Optimality of Proportional Reinsurance

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Proportional Reinsurance

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- A predetermined part of each and every risk τ_i is transferred to the reinsurer.
- τ_i of the premium is ceded to the reinsurer.
- Reinsurer pays τ_i of the loss if any.

Quota Share

- The ceding company cedes the same part of each and every risk, regardless of its size : $\tau_i = \tau$ for all risks i.

Link Between Capital and Quota Share

- Quota-share cession : τ.
- P : original premium.
- u : capital.
- $S = Y_1 + Y_2 + \cdots + Y_N$.
- S^{net} : the aggregate claims after reinsurance.
- ϵ = ruin probability = $P(S > u + P)$.

Link Between Capital and Quota Share

- $S^{net} = (1-\tau)Y_1 + \cdots + (1-\tau)Y_N = (1-\tau)S$.
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$$\begin{aligned}\epsilon^{net} &= P((1-\tau)S > u + (1-\tau)P) \\ &= P(S > u/(1-\tau) + P) \\ &< \epsilon\end{aligned}$$

- Effect similar to an increase of the capital.

Variable Quota Share

- Cession rate is varying according to subportfolios.
- Define k subportfolios.
- Define k cession rates : $\tau_1, \ldots, \tau_k$.

Surplus Reinsurance

- The insurer cedes that part of a risk that exceeds a predetermined retention : the line : R.
- Let SI_i be the insured sum of risk i.
- Cession rate :

$$\tau_i = \max\left(0, 1 - \frac{R}{SI_i}\right)$$

- In case of total loss, the retention pays :

$$(1-\tau_i)SI_i = \frac{R}{SI_i}SI_i = R.$$

Table of Lines

- Line is varying according to subportfolios.
- Define k subportfolios.
- Define k lines : $R_1, \ldots, R_k$.
- In practice qualitative definition of the danger of subportfolios implies different lines on these subportfolios.

Table of Lines

- Practitioner's rule 1 : let us do as if there were only total losses. Assume that the chance of making a loss (q) is different from a subportfolio to the other. Then choose the lines such that on average the loss in retention is the same :

$$R_1 \times q_1 = R_2 \times q_2 = R_3 \times q_3 = R_4 \times q_4.$$

Inverse claim probability method.

Table of Lines

- Practitioner's rule 2 : now let us account for the chance to reach the total loss. This is done by using the rate ($q\mathbb{E}X$) instead of the probability to make a loss (q). Assume that the rate is different from a subportfolio to the other. Then choose the lines such that on average the loss in retention is the same :

  $$R_1 \times rate_1 = R_2 \times rate_2 = R_3 \times rate_3 = R_4 \times rate_4.$$

  Inverse rate method.

- $X = \frac{C}{SI_j}$ is the damage rate.

Data Set

- Real-life data set. Leading Belgian insurance company. Contains $27\,551$ fire policies, covering industrial risks.
- The $27\,551$ policies are divided into four classes ($j = 1, 2, 3, 4$), depending on their claims probability (q_{ij}) as well as their relative claims severity (X_{ij}),$i = 1, \ldots, n_j$ where n_j is the number of policies in class j.

Model

- Knowing the sum insured SI_{ij}, we can obtain the loss amount : $C_{ij} = SI_{ij} \times X_{ij}$.
- We will assume the X_{ij} to be identically distributed within a given risk class ($j = 1, 2, 3, 4$) :
  $X_{ij} \sim X_j \,,\, i = 1, \ldots, n_j \,,\, j = 1, 2, 3, 4$.
- We also assume that the probability of making a loss is identical within a class :
  $q_{ij} = q_j \,,\, i = 1, \ldots, n_j \,,\, j = 1, 2, 3, 4$.

Bernegger's Model for Damage Rate

- For the density of X_j we will use the MBBEFD distribution class introduced by Bernegger (1997).
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$$\begin{aligned} b(c) &= e^{3.1-0.15c(1+c)} \\ g(c) &= e^{c(0.78+0.12c)} \end{aligned}$$
- Density function of X_j :

$$\begin{aligned} f(x) &= \frac{(b-1)(g-1)\ln(b)b^{1-x}}{\left((g-1)b^{1-x}+(1-gb)\right)^2}, \; 0 \le x < 1 \\ f(1) &= \frac{1}{g}. \end{aligned}$$

Data Set

- $c = 2, 3, 4, 5$ corresponds to the Swiss Re exposure curves 2, 3, 4 and the Lloyd's industrial exposure curve repectively.
- We will assume that we have the following characteristics for our portfolio :

$Class$	q	c
1	0.75%	2
2	1.00%	3
3	1.25%	4
4	1.50%	5

Data Set

- Regarding the sum insured, we have the following information at disposal :

$Class$	n	$\mathbb{E}$	σ	γ
1	3 933	13 457 022	10 752 926	8.51
2	17 472	12 034 729	7 960 092	2.23
3	3 121	11 826 858	9 119 825	4.62
4	3 025	10 879 648	7 826 747	11.98

Model

- Aggregate claims amount is given by

$$S^{ind} = \sum_{j=1}^{4} \sum_{i=1}^{n_j} D_{ij} C_{ij}$$

where

(a) D_{ij} is the indicator function taking value 1 when there is a claim and 0 when there is no claim. We have $\mathbb{P}[D_{ij} = 1] = q_j$.

(b) $C_{ij} = SI_{ij} X_{ij}$ is the loss value.

Distribution of S^{ind}

- Exact distribution of S^{ind} : possible (see e.g. Dhaene and Vandebroek (1995)) but difficult.
- Collective risk model as an approximation to the individual risk model. Also difficult.
- As the size of the porfolio is high, and its skewness less than 2 (see further for the calculations) we will concentrate on a parametric approximation, namely the shifted gamma distribution, that will reproduce the first three moments of the original distribution. We therefore need to obtain the first three moments of S^{ind}.

Shifted Gamma Distribution

- Shifted gamma distribution :

$$S \approx Z + x_0$$

where $Z \sim Gam(\alpha, \beta)$, i.e.

$$\begin{aligned} f_Z(x) &= \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \quad , \quad x > 0 \\ F_Z(x) &= \int_0^x f_Z(s) ds \end{aligned}$$

where $\Gamma(x)$ is the gamma function. By abuse of notation, we will also write $F(\alpha, \beta, x)$ the cumulative density function of Z.

Central Moments

- Central moments are given by

$$\mu = \sum_{j=1}^{4} [q_j \mathbb{E} X_j] \sum_{i=1}^{n_j} SI_{ij}$$

$$\mu_2 = \sum_{j=1}^{4} \left[q_j \mathbb{V}ar X_j + q_j(1-q_j)(\mathbb{E}X_j)^2\right] \sum_{i=1}^{n_j} SI_{ij}^2$$

$$\mu_3 = \sum_{j=1}^{4} \left[q_j \mathbb{E}X_j^3 - 3q_j^2 \mathbb{E}X_j \mathbb{E}X_j^2 + 2q_j^3(\mathbb{E}X_j)^3\right] \sum_{i=1}^{n_j} SI_{ij}^3$$

Shifted Gamma Distribution

$$
\begin{aligned}
\mu &= 293\,751\,934 \\
\sigma &= 57\,364\,022 \\
CV &= 0.20 \\
\gamma &= 0.6 \\
\alpha &= \frac{4}{\gamma^2} = 10.44 \\
\beta &= \frac{2}{\gamma\sigma} = 5.63\,10^{-8} \\
x_0 &= \mu - \frac{2\sigma}{\gamma} = 108\,404\,392
\end{aligned}
$$

Optimal Reinsurance

- de Finetti criterion minimizes the variance of the retained loss under the constraint that the expected gain is fixed.
- RORAC criterion maximizes the return on risk adjusted capital of the retained risk.

Loadings

- ξ : insurer's loading, accounting only for capital charge. All administrative expenses must be charged on top of that loading. Here $\xi = 5\%$.
- ξ^{Re} : reinsurer's loading, including the capital charge of the reinsurer as well as the administrative expenses. It is clear that the insurer pays for the administrative expenses of the reinsurer in the reinsurance premium. Here $\xi^{Re} = 7\%$.

de Finetti for Quota Share

- For a portfolio of n risks, de Finetti (1940) suggests to obtain the optimal cession rates by minimizing the variance of the gain of the retained portfolio under the constraint that the expected gain is known.
- The gain of the retained portfolio is

$$Z(\tau) = \sum_{i=1}^{n}((1+\xi_i)\mathbb{E}D_iC_i-(1+\xi_i^{Re})\tau_i\mathbb{E}D_iC_i-(1-\tau_i)D_iC_i).$$

where τ is the vector of cession percentages $\{\tau_1,\ldots,\tau_n\}$.

de Finetti for Quota Share

- The de Finetti problem is the following :

$$\min_{\tau} \mathbb{V}ar Z(\tau)$$

under the constraint that

$$\mathbb{E}Z(\tau) = k.$$

de Finetti for Quota Share

- de Finetti (1940) showed that the solution is given by

$$\tau_i = \max\left(0, 1 - \frac{\lambda \xi_i^{Re} \mathbb{E} D_i C_i}{\mathbb{V}ar D_i C_i}\right) \quad , \quad i = 1, \ldots, n,$$

  where λ is a constant given by the condition $\mathbb{E}Z(\tau) = k$.

- In practice not possible to use.

de Finetti for Variable Quota Share

- For a variable quota share treaty, de Finetti's result can be extended by using convex optimization to prove that the optimal lines are

$$\tau_{ij} = \tau_j = \min\left(1, \max\left(0, 1 - \frac{\lambda \sum_{i=1}^{n_j} \xi_{ij} \mathbb{E} D_{ij} C_{ij}}{2 \sum_{i=1}^{n_j} \mathbb{V}ar D_{ij} C_{ij}}\right)\right)$$

where λ is a constant given by $\mathbb{E}Z(\tau) = k$.

de Finetti for a Surplus Treaty with Table of Lines

- For a surplus treaty with table of lines, de Finetti's result can be extended by using convex optimization to prove that the optimal lines are

$$R_j = \frac{\lambda \sum_{i=1}^{n_j} \xi_{ij}^{Re} \mathbb{E}[D_{ij}C_{ij}]SI_{ij}}{2 \sum_{i=1}^{n_j} \mathbb{V}ar[D_{ij}C_{ij}]} \quad , \quad j = 1, 2, 3, 4$$

  where λ is a constant given by $\mathbb{E}Z(\tau) = k$.

de Finetti for a Surplus Treaty with Table of Lines

- The associated cession rates are

$$\tau_{ij} = \min\left(1, \max\left(0, 1 - \frac{R_j}{SI_{ij}}\right)\right).$$

- On the reasonable assumption that the X_{ij} and D_{ij} are identically distributed within the class j and that the reinsurance loading is the same for each risk within the class j, the formula is reduced to

$$R_j = \frac{\lambda \xi_j^{Re} \mathbb{E}[D_j X_j]}{2\mathbb{V}ar[D_j X_j]} \quad , \quad j = 1, 2, 3, 4$$

where λ is a constant given by $\mathbb{E}Z(\tau) = k$.

RORAC

- Retained risk of the cedant : $S^R = S - S^{Re}$.
- Required solvency level, RSL, is given by the Tail Value at Risk at the level $\epsilon = 99\%$.
- Using our shifted gamma approximation, we have

$$\begin{aligned} RSL &= \mathbb{E}[S^R | S^R > VaR_{S^R}(\epsilon)] \\ &= \mathbb{E}[Z|Z > VaR_Z(\epsilon)] + x_0 \\ &= \frac{\alpha}{\beta}\frac{1}{1-\epsilon}(1 - F(\alpha+1, \beta, VaR_Z(\epsilon))) + x_0 \end{aligned}$$

where $VaR_Z(\epsilon) = F^{-1}(\alpha, \beta, \epsilon)$.

RORAC

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- The retained premium is equal to

$$P^R = (1+\xi)\mathbb{E}S - (1+\xi^{Re})\mathbb{E}S^{Re}.$$

- The risk adjusted capital is obtained by deducting the retained premium from RSL. In other words, the risk adjusted capital is the required solvency level minus the premium that is borrowed from the policyholders plus the premium that is charged by the reinsurers :

$$RAC = RSL - P^R$$

- $$RORAC = \frac{P^R - \mathbb{E}S^R}{RAC}$$

Numerical Application

- For the original (i.e. before any reinsurance) portfolio, we obtain the following :

$$\begin{aligned}
\mathbb{E}S = \mathbb{E}S^R &= 293\,751\,934 \\
CV &= 0.20 \\
\gamma &= 0.62 \\
VaR &= 452\,547\,891 \\
RSL = TVaR &= 483\,141\,978 \\
P = P^R &= 308\,439\,531 \\
RAC &= 174\,702\,447 \\
RORAC &= 8.41\%.
\end{aligned}$$

RORAC in Function of the Line of a Surplus Treaty

$Case$	$Line$	CV	γ	$RORAC$	Expected gain	$\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$
1	5 000 000	0.16	0.24	4.16%	2 080 641	61.31%
2	7 500 000	0.16	0.24	7.58%	5 222 627	46.03%
3	10 000 000	0.16	0.25	9.05%	7 714 795	33.91%
4	12 500 000	0.17	0.26	9.71%	9 583 949	24.82%
5	15 000 000	0.17	0.28	9.98%	10 961 666	18.12%
6	17 500 000	0.17	0.29	10.06%	11 973 326	13.20%
7	20 000 000	0.18	0.30	10.06%	12 715 622	9.59%
8	22 500 000	0.18	0.31	10.00%	13 266 739	6.91%

RORAC in Function of the Cession of a Quota Share Treaty

$Case$	τ	CV	γ	$RORAC$	$\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$
1	61.31%	0.20	0.62	2.92%	61.31%
2	46.03%	0.20	0.62	5.38%	46.03%
3	33.91%	0.20	0.62	6.57%	33.91%
4	24.82%	0.20	0.62	7.22%	24.82%
5	18.12%	0.20	0.62	7.61%	18.12%
6	13.20%	0.20	0.62	7.86%	13.20%
7	9.59%	0.20	0.62	8.02%	9.59%
8	6.91%	0.20	0.62	8.14%	6.91%

RAROC in Function of the Cession of a SP with ToL (inverse rate method)

$Case$	R_1	R_2	R_3	R_4	CV	γ	$RORAC$
1	2 792 144	5 430 844	11 891 468	25 987 731	0.16	0.29	4.06%
2	4 373 473	8 506 598	18 626 192	40 705 865	0.16	0.28	7.47%
3	6 066 679	11 799 959	25 837 392	56 465 292	0.16	0.28	8.93%
4	7 857 669	15 283 513	33 465 040	73 134 831	0.17	0.29	9.58%
5	9 739 358	18 943 481	41 478 968	90 648 548	0.17	0.30	9.86%
6	11 697 749	22 752 639	49 819 564	108 876 170	0.17	0.31	9.96%
7	13 736 088	26 717 298	58 500 649	127 847 900	0.18	0.32	9.96%
8	15 858 279	30 845 054	67 538 854	147 600 082	0.18	0.33	9.92%

RORAC in Func. of Cession of a SP with ToL (de Finetti's Optimal Table)

$Case$	R_1	R_2	R_3	R_4	CV	γ	$RORAC$
1	3 949 974	5 155 430	7 327 325	11 286 824	0.15	0.24	4.22%
2	6 007 752	7 841 203	11 144 567	17 166 807	0.16	0.25	7.68%
3	8 113 889	10 590 093	15 051 518	23 184 974	0.16	0.26	9.15%
4	10 247 187	13 374 433	19 008 852	29 280 752	0.17	0.27	9.79%
5	12 397 936	16 181 549	22 998 558	35 426 392	0.17	0.28	10.05%
6	14 573 268	19 020 751	27 033 868	41 642 281	0.17	0.29	10.13%
7	16 743 363	21 853 117	31 059 460	47 843 201	0.18	0.30	10.11%
8	18 964 227	24 751 746	35 179 233	54 189 193	0.18	0.31	10.05%

RORAC in Function of the Optimal Variable Quota Share Cession

$Case$	τ_1	τ_2	τ_3	τ_4	CV	γ	$RORAC$	$\frac{\mathbb{E}S^{Re}}{\mathbb{E}S}$
1	74.71%	57.94%	45.15%	3.49%	0.19	0.54	3.13%	61.31%
2	64.24%	40.51%	22.44%	0.00%	0.19	0.51	5.82%	46.03%
3	55.89%	26.63%	4.33%	0.00%	0.19	0.50	7.11%	33.91%
4	49.29%	15.64%	0.00%	0.00%	0.19	0.49	7.81%	24.82%
5	44.30%	7.35%	0.00%	0.00%	0.19	0.49	8.23%	18.12%
6	40.64%	1.26%	0.00%	0.00%	0.19	0.49	8.49%	13.20%
7	31.39%	0.00%	0.00%	0.00%	0.19	0.51	8.59%	9.59%
8	22.62%	0.00%	0.00%	0.00%	0.19	0.53	8.59%	6.91%

Pure Application of de Finetti

1. surplus with one line
2. surplus with table of lines corresponding to the quota share treaty
3. surplus with table of lines corresponding to the variable quota share (the lines are chosen such that the global cession for the subportfolio is the same for both covers)
4. surplus with table of lines obtained by the inverse rate method
5. surplus with table of lines obtained by the inverse claims probability method

Pure Application of de Finetti

Case	R_1	R_2	R_3	R_4	$\mathbb{E}$	σ
1	7 304 175	7 304 175	7 304 175	7 304 175	5 000 000	24 858 743
2	7 989 249	7 065 148	6 963 402	6 167 660	5 000 000	25 111 701
3	4 886 924	8 036 122	12 533 770	333 398 280	5 000 000	24 700 617
4	4 246 111	8 258 874	18 083 771	39 520 454	5 000 000	24 913 398
5	9 084 700	6 813 525	5 450 820	4 542 350	5 000 000	25 693 734

Conclusion

- Quota share reinsurance is of interest to the ceding company when the loading of the reinsurer is smaller than the loading of the insurer.
- This is possible if one refers to the diversification possibilities that are offered to the reinsurer. So one may argue that less capital needs to be remunerated with the reinsurer's position. On the other hand, one may argue that the reinsurer's shareholders may require a higher cost of capital due to the agency costs that apply when underwriting a business that is less known than in the situation of the insurer. This means that ceding companies should provide as much information as possible to reinsurers in order to reduce these agency costs.

Conclusion

- We have also observed that surplus reinsurance with a table of lines based on the inverse probability method, or inverse rate method, is not, in our numerical example, optimal when compared to surplus reinsurance with one single line. This goes against the traditional belief of practitioners.
- Reinsurer's loading would most probably not remain constant in case of surplus treaties with increasing retentions.
- High retentions imply large volatility to the reinsurer. Therefore higher capital charges.
- Reinsurer has fixed administrative costs.