A Method for Efficient Simulation of the Collective Risk Model

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Overview

- Conditional Aggregate Distribution (CAD) Method
- Convergence Theorem
- Mixed Poisson Distributions

Conditional Aggregate Distribution (CAD) Method

Collective Risk Model (CRM)

- Claim sizes are independent and identically distributed X(i)
- Claim counts N are independent from the X(i)
- Total claims Z=X(1) + … + X(N)

- EZ=EX EN
- VZ=VZ EN+VN $(EX)^2$

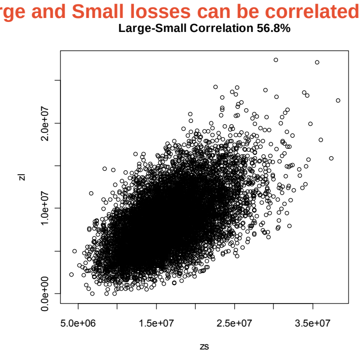
Simulating the CRM

- Problem: When EN is large it can take a long time to simulate X(1),…,X(N)
- Common Solution:
  - Split claims sizes into large claim and small claims
  - Simulate large counts N_L and large claims sizes Y(k) with a CRM
    - Z_L=Y(1) + … Y(N_L)
  - Simulate total small claims in aggregate
  - Small and Large might be left independent or correlated with a copula or other mechanism

Correlation of Small and Large Losses for CRM

$$\rho(Z_S, Z_L) = \frac{q(1-q)\mathrm{E}[X_S]\mathrm{E}[X_L](\sigma^2(N) - \mathrm{E}[N])}{\sigma(Z_S)\sigma(Z_L)}$$

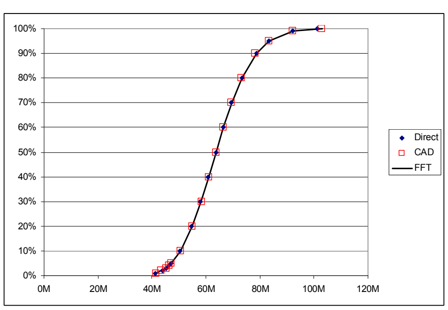
CRM Large and Small losses can be correlated



Conditional Aggregate Distribution (CAD_k) Algorithm

1. Simulate total counts N
2. Simulate large counts N_L conditional on N
   - N_L~Binomial because N and X(k) are independent and the X(k) are iid.
   - N_S=N-N_L
3. Simulate large sizes Y(1)…Y(N_L)
   - $F_Y(x)=F_X(x)/(1-F(\tau))$ (τ is the threshold between large and small losses)
   - Z_L =Y(1)+…+ Y(N_L)
4. Simulate aggregate small losses conditional on N_S
   - Simulate an approximation to $Z_S|N_S$ by drawing from a distribution that matches the first k moments of $Z_S|N_S$.
   - Call this $\tilde{Z}_S$ the CAD distribution
5. Deliver {$\tilde{Z}_S$, Z_L (Y(1),…, Y(N_L)}

CAD Method approximates CRM nicely



Summary of CAD Features

- Advantages of CAD_k –
  - Fast, easy to program.
  - Preserves first k moments and Pearson correlation (k>=2).
  - Provides structural method for modeling dependence of small, large losses.
- Also, apparently converges.
- That is, can observe that CAD provides good fits to Z_S.
- Here, we assume severity distribution is fixed with distribution of N depending consistently on λ.
- Fits are good even for "middling" values of λ.

Convergence Theorem

Mixed Poisson

- N is a mixed Poisson if N~Poisson(Λ), where Λ is a random parameter.
- Write $\Lambda=\lambda G$, where EG=1 and VarG=c.
  - Eg – Negative Binomial, G~Gamma[1/c,c]
  - G is the mixing distribution, c the contagion parameter.
  - If Z is a mixed Poisson CRM, then so are Z_S, Z_L, with same G.
  - $\rho(Z_S, Z_L)=c/[\nu(Z_S)\nu(Z_L)]$, ν =c.v.

"Severity is Irrelevant"

- Limiting behavior of a mixed Poisson CRM is controlled by the mixing distribution:
- Theorem (Mildenhall): If Z is a mixed Poisson CRM then $Z/EZ \to G$ as $\lambda \to \infty$.
- This is "weak convergence", aka "convergence in distribution"

Convergence Theorem

- Significance to CAD method: Suppose we use Gamma as the CAD-family $\tilde{Z}_S$ approximating Z_S.
- Then $\tilde{Z}_S$ =Gamma[N_Sa,β]=$\Sigma_{1,\ldots N\_S}$(Gamma[a,β]), a,β constants.
- This converges to G by Theorem (after normalizing by the mean, of course).
- **Generalized Convergence Theorem**: N_λ mixed Poisson with mixing dist G, Y_n random variables s.t. $EY_n=nm$, and $Var(Y_n)<=n^j s^2$ for 0<=j<2. Then $Y_{N\_\lambda}/(\lambda m) \to G$ as $\lambda \to \infty$.
- Y_{N_λ} is defined by $Y_{N\_\lambda} |(N_\lambda =n)= Y_n$.
- Note: variance won't converge for j>=2.

"Severity, CAD distribution are irrelevant"

- **Example 1** – $Y_n=X_1+\ldots+X_n$, X_i iid. Then EY_n=nEX, $Var(Y_n)$=nVarX, and Y_{N_λ} is CRM so get Mildenhall's theorem.
- **Example 2** – For CAD_k with k>=2, set $Y_n= \tilde{Z}_S |N_S=n$. Then Y_n satisfies theorem with j=1, and $Y_{N\_\lambda} = \tilde{Z}_S$, so $\tilde{Z}_S$ converges to G (as does Z_S).
- **Example 3** – Set $Y_n = \tilde{Z}_S + Z_L|(N_S=n-B, N_L=B)$, where B~Bin(n,q). This shows that CAD_k converges in an overall sense for k>=1.
- Even a CAD_1 method will converge as long as the variance is under control (j<2).
- Convergence in evidence for moderately sized portfolios.

Convergence Theorem

- Proof of Theorem:
- Can take m=1. Must show that Lim $_{\lambda\to\infty}$ $\phi_{Y\_N\_\lambda/\lambda}(t)=\phi_G(t)$, where ϕ is the characteristic function $\phi_X(t) = Ee^{itX.}$
- Bounded Convergence Theorem allows switching Lim and E·.
- Poisson characteristic function: $e^{\lambda(e\text{^}it-2)}$.
- Expand ϕ in Taylor series.
- Fact (Durret): $\phi_X(t)=1+itEX+E[X^2]O(t^2)$.

Convergence Theorem

- Last Line of Proof:

$\text{Lim}\phi_{Y\_N\_\lambda/\lambda}(t)=E[\text{Lim}E[\phi_{Y\_n}(t/\lambda)|G,N_{\lambda G}=n]]$

$=\text{Lim}E[e^{i(t/\lambda)n}|G,N_{\lambda G}=n]=\text{Lim}E[e^{\lambda G(e\text{^}it-1)}]$

$=\text{Lim}E[e^{\lambda G(it/\lambda+O((t/\lambda)\text{^}2)}]=\text{Lim}E[e^{Git}]=\phi_G(t)$.

- 1^{st} "=" is BCT, 2^{nd} "=" is from earlier steps eliminating other terms with Durrett fact and j<2; 3^{rd} is Poisson char. fcn.; 4^{th} is Durrett fact applied to X=1.

Mixed Poisson

More Mixed Poisson

- So, choice of mixing distribution G is important - Controls characteristics of Z ($\tilde{Z}$), Z_S ($\tilde{Z}_S$), and even Z_L, to a lesser extent.
- Choices other than Gamma are allowable.
- Choice of G might reflect an assumption about skewness.
- Note, choosing Gamma is such an assumption ("skew-nu" ratio = 2).
- Paper gives parameterizations for many possibilities:
  - Usual suspects – Gamma, Lognormal (Uniform, Inverse Gaussian).
  - High-Skew – Pareto, (shifted) Exponential.
  - Discrete – Discrete Uniform, Poisson, Binomial
  - Component and shifted versions
- Adding shift drives up skewness and effective minimum value.

CAD with Limited Information

- Generally, do not need full severity distribution to run CAD.
- Start with: EZ (EZ_S), $\nu(Z)$ ($\nu(Z_S)$), $\lambda(N_L)$, severity distribution for Z_L.
- *Consistent* choice of λ,c is then enough to run a CAD_2 model.
- That is, can derive formulas for $E[Z_S|N_S]$, $\nu(Z_S|N_S)$ that do not involve small loss severity.
- Consistent means you do not obtain a result like $EX_S>EX_L$.

CAD with Limited Information

- Derivation of Equations
  - $E[Z_S|N_S]=N_S EX_S=N_S EZ_S/\lambda(N_S)=N_S(EZ-EZ_L)/(\lambda(1-q))$ $[q=\lambda(N_L)/\lambda]$
  - $\nu(Z_S|N_S)=\nu(X_S)/sqrt(N_S)=sqrt([\lambda(N_S)(\nu^2(Z_S)-c)-1]/N_S)=\ldots$

$=sqrt([\lambda(1-q)[(EZ)^2(\nu(Z)^2-c)-(EZ_L)^2(\nu(Z_L)^2-c)]-(EZ-EZ_L)^2]/(N_S(EZ-EZ_L)^2)$

Reinsurance Example

- Loss Assumptions:
  - Large loss threshold τ = \$200k, max. loss = \$1m,
  - EZ=\$25m, $\nu(Z)=0.28$, $\lambda(N_L)=21.5$
  - Empirical distribution for X_L, $\lambda=500$, c=0.0625.
  - Coverage on $Z_{XoL}=Z_L- \tau N_{L,}$ $Z_{Net}=Z_S+ \tau N_L$
- Coverage
  - Section 1 – Stop-Loss \$25m xs \$20m on $Z_{Net}+50\%Z_{XoL}$.
  - Section 2 – Coverage on remaining 50% of Z_{XoL}.
  - Z_{XoL} limited to \$12.5m per section.
  - Premium = \$10m of which \$1.5m is R\I margin.
  - Remainder to EA, PC = 100% of residual EA.

Reinsurance Example

- Used Igloo software to analyze cover with nine different assumptions for the mixing distribution G.
- Analysis based on full contract cash flows – premium at time 0, mid-tail type payout patterns, discount rate = 3%.
- Rich set of outputs – Summary and percentile statistics, for NPV(Loss), NPV(PC), NPV(Income), Prob(Negative NPV), ERD, TVaR capital, RoRAC return metric.
- Also, many charts.
- Conclusion: Though results do not vary greatly, could be enough to produce different accept/reject decisions. Uniform mixing distribution is somewhat of an outlier.

Multiple Lines of Business

- Mixed Poisson used for "common shock" correlation model.
- Here, mixing distributions are of the form $G_i=G_1[c_1]G_{2,i}[c_{2,i}]$, where G1 is the common component and the $G_{2,i}$ are the line-specific components.
- Independent ("straight") product with $c_i=c_1+c_{2,i}+c_1c_{2,i}$.
- Can also use "twisted product": $G_i=G_1[c_1]G_{2,i}[c_{2,i}/G1]$ (Eg: ISO Risk Load Negative Binomial with parameter risk model).
- For twisted product $c_i=c_1+c_{2,i}$.
- Each formulation results in correlations $\rho_{ij}=c_1/(\nu_i\nu_j)$.
- Multiline CAD with twisted product common shock – Can also impose correlation at CAD step.