	Y from a distribution from the exponential family

Generalized linear models

- Each observation i from distribution with mean μ_i



Generalized linear models

$$E[Y_i] = \mu_i = g^{-1}(\sum_j X_{ij}\beta_j + \xi_i)$$

$$Var[Y_i] = \phi . V(\mu_i)/\omega_i$$

Generalized linear models

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\,\mathbf{X}.\underline{\beta} + \underline{\xi}\,)$$

$$Var[\underline{Y}] = \phi.V(\underline{\mu}) / \underline{\omega}$$

Generalized linear models

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\,\mathbf{X}.\underline{\beta}\qquad)$$

Observed thing (data)

Some function (user defined)

Some matrix based on data (user defined) as per linear models

Parameters to be estimated (the answer!)

What is $g^{-1}(\mathbf{X}.\underline{\beta})$?

$$\underline{Y} = g^{-1}(\mathbf{X}.\underline{\beta}) + \text{error}$$

Assuming a model with three categorical factors, each observation can be expressed as:

$$Y_{ijk} = g^{-1}(\alpha + \beta_i + \gamma_j + \delta_k) + \text{error}$$

$$\beta_2 = \gamma_1 = \delta_3 = 0$$

age is in group i

sex is in group j

car is in group k

What is $g^{-1}(X.\underline{\beta})$?

- $g(x) = x \quad \Rightarrow \quad Y_{ijk} = \alpha + \beta_i + \gamma_j + \delta_k + \text{error}$

- $g(x) = \ln(x) \quad \Rightarrow \quad Y_{ijk} = e^{(\alpha + \beta_i + \gamma_j + \delta_k)} + \text{error}$

  $= A.B_i.C_j.D_k \quad + \text{error}$

  where $B_i = e^{\beta_i}$ etc

- Multiplicative form common for frequency and amounts

Multiplicative model

$ 207.10 x

Age	Factor
17	2.52
18	2.05
19	1.97
20	1.85
21-23	1.75
24-26	1.54
27-30	1.42
31-35	1.20
36-40	1.00
41-45	0.93
46-50	0.84
50-60	0.76
60+	0.78

Group	Factor
1	0.54
2	0.65
3	0.73
4	0.85
5	0.92
6	0.96
7	1.00
8	1.08
9	1.19
10	1.26
11	1.36
12	1.43
13	1.56

Sex	Factor
Male	1.00
Female	1.25

Area	Factor
A	0.95
B	1.00
C	1.09
D	1.15
E	1.18
F	1.27
G	1.36
H	1.44

E(losses) = $ 207.10 x 1.42 x 0.92 x 1.00 x 1.15 = $ 311.14

Generalized linear models

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\mathbf{X}.\underline{\beta} + \underline{\xi})$$

"Offset"

Eg $\underline{Y}$ = claim *numbers*

Smith: Male, 30, Ford, 1 years, 2 claims

Jones: Female, 40, VW, **½** year, 1 claim

What is ξ ?

- $g(x) = \ln(x)$
- $\xi_{ijk} = \ln(\text{exposure}_{ijk})$
- $E[Y_{ijk}] = e^{(\alpha + \beta_i + \gamma_j + \delta_k + \xi_{ijk})}$

$$= A.B_i.C_j.D_k.\ e^{(\ln(\text{exposure}_{ijk}))}$$

$$= A.B_i.C_j.D_k.\ \text{exposure}_{ijk}$$

Restricted models

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\mathbf{X}.\underline{\beta} + \underline{\xi})$$

Offset

- Constrain model (eg territory, ABS discount)
- Other factors adjusted to compensate

Generalized linear models

$$E[\underline{Y}] = \underline{\mu} = g^{-1}(\mathbf{X}.\underline{\beta} + \underline{\xi})$$

$$Var[\underline{Y}] = \phi.V(\underline{\mu}) / \underline{\omega}$$

Generalized linear models

$$\mathrm{Var}[\underline{Y}] = \phi.V(\underline{\mu}) \,/\, \underline{\omega}$$

Normal: $\phi = \sigma^2$, $V(x) = 1 \Rightarrow \mathrm{Var}[\underline{Y}] = \sigma^2.\underline{1}$

Poisson: $\phi = 1$, $V(x) = x \Rightarrow \mathrm{Var}[\underline{Y}] = \underline{\mu}$

Gamma: $\phi = k$, $V(x) = x^2 \Rightarrow \mathrm{Var}[\underline{Y}] = k\underline{\mu}^2$

The scale parameter

$$\mathrm{Var}[\underline{Y}] = \phi . V(\underline{\mu}) / \underline{\omega}$$

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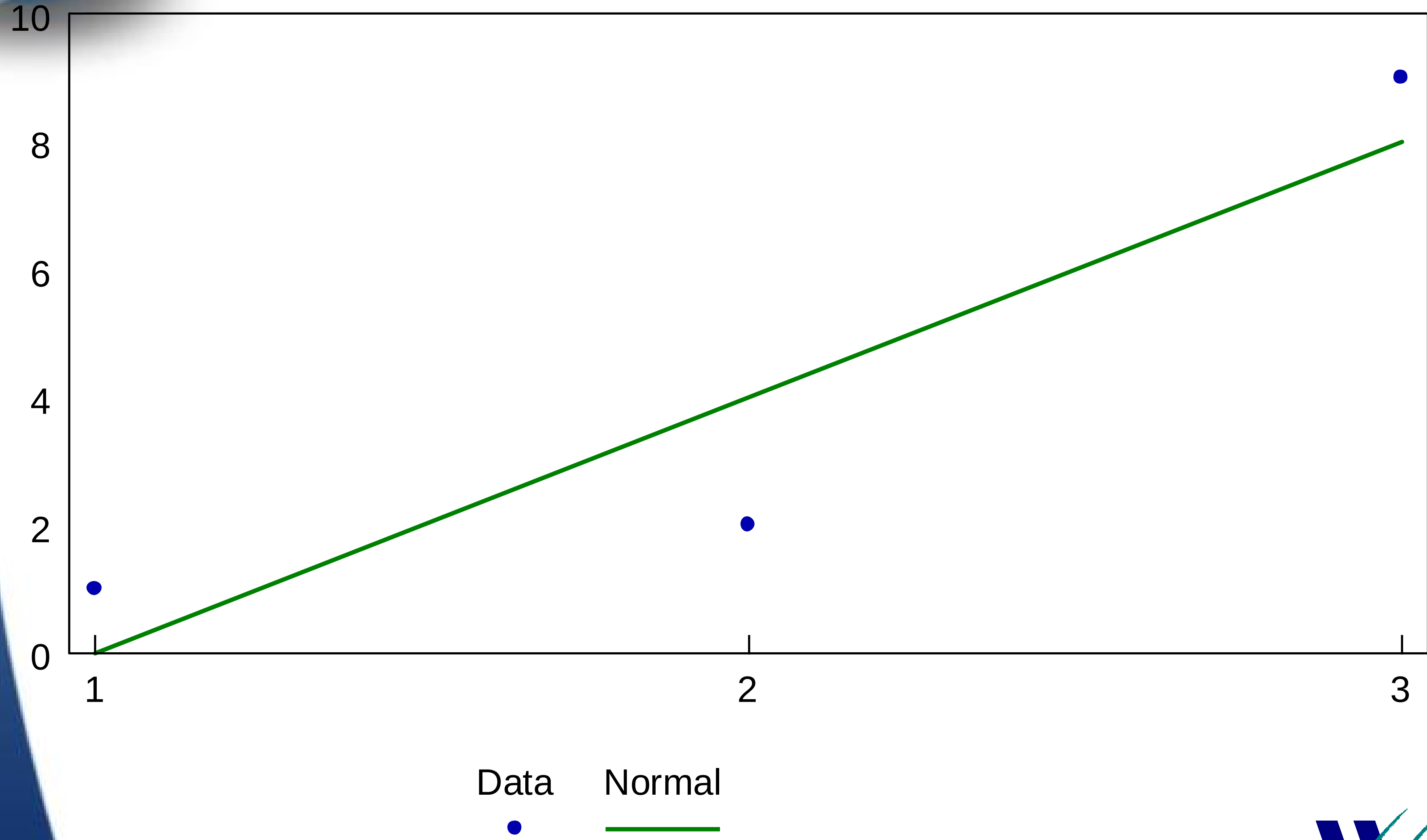
Example of effect of changing assumed error - 1



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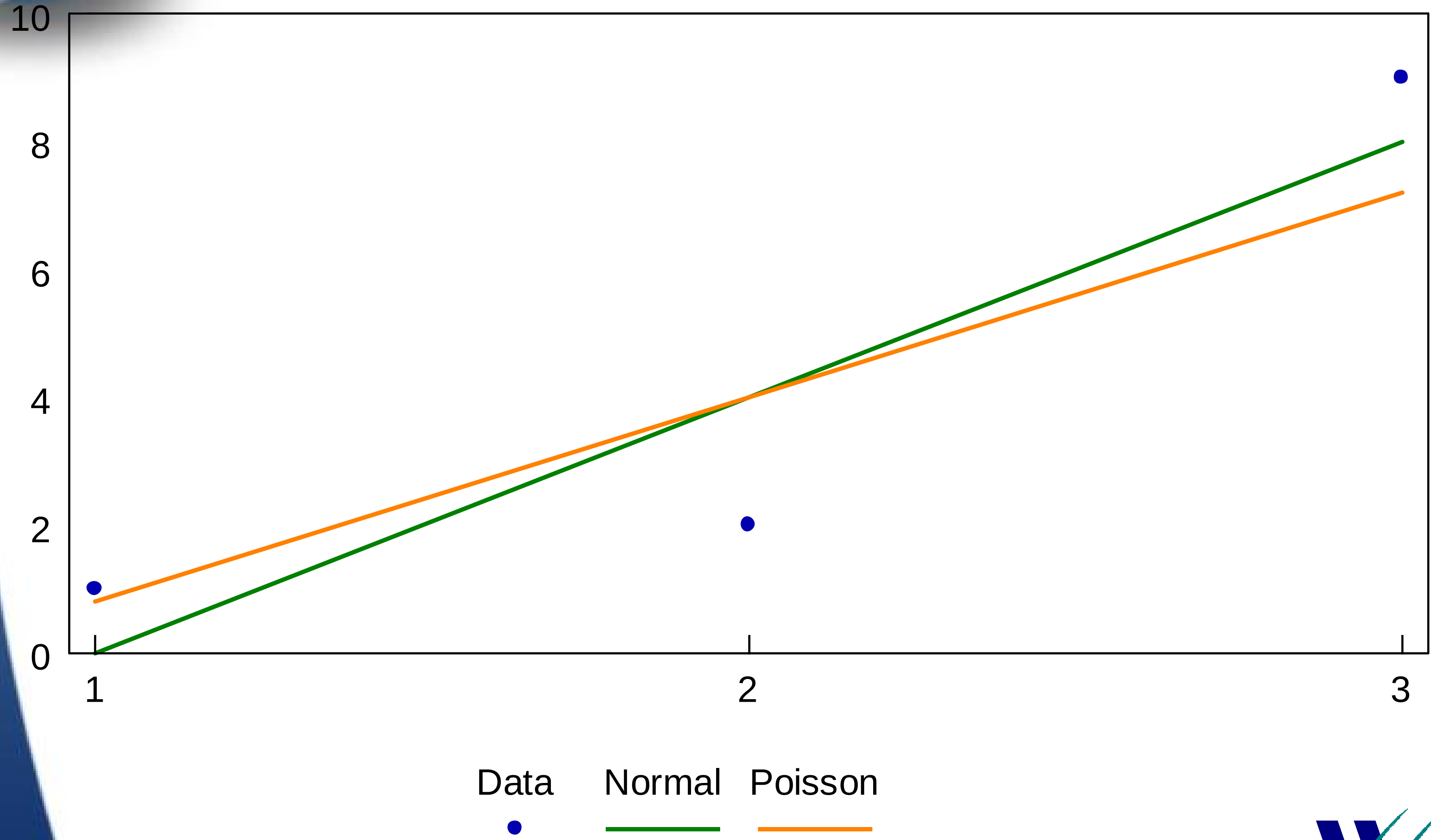

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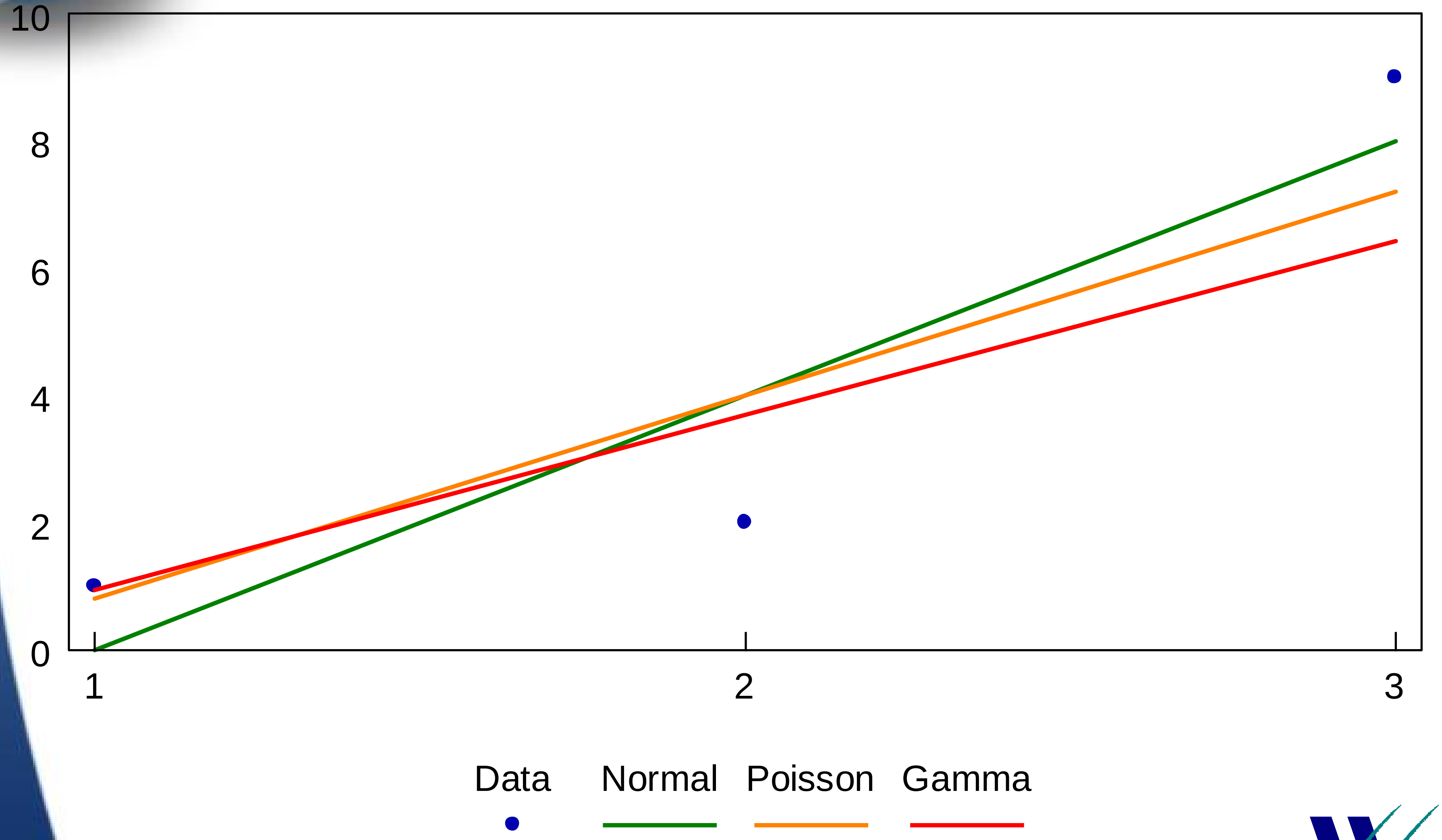
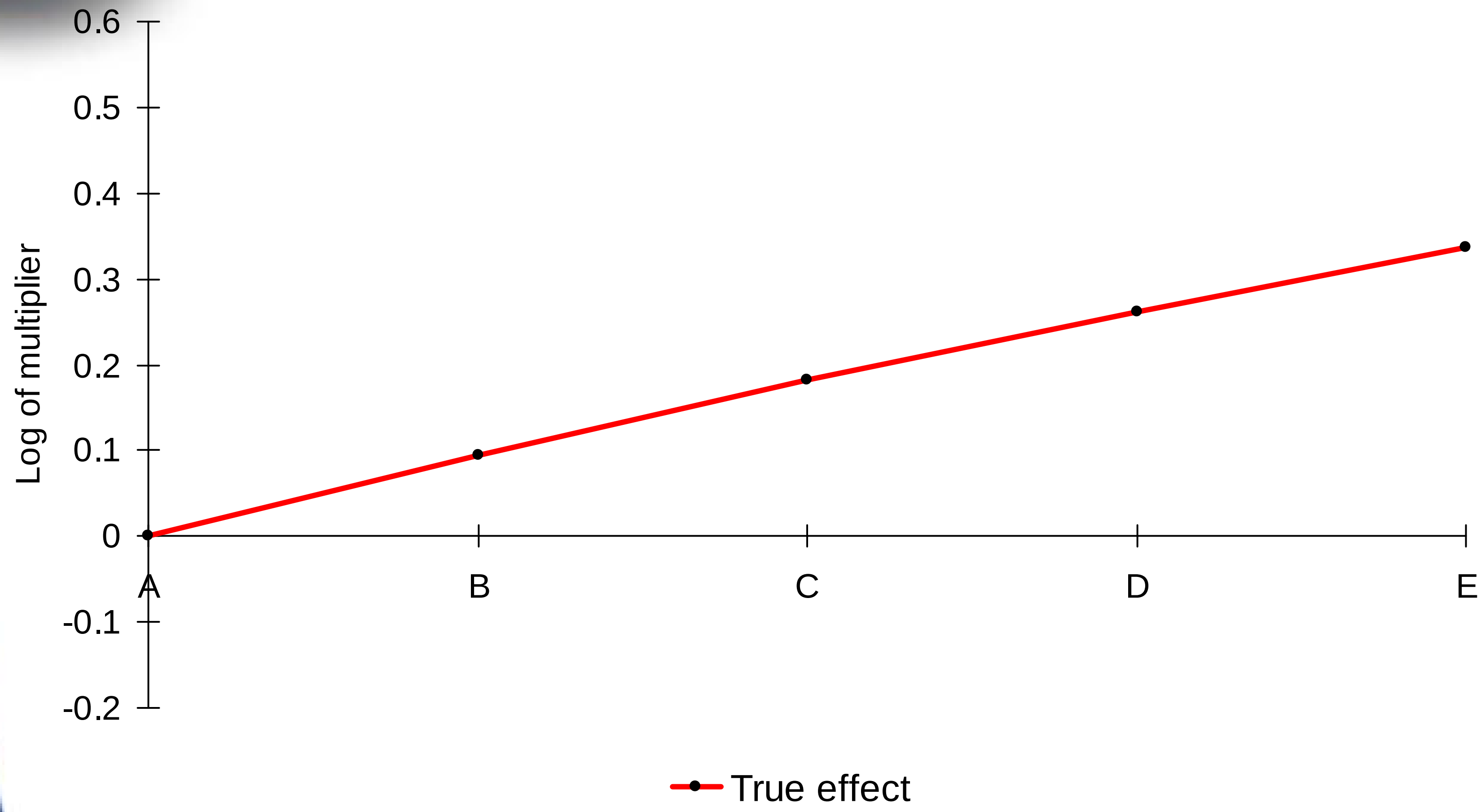


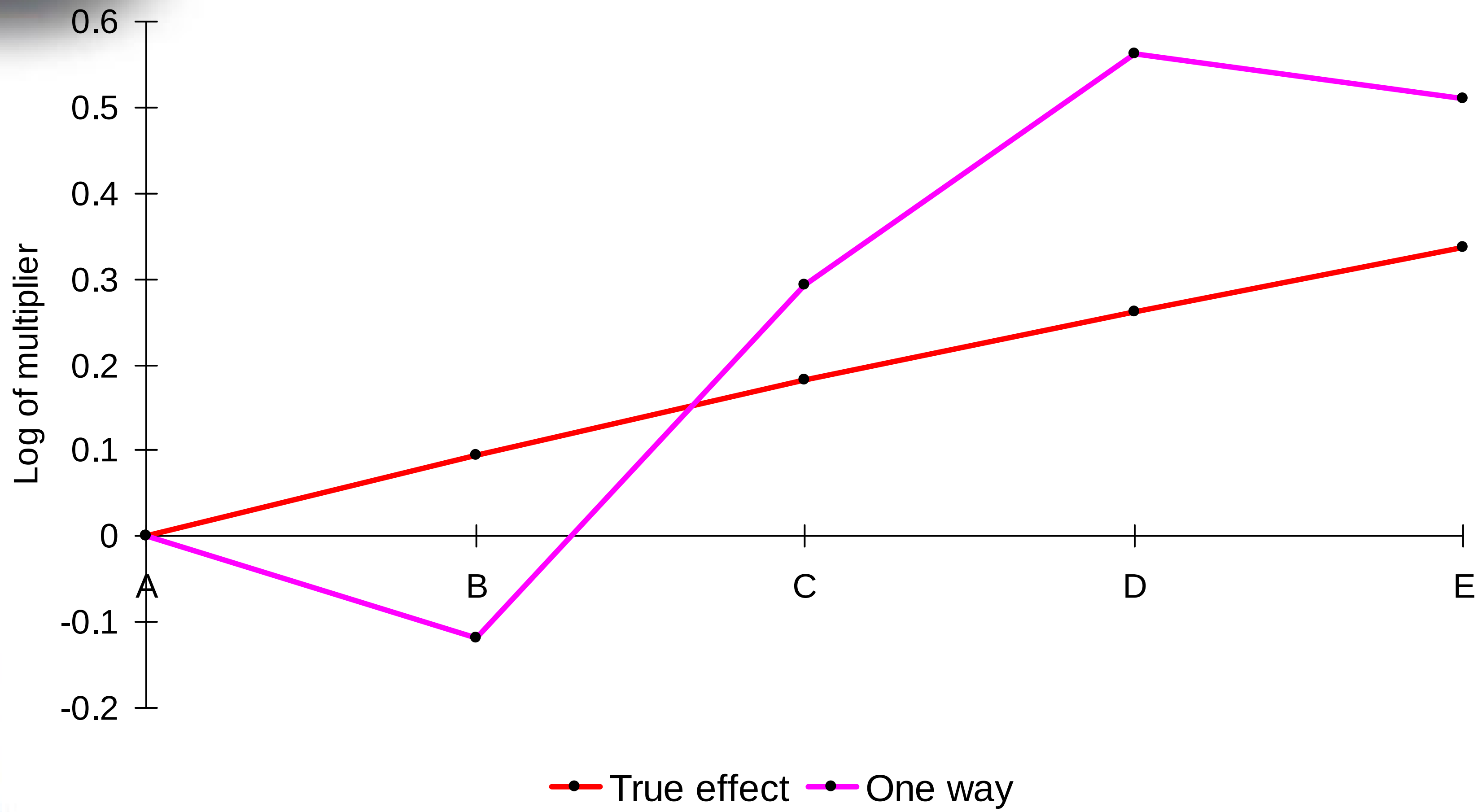
Example of effect of changing assumed error - 2

- Example portfolio with five rating factors, each with five levels A, B, C, D, E
- Typical correlations between those rating factors
- Assumed true effect of factors
- Claims randomly generated (with Gamma)
- Random experience analyzed by three models

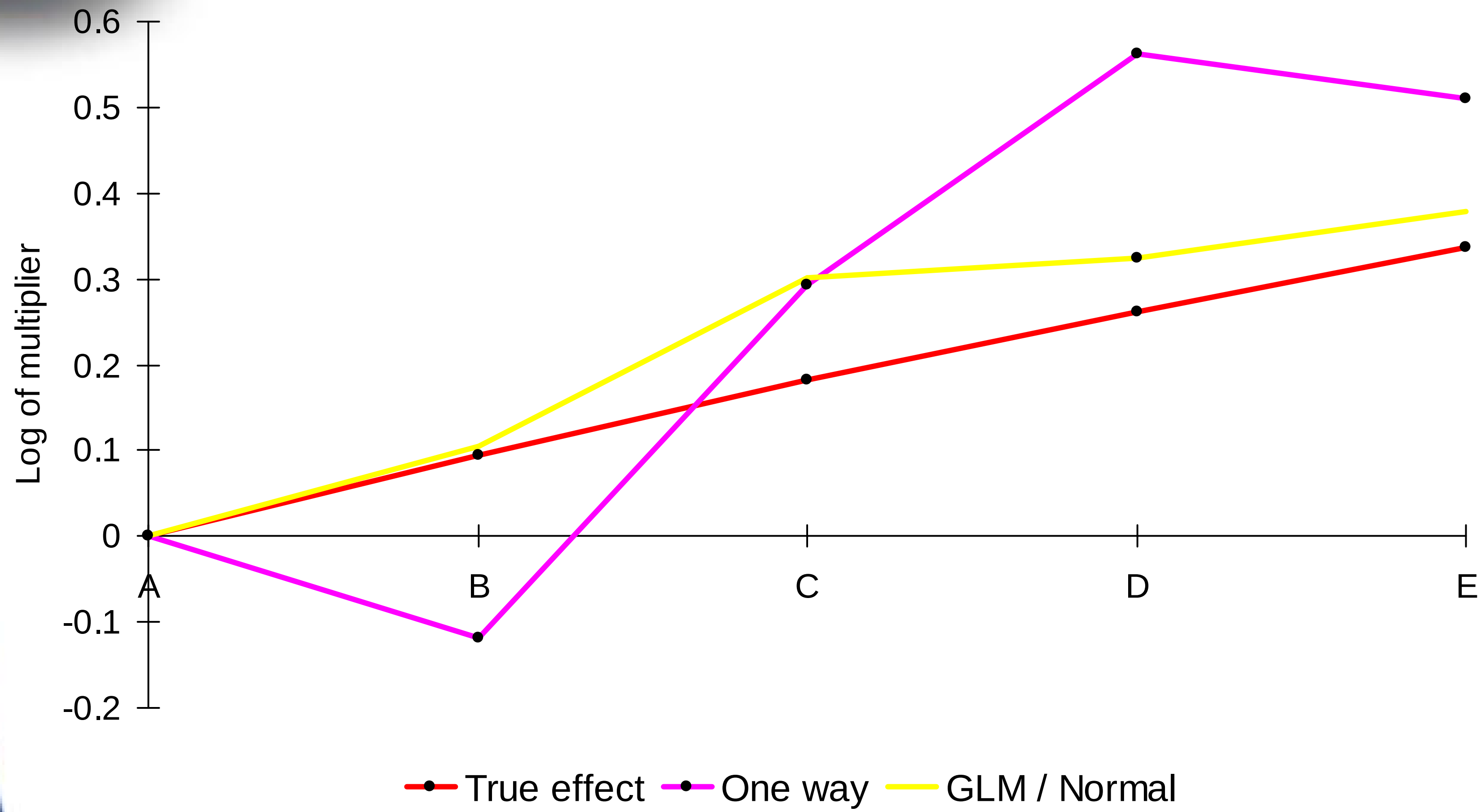
Example of effect of changing assumed error - 2



Example of effect of changing assumed error - 2



Example of effect of changing assumed error - 2



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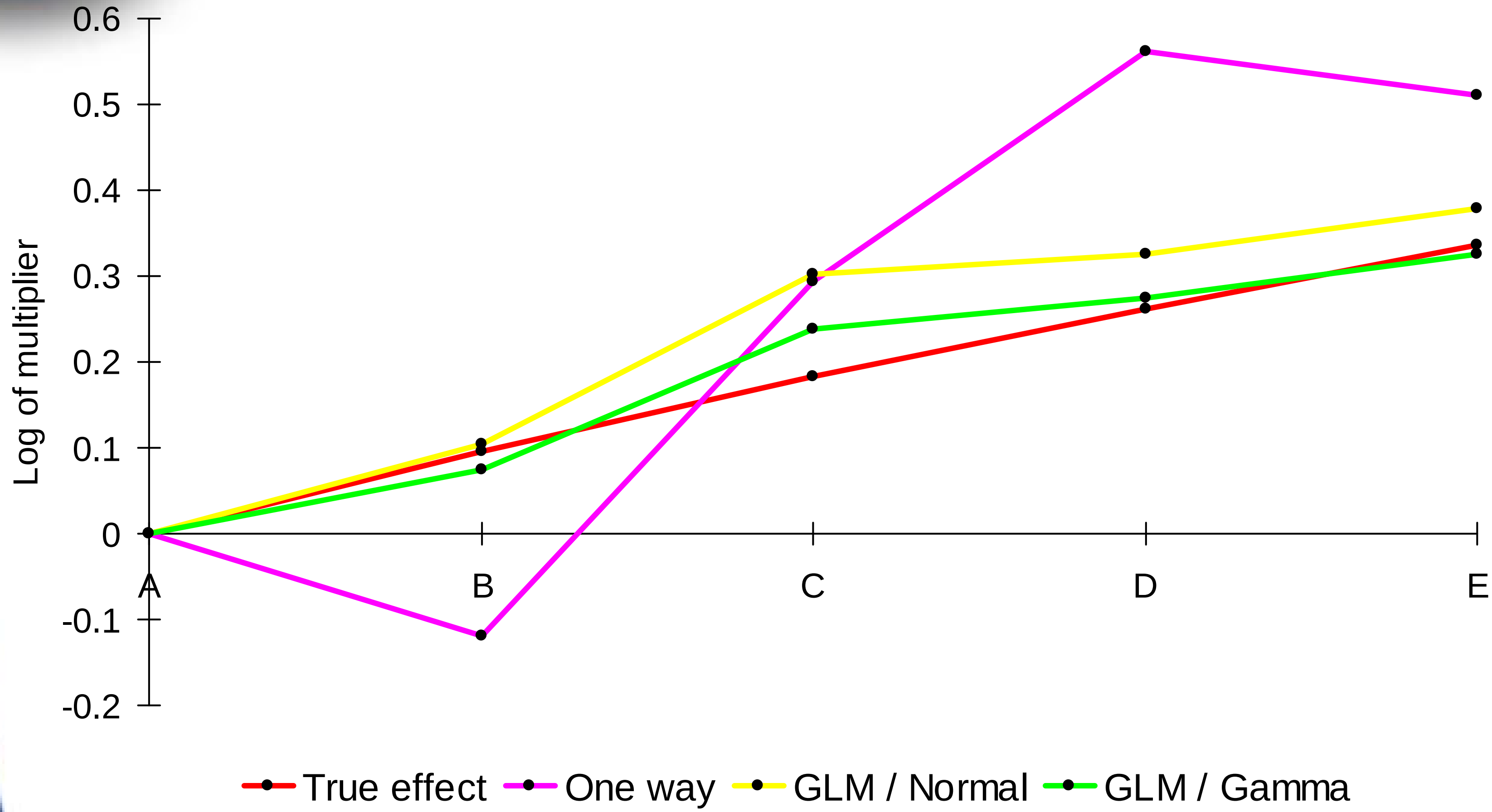

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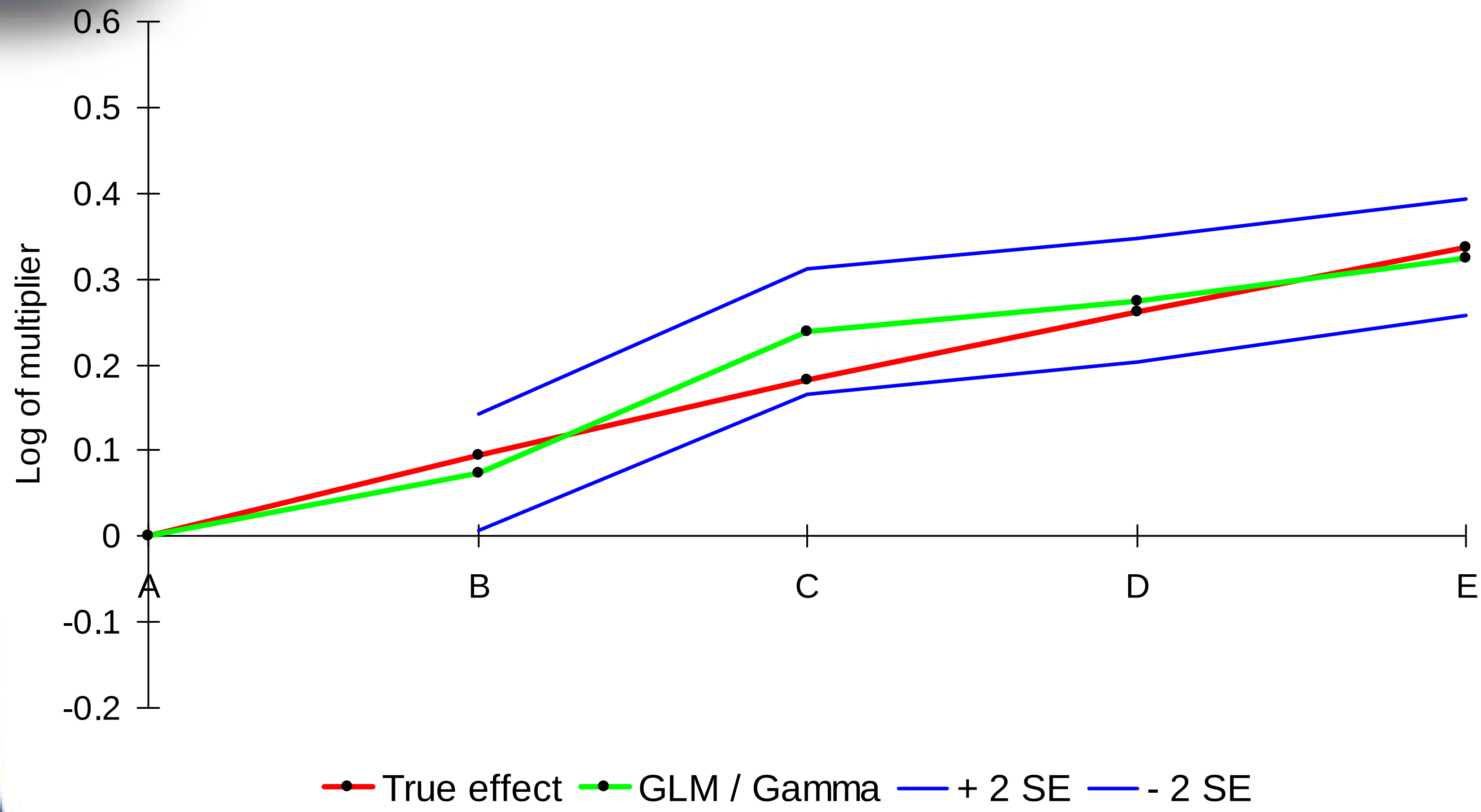


Prior weights

$$\text{Var}[\underline{Y}] = \phi.V(\underline{\mu}) \,/\, \underline{\omega}$$

- Exposure
- Other credibility

Eg $\underline{Y}$ = claim *frequency*

Smith: Male, 30, Ford, 1 years, 2 claims, 100%

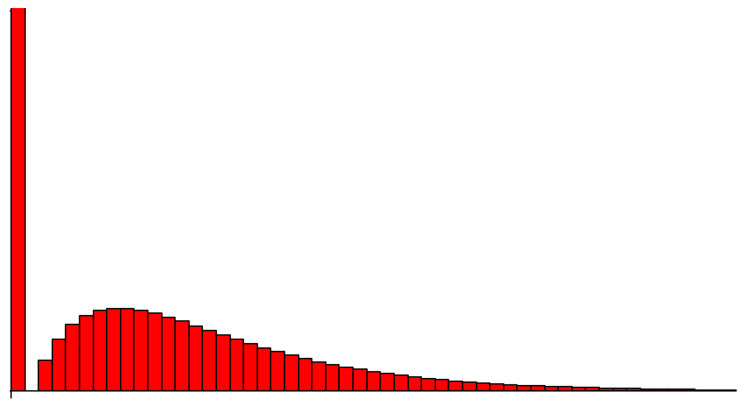
Jones: Female, 40, VW, ½ year, 1 claim, 100%

Typical model forms

Y	Claim frequency	Claim number	Average claim amount	Probability (eg lapses)
g(x)	ln(x)	ln(x)	ln(x)	ln(x/(1-x))
Error	Poisson	Poisson	Gamma	Binomial
ϕ V(x)	1 x	1 x	estimate x^2	1 x(1-x)
ω	exposure	1	# claims	1
ξ	0	ln(exposure)	0	0

Tweedie distributions

- Incurred losses have a point mass at zero and then a continuous distribution
- Poisson and gamma not suited to this



- Tweedie distribution has point mass and parameters which can alter the shape to be like Poisson and gamma above zero

$$f_Y(y;\theta,\lambda,\alpha)=\sum_{n=1}^{\infty}\frac{\left\{(\lambda\omega)^{1-\alpha}\kappa_\alpha(-1/y)\right\}^n}{\Gamma(-n\alpha)n!y}.\exp\left\{\lambda\omega[\theta_0 y-\kappa_\alpha(\theta_0)]\right\}\quad \text{for } y>0$$

$$p(Y=0)=\exp\left\{-\lambda\omega\kappa_\alpha(\theta_0)\right\}$$

Generalized linear models

$$\mathrm{Var}[\underline{Y}] = \phi . V(\underline{\mu}) \,/\, \underline{\omega}$$

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Tweedie: $\phi = k$, $V(x) = x^p \Rightarrow \mathrm{Var}[\underline{Y}] = k\underline{\mu}^p$

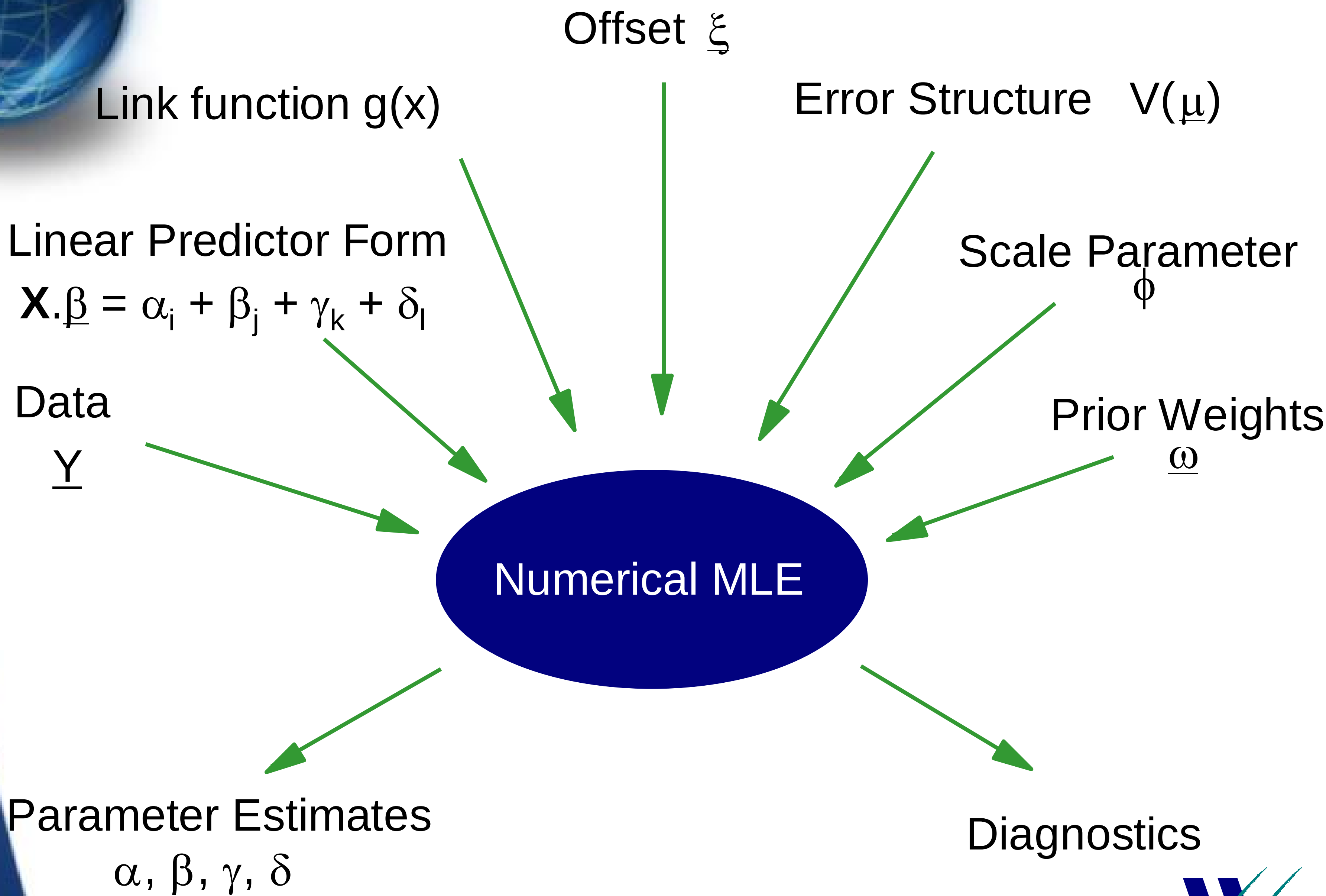
Tweedie distributions

Tweedie: $\phi = k$, $V(x) = x^p \Rightarrow \mathrm{Var}[\underline{Y}] = k\underline{\mu}^p$

- Defines a valid distribution for $p<0$, $1<p<2$, $p>2$
- Can be considered as Poisson/gamma process for $1<p<2$
- Typical values of p for insurance incurred claims around, or just under, 1.5

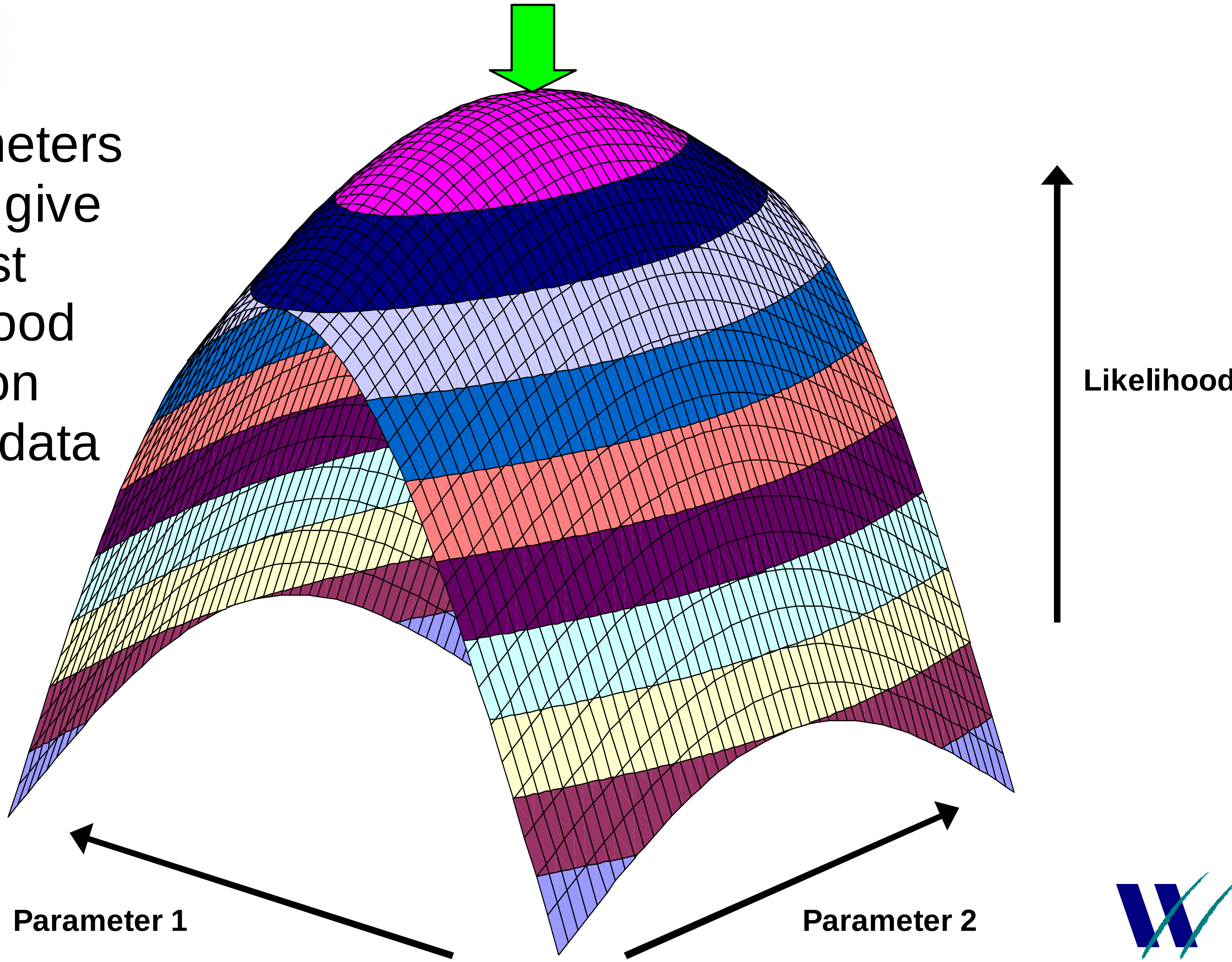
Tweedie conclusions

- Helpful when important to fit to pure premium
- Often similar results to traditional approach but differences may occur if numbers and amounts models have effects which are both large and insignificant
- No information about whether frequencies or amounts are driving result


Offset $\underline{\xi}$
Link function g(x)
Error Structure V($\underline{\mu}$)
Linear Predictor Form
$\mathbf{X}.\underline{\beta} = \alpha_i + \beta_j + \gamma_k + \delta_l$
Scale Parameter
ϕ
Data
$\underline{Y}$
Prior Weights
$\underline{\omega}$
Numerical MLE
Parameter Estimates
$\alpha, \beta, \gamma, \delta$
Diagnostics

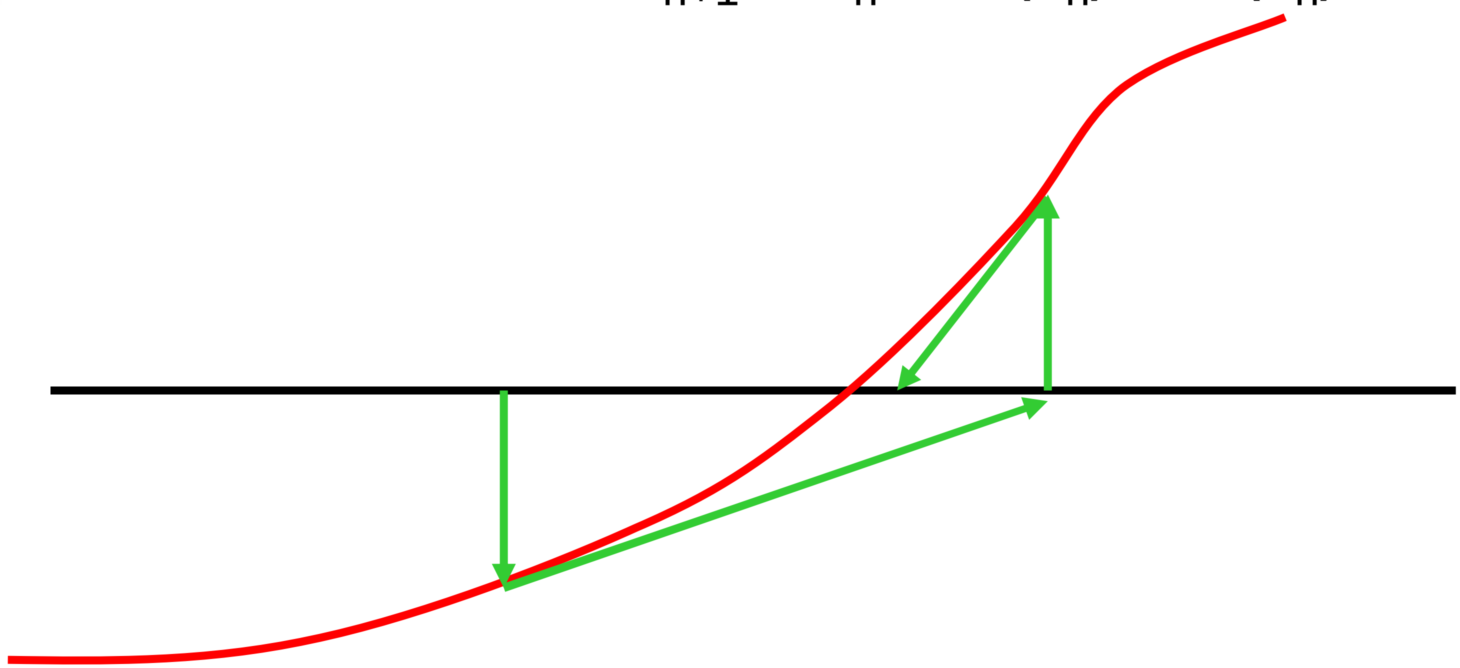
Maximum likelihood estimation

- Seek parameters which give highest likelihood function given data



Newton-Raphson

- In one dimension: $x_{n+1} = x_n - f'(x_n) / f''(x_n)$



- In n dimensions: $\underline{\beta}_{n+1} = \underline{\beta}_n - \mathbf{H}^{-1}.\underline{s}$

where $\underline{\beta}$ is the vector of the parameter estimates (with *p* elements), $\underline{s}$ is the vector of the first derivatives of the log-likelihood and **H** is the (*p*p*) matrix containing the second derivatives of the log-likelihood

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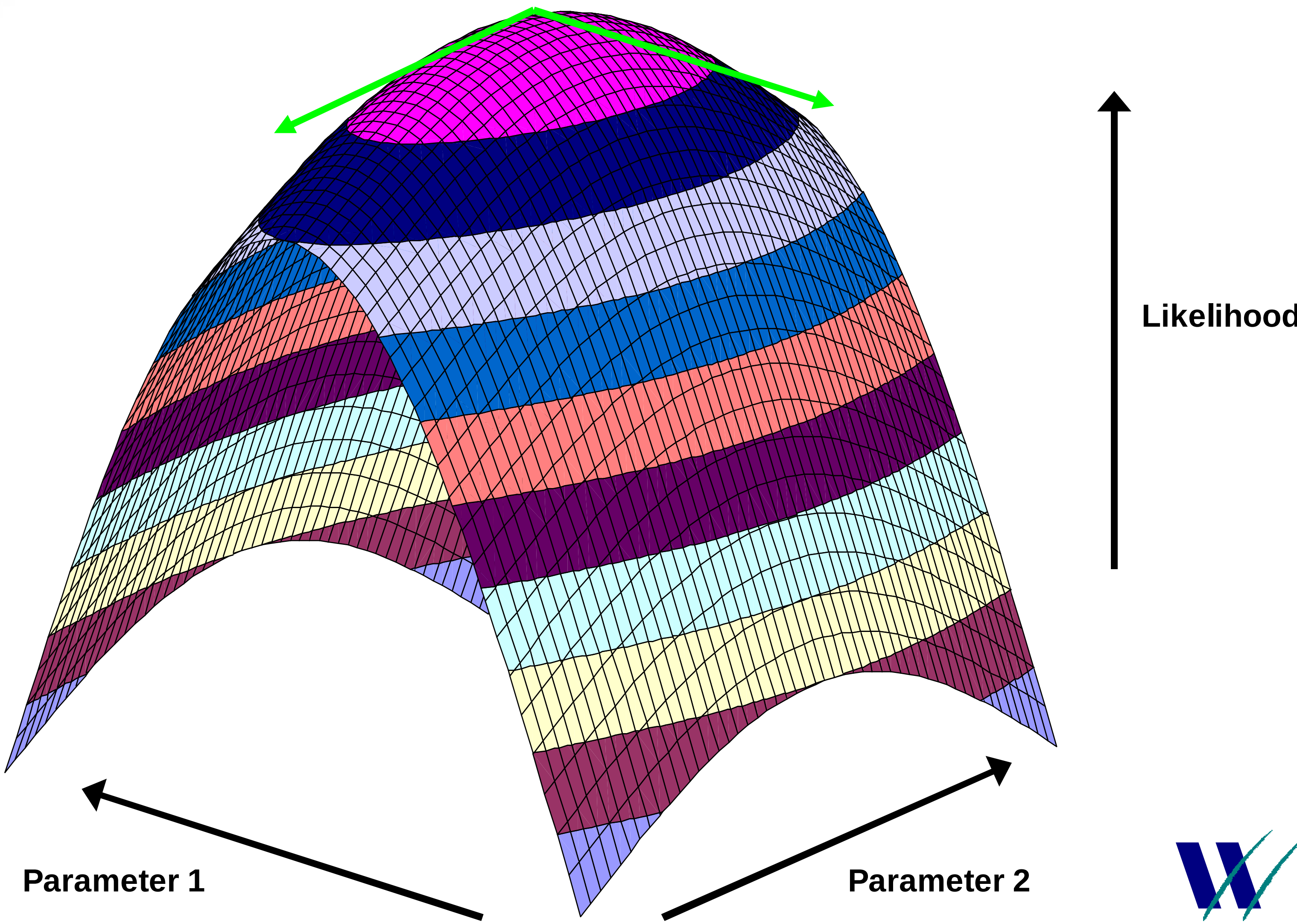
Model testing

- Use only those variables which are predictive
  - standard errors of parameter estimates
  - F tests / χ^2 tests on deviances

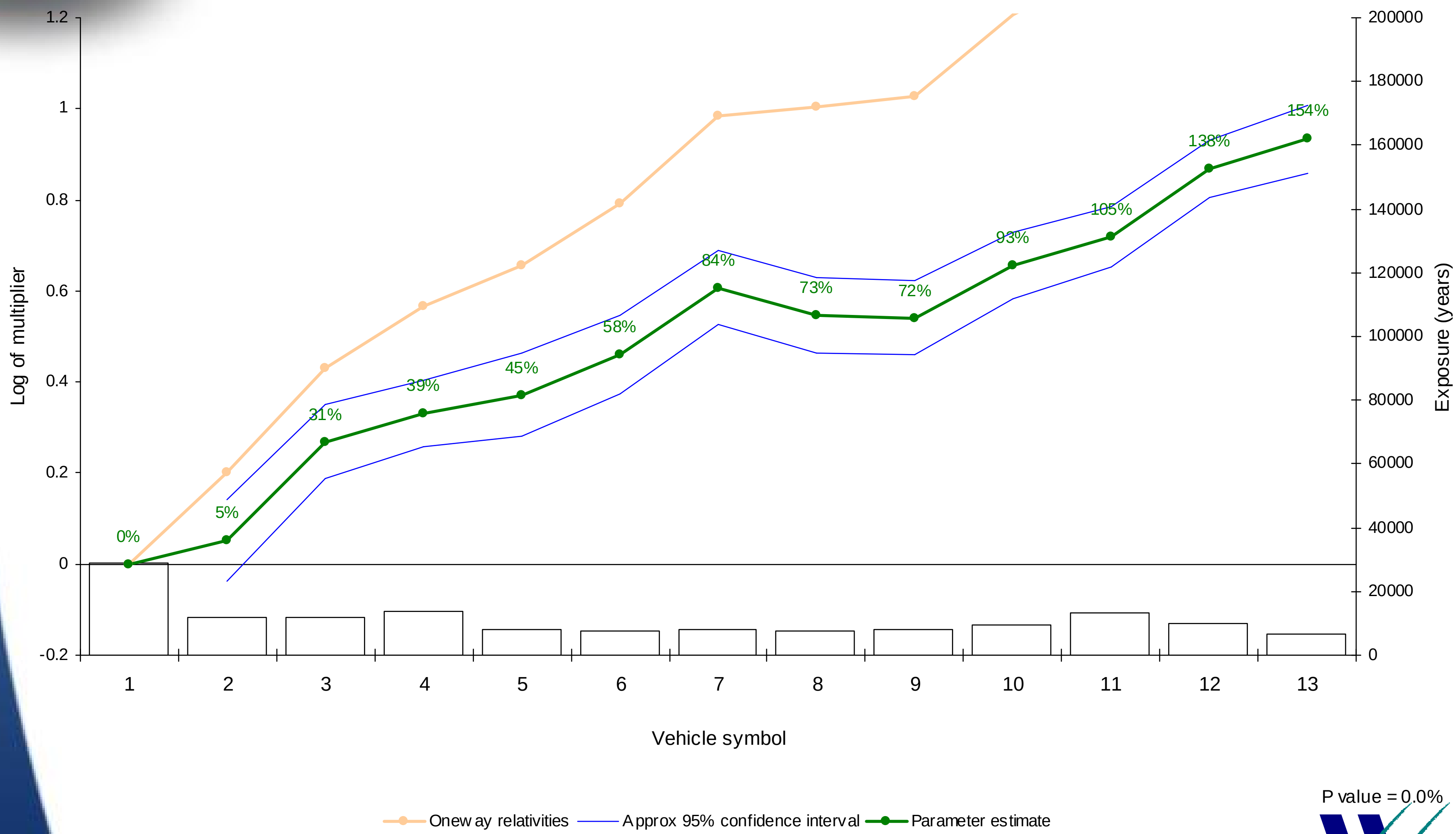
- Make sure the model is reasonable
  - histogram of deviance residuals
  - residual vs fitted value
  - Box Cox link function investigation

Standard errors

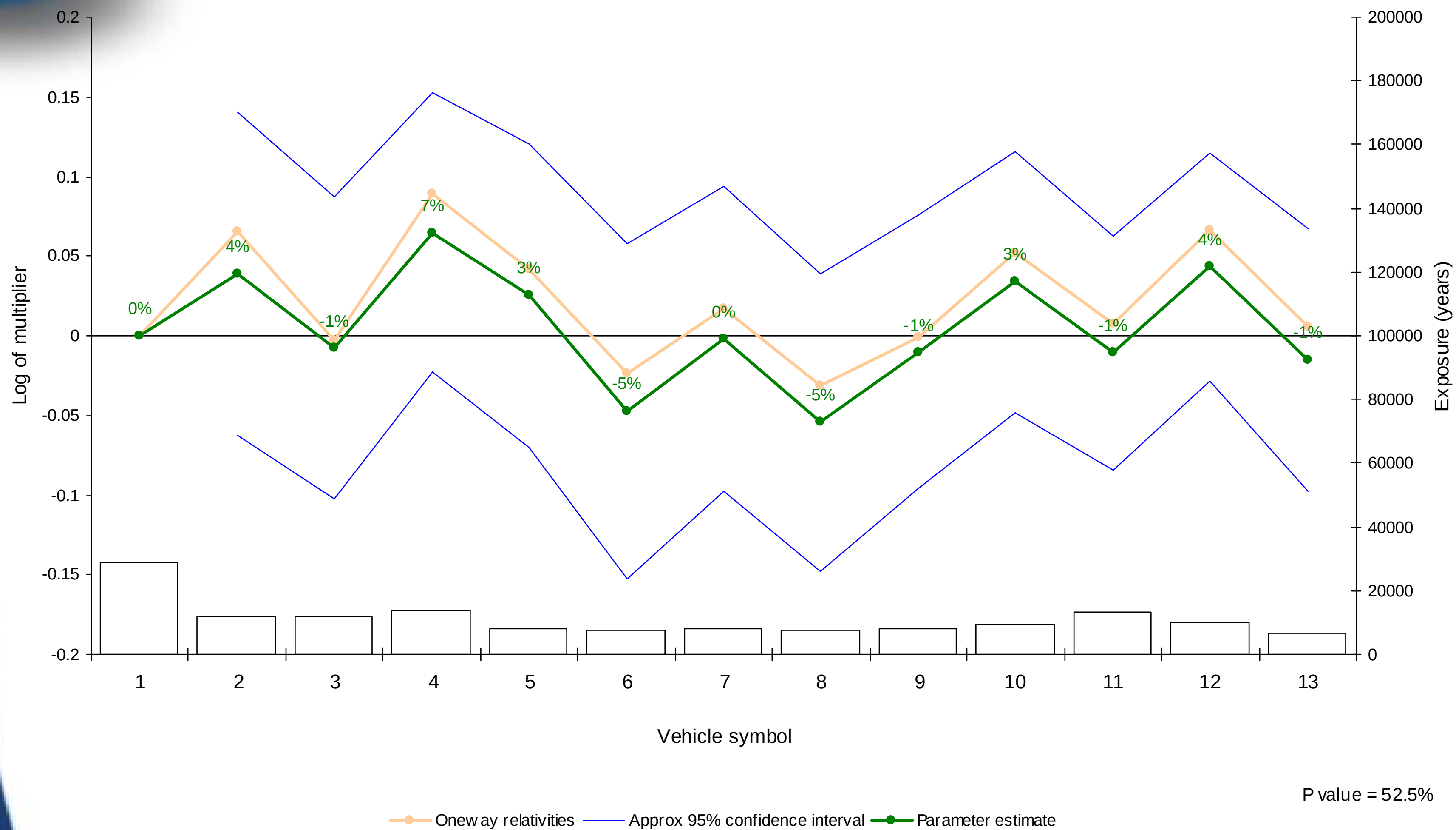
- Roughly speaking, for a parameter p: SE = -1 / (∂^2/ ∂p^2 Likelihood)



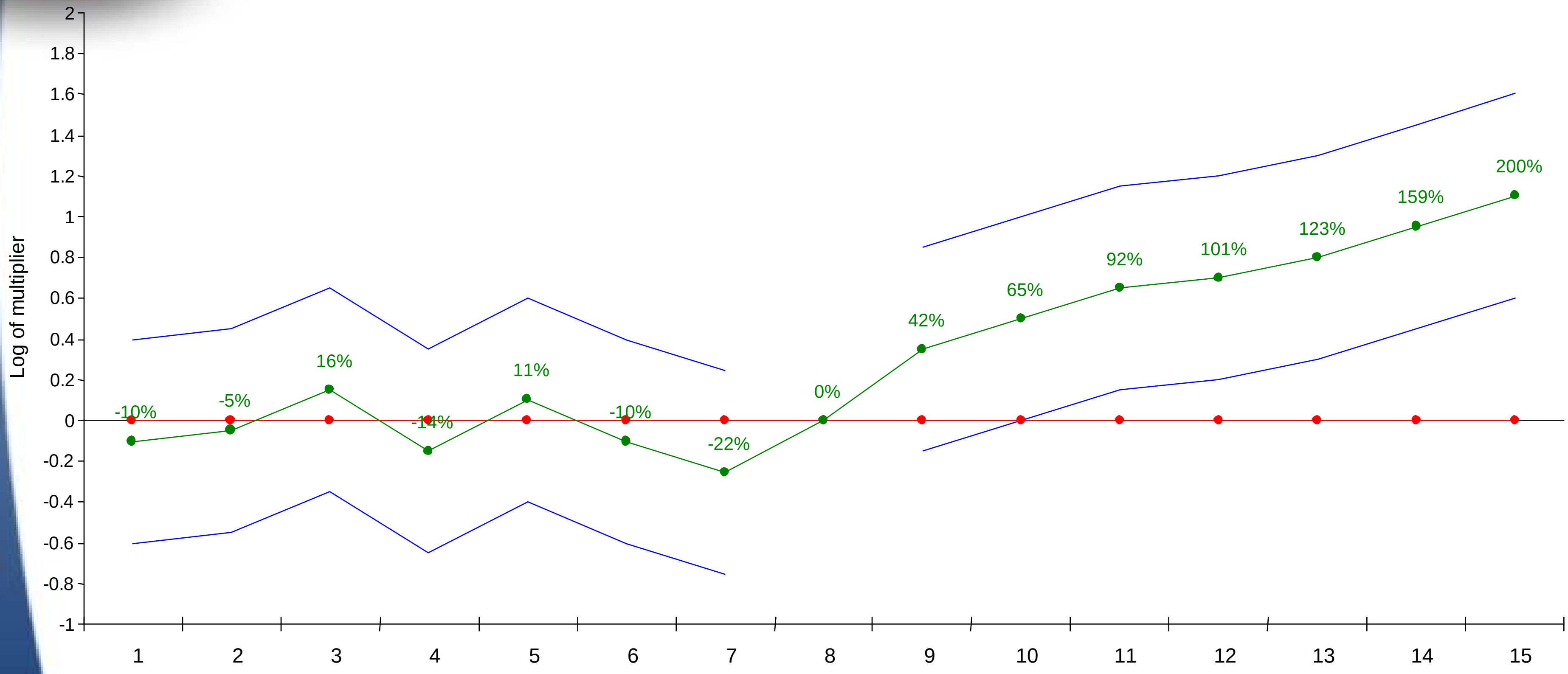
GLM output (significant factor)



GLM output (insignificant factor)



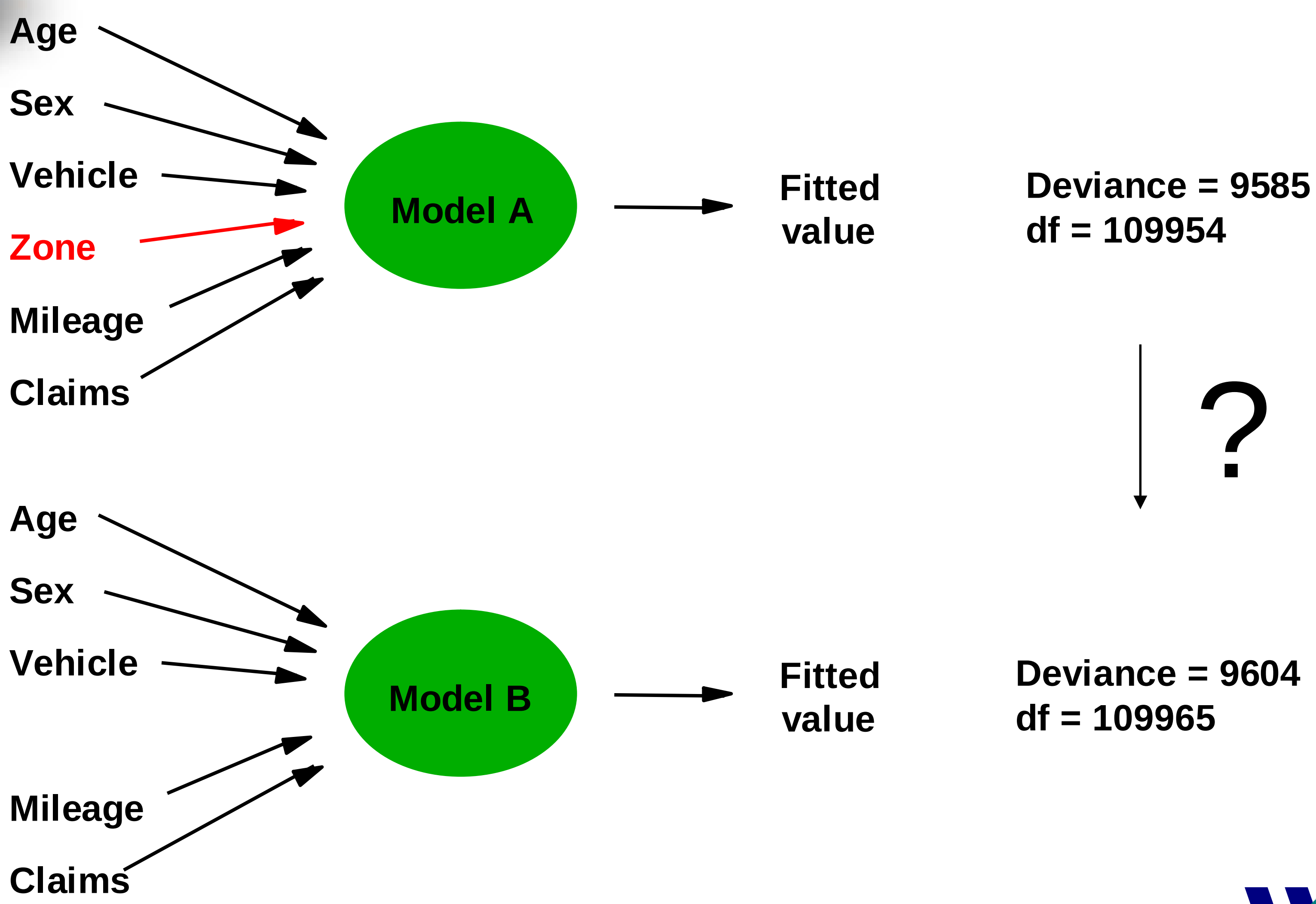
Awkward cases



Deviances

- Single figure measure of goodness of fit
- Try model with & without a factor
- Statistical tests show the theoretical significance given the extra parameters

Deviances


Age
Sex
Vehicle
Zone
Mileage
Claims
Model A
Fitted value
Deviance = 9585
df = 109954
?
Age
Sex
Vehicle
Mileage
Claims
Model B
Fitted value
Deviance = 9604
df = 109965

Deviances

- If ϕ known, scaled deviance S output

$$S = \sum_{u=1}^{n} 2\, \omega_u / \phi \int_{\mu_u}^{Y_u} (Y_u - \zeta) / V(\zeta)\, d\zeta$$

$$S_1 - S_2 \sim \chi^2_{d_1 - d_2}$$

- If ϕ unknown, unscaled deviance D = ϕ.S output

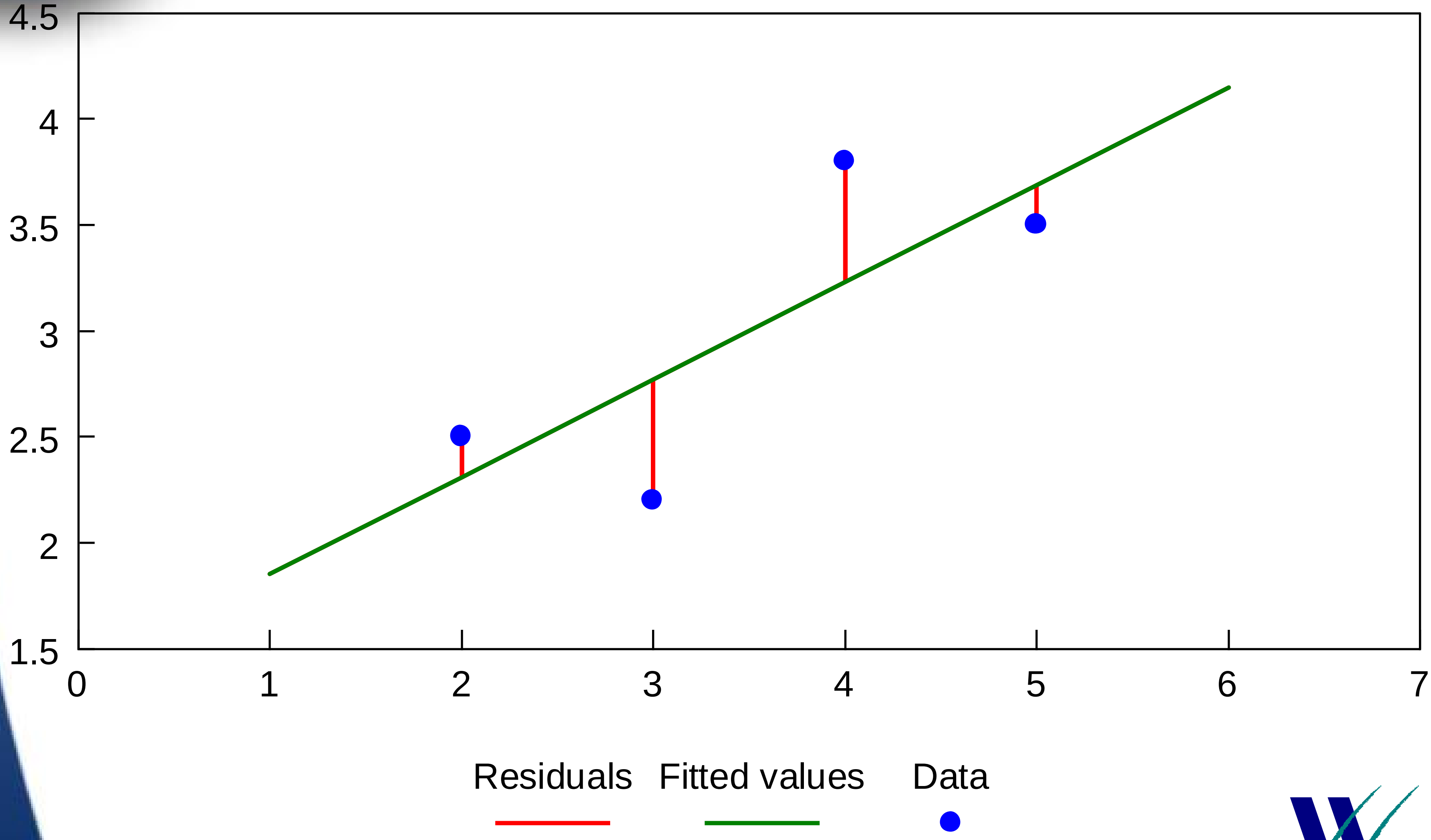
$$\frac{(D_1 - D_2)}{(d_1 - d_2)\, D_3 / d_3} \sim F_{d_1 - d_2,\, d_3}$$

Model testing

- Use only those variables which are predictive
  - standard errors of parameter estimates
  - F tests / χ^2 tests on deviances

- Make sure the model is reasonable
  - histogram of deviance residuals
  - residual vs fitted value
  - Box Cox link function investigation

Residuals



Residuals

- Several forms, eg
  - standardized deviance

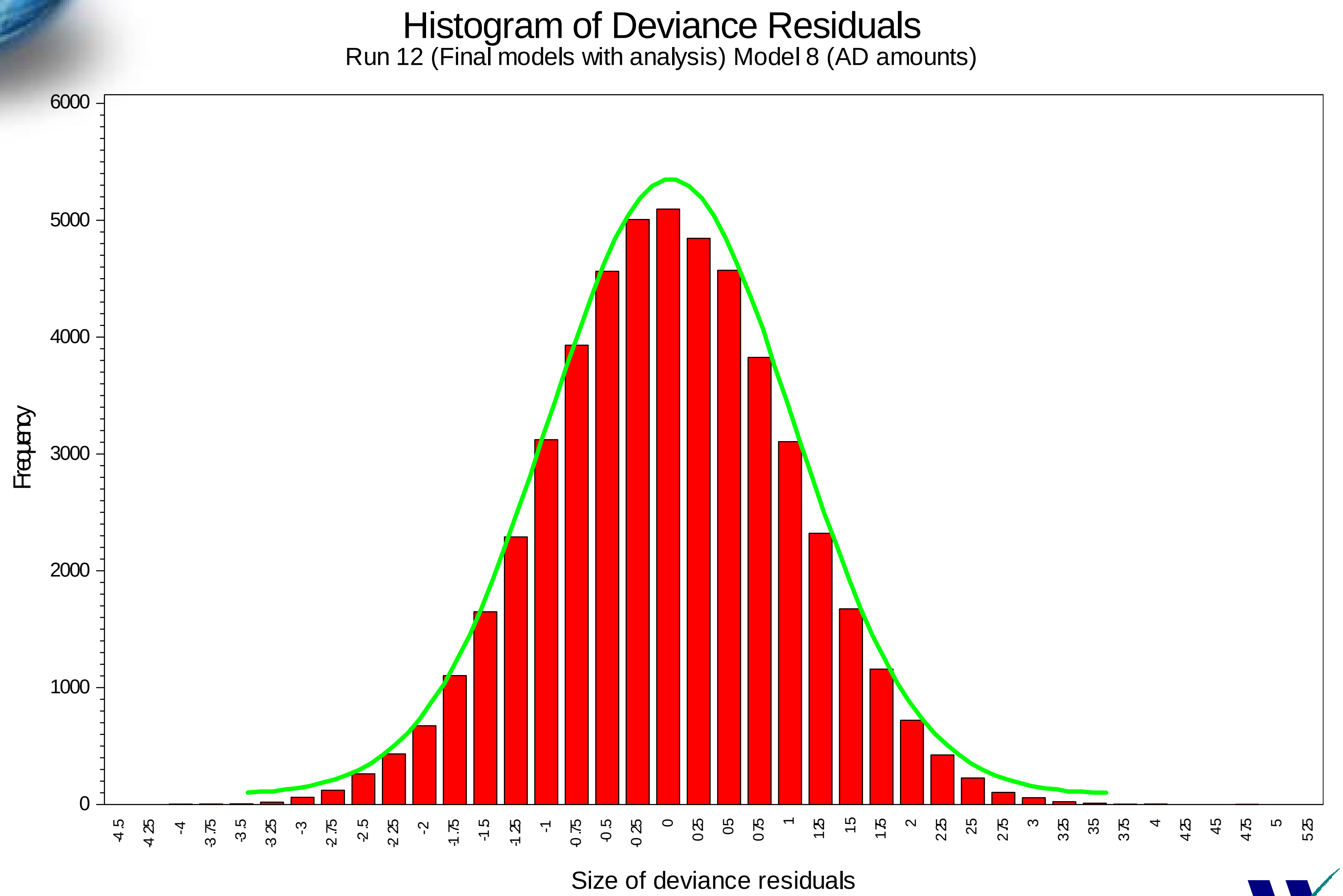
$$\text{sign}\,(Y_u-\mu_u)\,/\,(\,\phi\,(1-h_u)\,)^{1/2}\sqrt{2\,\omega_u\int_{\mu_u}^{Y_u}(\,Y_u-\zeta\,)\,/\,V(\zeta)\,d\zeta}$$

  - standardized Pearson

$$\frac{Y_u - \mu_u}{(\,\phi.V(\mu_u).(1-h_u)\,/\,\omega_u\,)^{1/2}}$$

- Standardized deviance - Normal (0,1)
- Numbers/frequency residuals problematical

Residuals



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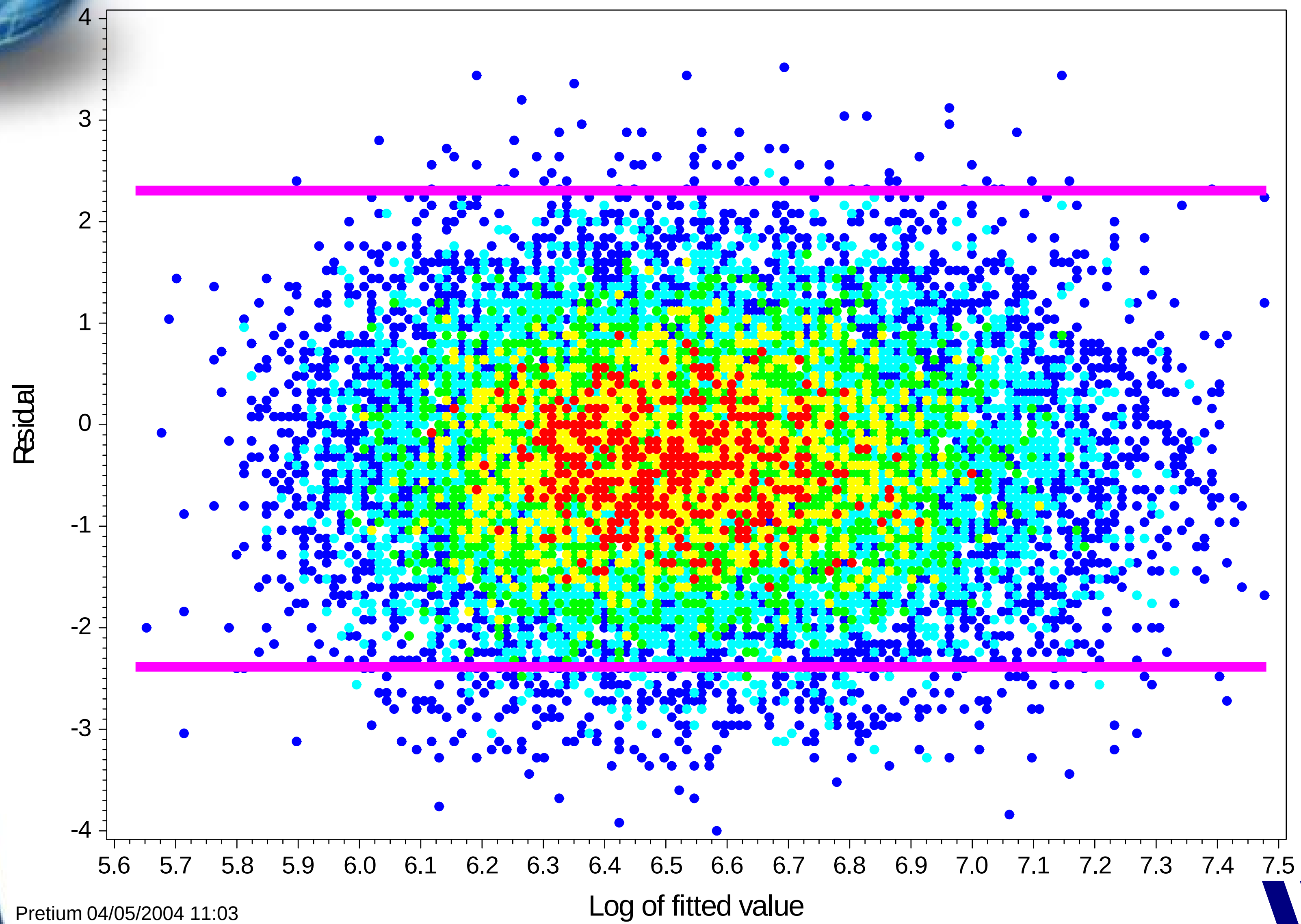
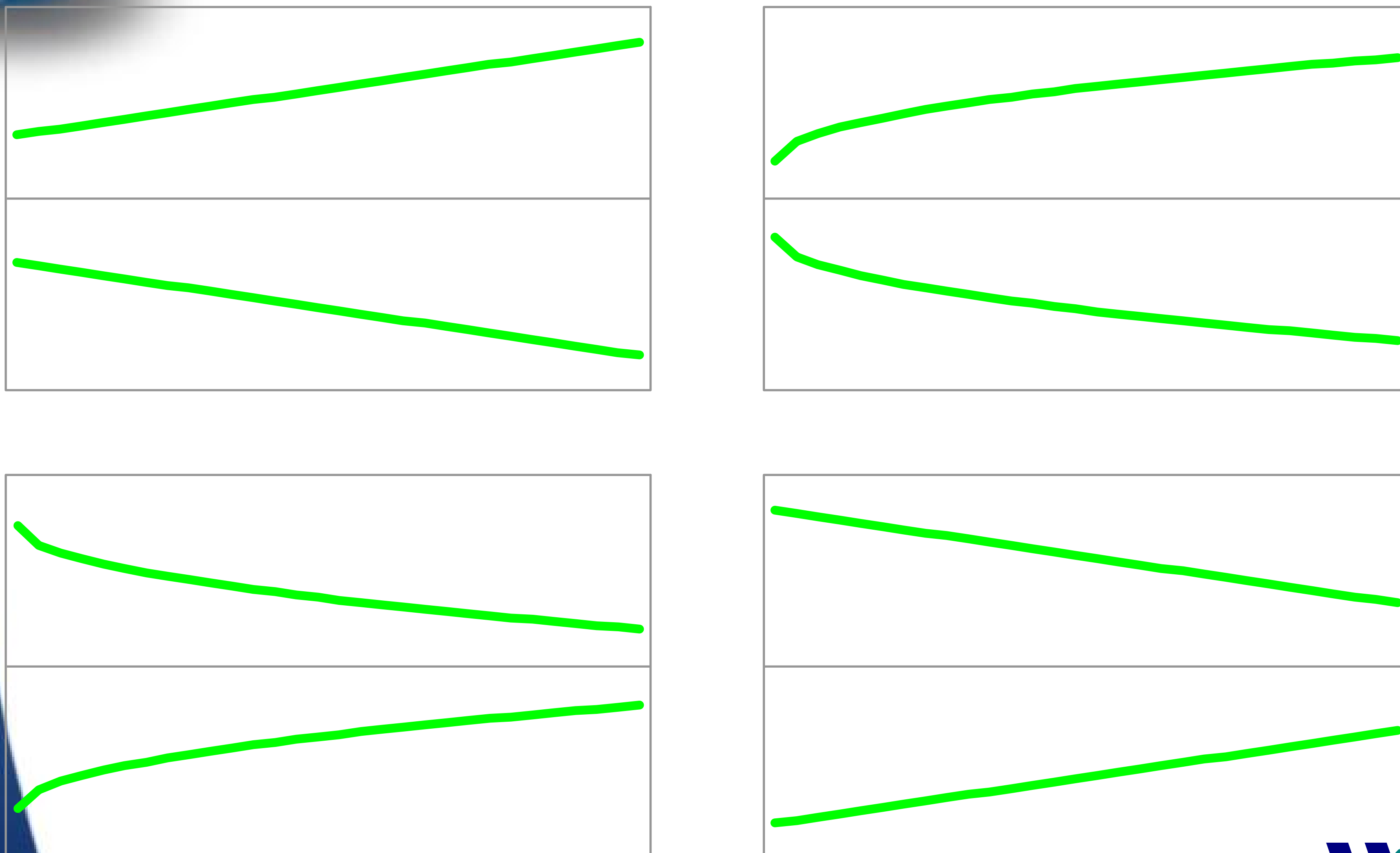
Residuals

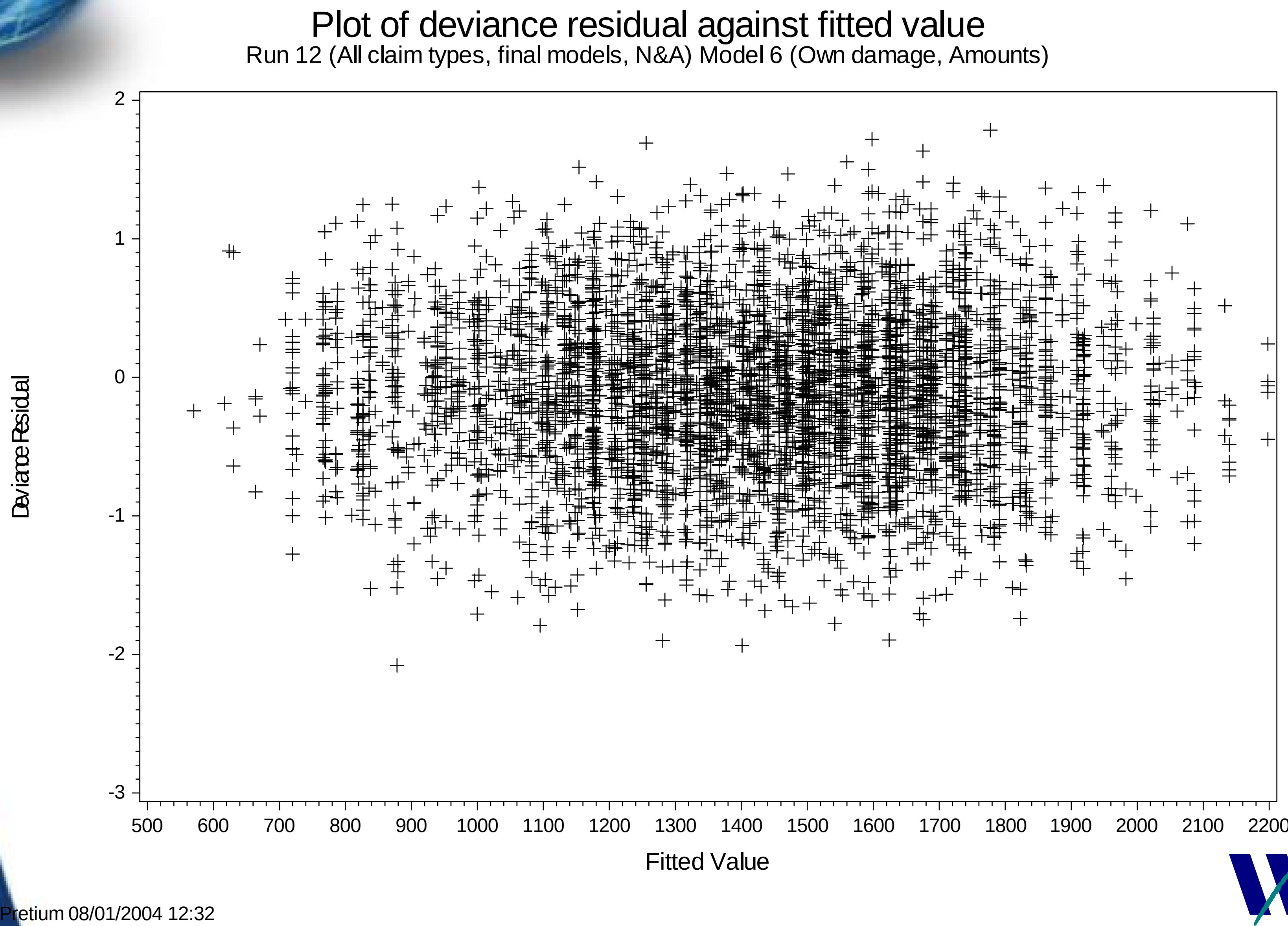


4
3
2
1
0
-1
-2
-3
-4
Residual
5.6 5.7 5.8 5.9 6.0 6.1 6.2 6.3 6.4 6.5 6.6 6.7 6.8 6.9 7.0 7.1 7.2 7.3 7.4 7.5
Log of fitted value

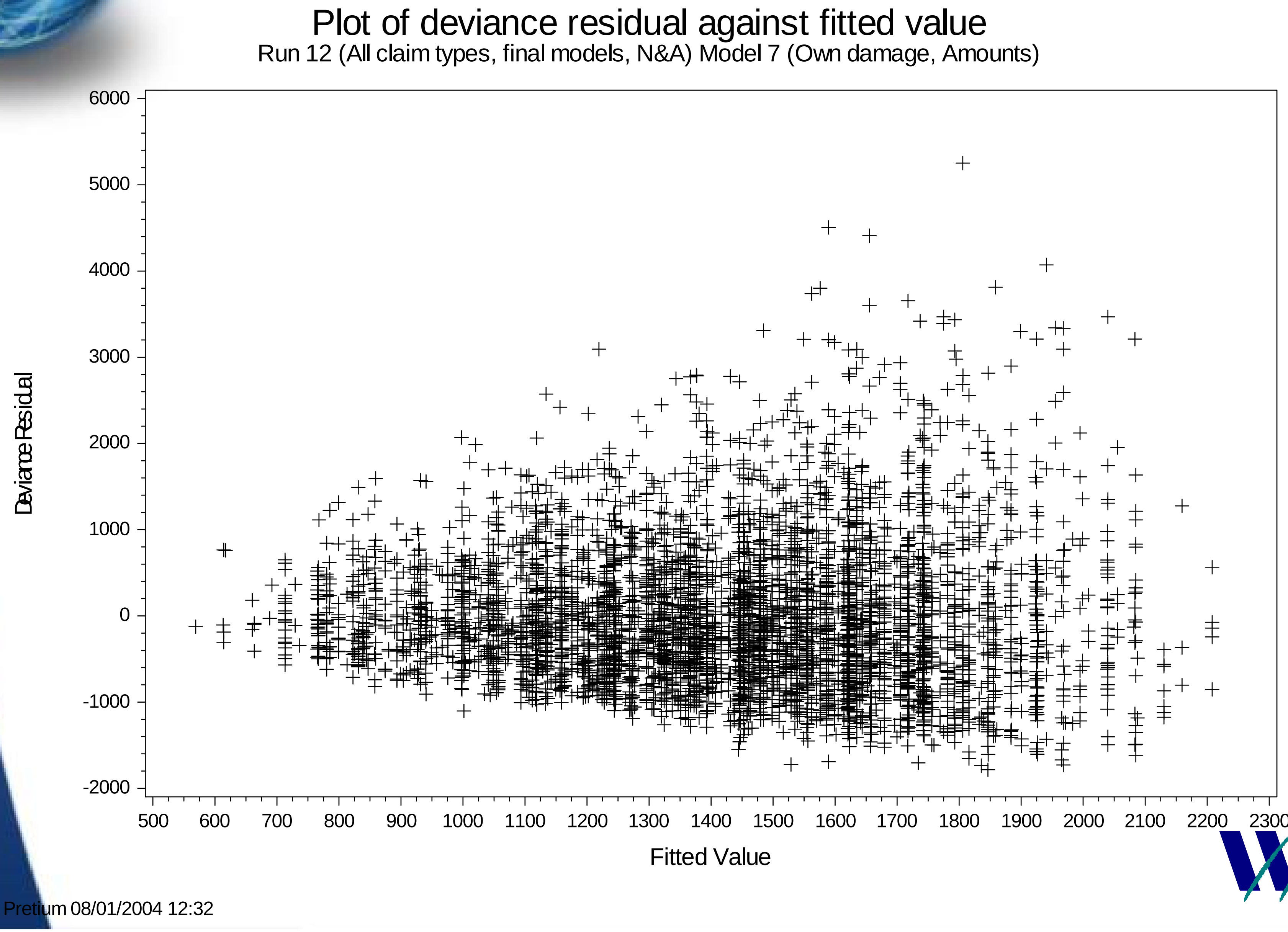
Residuals



Gamma data, Gamma error



Gamma data, Normal error



Agenda

- Formularization of GLMs
  - linear predictor, link function, offset
  - error term, scale parameter, prior weights
  - typical model forms
- Model testing
  - use only variables which are predictive
  - make sure model is reasonable
- Aliasing

Aliasing and "near aliasing"

- Aliasing
  - the removal of unwanted redundant parameters
- Intrinsic aliasing
  - occurs by the design of the model
- Extrinsic aliasing
  - occurs "accidentally" as a result of the data

Intrinsic aliasing

$\mathbf{X}.\underline{\beta} = \alpha + \beta_1$ if <u>age</u> 20 - 29

~~$+ \beta_2$ if <u>age</u> 30 - 39~~

$+ \beta_3$ if <u>age</u> 40 +

~~$+ \gamma_1$ if <u>sex</u> male~~

$+ \gamma_2$ if <u>sex</u> female

- "Base levels"

Intrinsic aliasing

Extrinsic aliasing

- If a perfect correlation exists, one factor can alias levels of another
- Eg if doors declared first:

Exposure: # Doors → / Color ↓	2	3	4 (Selected base)	5	Unknown
Red (Selected base)	13,234	12,343	13,432	13,432	0
Green	4,543	4,543	13,243	2,345	0
Blue	6,544	5,443	15,654	4,565	0
Black	4,643	1,235	14,565	4,545	0
Unknown (Further aliasing)	0	0	0	0	3,242

- This is the only reason the order of declaration can matter (fitted values are unaffected)

Extrinsic aliasing

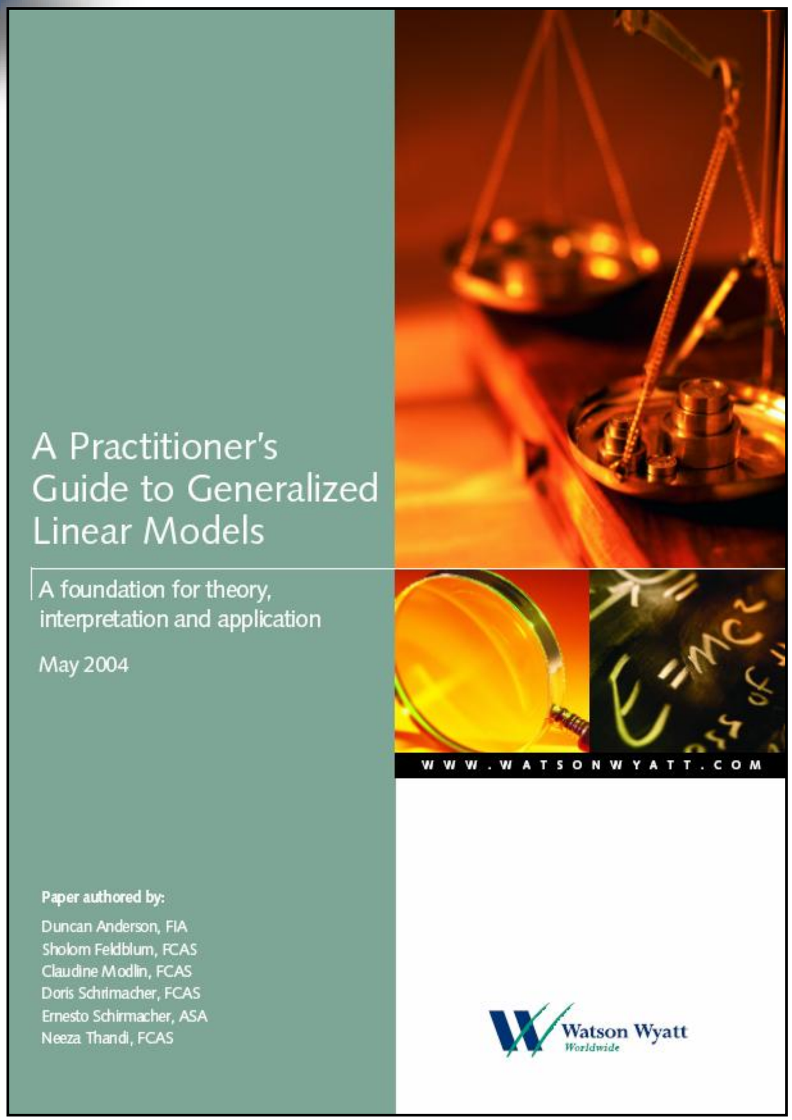
"Near aliasing"

- If two factors are almost perfectly, but not quite aliased, convergence problems can result as a result of low exposures (even though one-ways look fine), and/or results can become hard to interpret

Exposure: # Doors → / Color ↓	2	3	4 (Selected base)	5	Unknown
Red (Selected base)	13,234	12,343	13,432	13,432	0
Green	4,543	4,543	13,243	2,345	0
Blue	6,544	5,443	15,654	4,565	0
Black	4,643	1,235	14,565	4,545	2
Unknown	0	0	0	0	3,242

- Eg if the 2 black, unknown doors policies had no claims, GLM would try to estimate a very large negative number for unknown doors, and a very large positive number for unknown color

"A Practitioner's Guide to Generalized Linear Models"



- CAS 2004 Discussion Paper Program
- Copies available at www.watsonwyatt.com/glm

The Matrix Inverted: A Primer in GLM Theory

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