Notes on Using Property Catastrophe Model Results

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Agenda

- Popular Cat Models
- OEP, Return Period, AEP and PML
- OEP and the Collective Risk Model
- Simulation of Cat Losses from RMS-style ELT
- Model Blending
- When is the AEP like the OEP?

Popular Cat Models

AIR
Year Event Loss Table (YELT)

Year	Event ID	Loss
1	1	100
3	2	500
3	3	300
4	4	100

Mean=250=(100+500+300+100)/4

Std=320={ $(100)^2+(500+300)^2+(100)^2)/4-250^\wedge 2$ }$^{0.5}$

RMS
Event Loss Table (ELT)

Event ID	Rate	Mean	Sdi	Sdc	Exposure
1	.10	500	500	500	10,000
2	.10	300	400	800	5,000
3	.50	200	300	400	4,000

The total standard deviation for each event is split into two additive components. This is an approximation used by RMS to facilitate combining the correlated pieces of the event.

Individual events are approximated assuming:
Poisson event count with Poisson Mean=[Rate]
Beta event size with Size mean=[Mean] and Size std=[Sdi]+[Sdc]

Mean=180=(.1)(500)+(.1)(300)+(.5)(200)

$$\text{Std}=737=\{\ (.1)((500+500)^2+500^2)\ \ (.1)((400+800)^2+300^2)\ \ (.5)((300+400)^2+200^2)\ \}^{0.5}$$

OEP, Return Period, AEP, and PML

The Occurrence Exceedance Probability (OEP) describes the distribution of the largest event in a year.

Year, Event Loss Detail

Year	Event ID	Loss
1	1	100
3	2	500
3	3	300
4	4	100

Year, Largest Event

Year	Largest Event
1	100
2	0
3	500
4	100

The OEP, *O(x)* is the probability that the largest event in a year exceeds *x*.

PML is the dollar amount associated with the OEP

Return Period is the expected number of years between events that exceed *x*.

Year	Largest Event
1	100
2	0
3	500
4	100

PML x	OEP $O(x)$	Return Period $r=1/O(x)$
0	75%	1.33
100	25%	4.00
500	0%	Inf

The columns pairs {PML,OEP} or {PML, Return Period} are often referred to as the PML curve or the OEP curve.

The PML curve or the OEP curve

PML *x*	OEP *O(x)*	Return Period *r=1/O(x)*
0	75%	1.33
100	25%	4.00
500	0%	Inf

Aggregate Exceedance Probability (AEP) describes the distribution of the annual event claim total

Year, Event Loss Detail

Year	Event ID	Loss
1	1	100
3	2	500
3	3	300
4	4	100

Year, Event Total

Year	Largest Event
1	100
2	0
3	800=(500+300)
4	100

Event Severity, Largest Annual Event and Annual Event Total

Random Variable	Cumulative distribution function (cdf)	Note	Notation
Event Severity	$F(x)$	Claim size component of a frequency-severity model	X_i, $i=1,\dots N$
Largest Annual Event	$F_M(x)$	$OEP=O(x)=1-F_M(x)$	$M=\max(X_1,\dots,X_N)$
Annual Event Total	$F_Z(x)$	$AEP=A(x)=1-F_Z(x)$	$Z=X_1+,\dots,+X_N$

OEP and the Collective Risk Model

OEP, Event Size and Event Counts

O(x)=OEP

$$
\begin{aligned}
O(x) &= \Pr(M > x) \text{ where } M = \max(X_1, ..., X_N) \\
&= 1 - F_M(x) \\
&= 1 - \Pr(X_i \leq x \text{ for } i = 1, ..., N) \\
&= 1 - \mathrm{E}_N(F_X(x)^N) = 1 - \mathrm{PGF}(F_X(x))
\end{aligned}
$$

$F_M(x)$=Cumulative distribution function (cdf) for the largest annual event

$F_X(x)$=Cdf for the Event Size distribution

PGF=Probability Generating function for the Event Count distribution N. $PGF_N(t)=E(t^N)$

OEP, Event Size and Event Counts

$$F_M(x) = \mathrm{PGF}(F_X(x))$$

$$O(x) = 1 - \mathrm{PGF}(F_X(x))$$

We can write the Event Size distribution as a function of the OEP. When the Event Counts are Poisson we can do this explicitly

With independent Event Counts and independent identically distributed Event Sizes

$$F_X(x) = \mathrm{PGF}^{-1}(1 - O(x))$$

For the Poisson Event Count distribution:

$$\mathrm{PGF}(t) = \exp(\lambda(1-t))$$

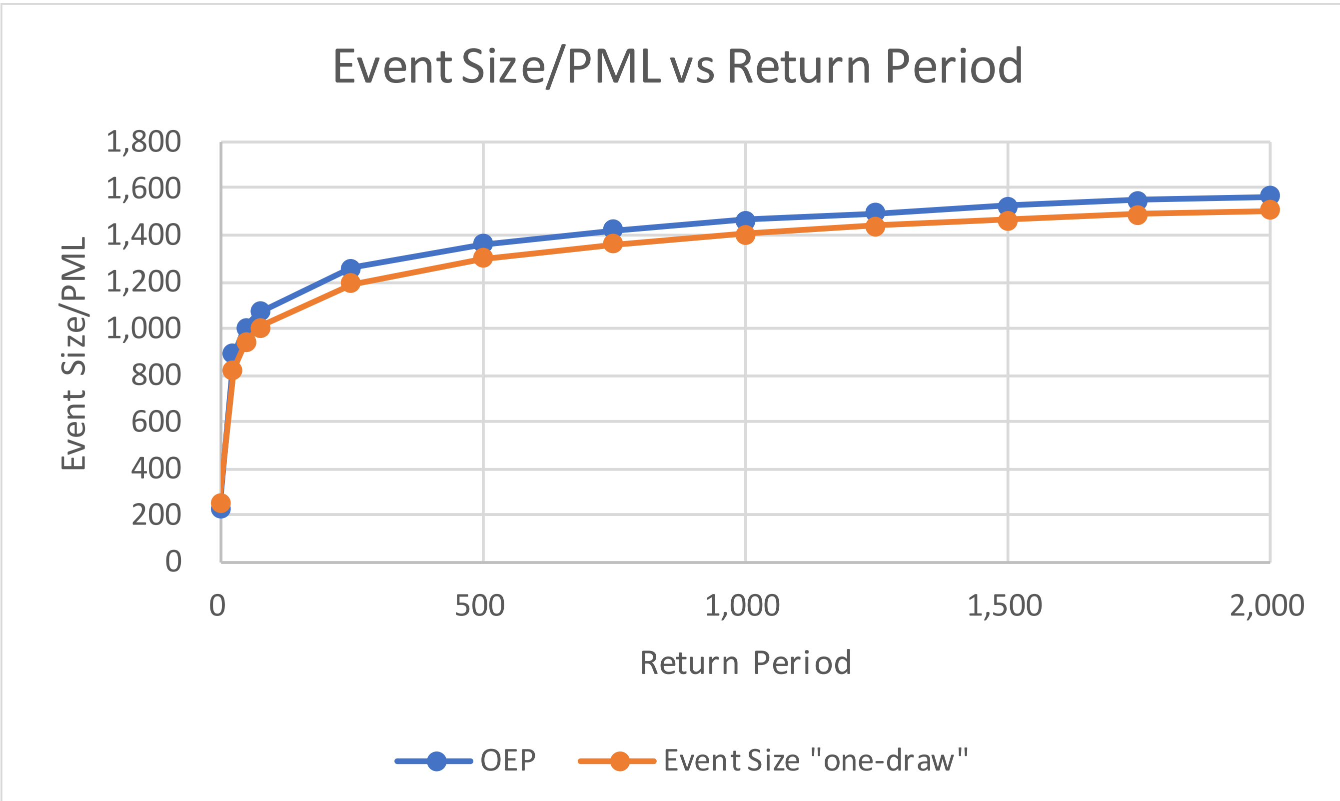
$$\mathrm{PGF}^{-1}(s) = 1 - \frac{\log(s)}{\lambda} \quad \text{and,}$$

$$F_X(x) = 1 + \frac{\log(1 - O(x))}{\lambda}$$

Sample OEP Conversion to Event Size Cumulative Distribution Function (cdf)

Return Period	Event Size or PML	OEP	Event Count	Cdf
R	x	$O(x)=1/R$	λ	$F(x)=1+\ln(1-O(x))/\lambda$
1.5	224	66.67%	1.5	26.76%
2.0	354	50.00%		53.79%
5.0	592	20.00%		85.12%
10.0	728	10.00%		92.98%
25.0	889	4.00%		97.28%
50.0	1,003	2.00%		98.65%
75.0	1,068	1.33%		99.11%
100.0	1,114	1.00%		99.33%
250.0	1,255	0.40%		99.73%
500.0	1,360	0.20%		99.87%
1,000.0	1,464	0.10%		99.93%
2,000.0	1,566	0.05%		99.97%

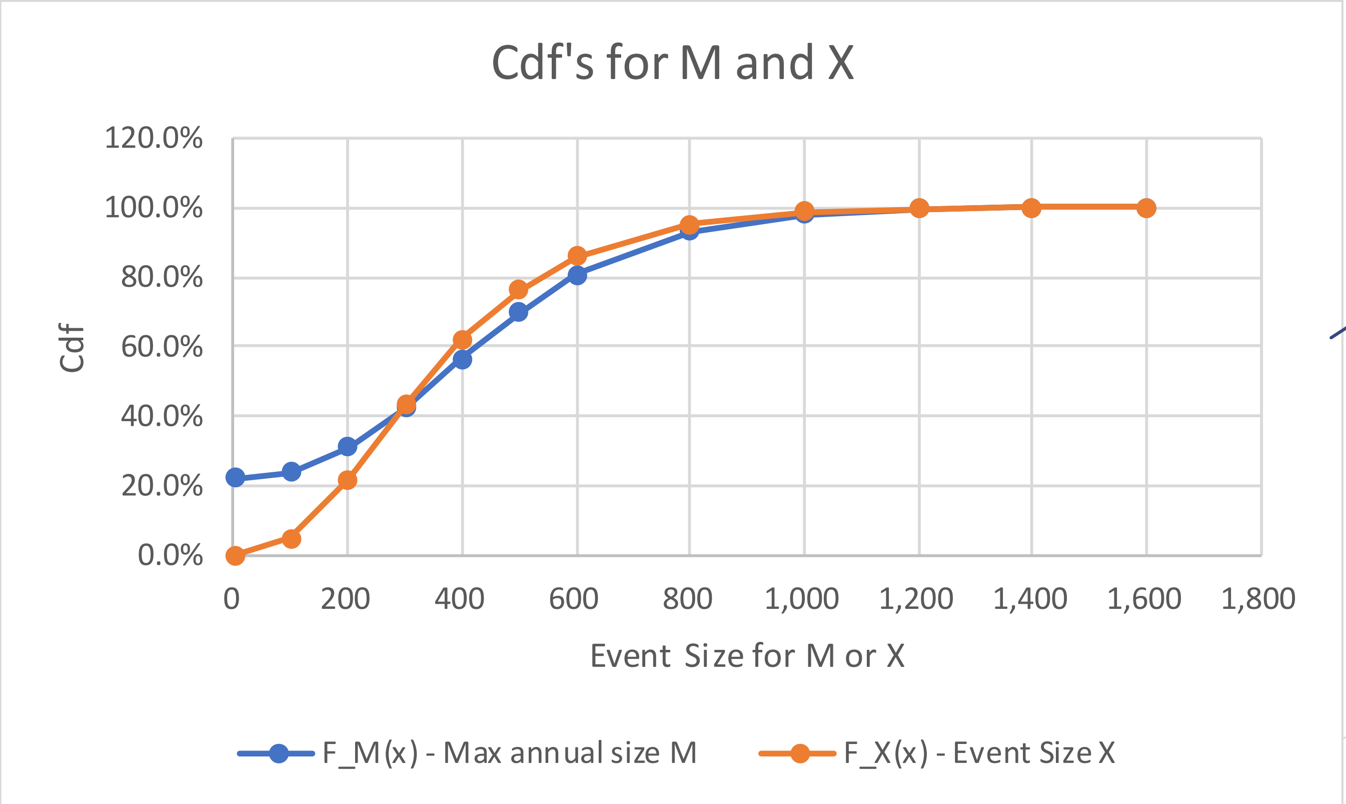
Comparison of Return Periods for Maximum Annual Loss M and Event Size X



Return Period for the OEP curve is $r=1/O(x)=1/(1-F_M(x))$

Return Period for the Event Size curve is $r=1/(1-F_X(x))$

Comparison of Cdf's for Maximum Annual Loss M and Event Size X



Recall that $F_M(x)=PGF(F_X(x))$

Simulation of Cat Losses from RMS-style ELT

Draw the Number of Events N from a Poisson Distribution with mean= $\lambda = \sum Rate_i$

Event ID	Rate	Mean	Sdi	Sdc	Exposure
1	.10	500	500	500	10,000
2	.10	300	400	800	5,000
3	.50	200	300	400	4,000

$$\lambda = (.1 + .1 + .5) = .7$$
$$N \sim Poisson(\lambda)$$

For each event, draw a random row R from the ELT in proportion to the rates.

Event ID	Rate	Mean	Sdi	Sdc	Exposure
1	.10	500	500	500	10,000
2	.10	300	400	800	5,000
3	.50	200	300	400	4,000

$$U \sim \text{Uniform}(0,1)$$

$$R = \min\left(r : U \leq \sum_{i=1}^{r} Rate_i / \lambda\right)$$

The size of the event is drawn from a Beta distribution with parameters computed from the ELT

$$a_R = \left(\frac{\mathrm{Mean_R}}{\mathrm{Sdi_R} + \mathrm{Sdc_R}}\right)^2 \left(1 - \frac{\mathrm{Mean_R}}{\mathrm{Exposure_R}}\right) - \frac{\mathrm{Mean_R}}{\mathrm{Exposure_R}}$$

$$b_R = a_R \left(\frac{\mathrm{Exposure_R}}{\mathrm{Mean_R}} - 1\right)$$

$$X \sim (\mathrm{Exposure}_R)(\mathrm{Beta}(a_R, b_R))$$

Excel formula:
X=BETA.INV(RAND(),a_R,b_R)*$\mathrm{Exposure}_R$

When the ELT has sub-lines there are additional steps.

		Personal Lines				Commercial Lines			
Event ID	Rate	Mean	Sdi	Stc	Exposure	Mean	Sdi	Sdc	Exposure
1	0.1	300	400	300	3000	200	300	200	1000
2	0.1	100	371	267	1000	200	150	533	4000
3	0.5	100	224	200	2000	100	200	200	2000

- Aggregate the two sublines and apply the simulation recipe
- Allocate the simulated losses to the sublines

Aggregate the sub-lines

Event ID	Rate	Mean	Sdi	Stc	Exposure
1	0.1	300+200	$\sqrt{400^2+300^2}$	300+200	3000+1000
2	0.1	100+200	$\sqrt{371^2+150^2}$	267+533	1000+4000
3	0.5	100+100	$\sqrt{224^2+200^2}$	200+200	2000+2000

Once the sub-line table is aggregated the event size *X* can be simulated using the earlier recipe.

$$\mathrm{Mean}_R = \sum_k \mathrm{Mean}_{R,k}$$

$$\mathrm{Exposure}_R = \sum_k \mathrm{Exposure}_{R,k}$$

$$\mathrm{Sdi}_R = \sqrt{\sum_k \mathrm{Sdi}^2_{R,k}}$$

$$\mathrm{Sdc}_R = \sum \mathrm{Sdc}_{R,k}$$

k indexes the sublines.

Allocate the simulated losses in proportion to the subline means.

$$X_k = X \frac{\text{Mean}_{R,k}}{\text{Mean}_{R}}$$

Model Blending

ELT/YELT Blending

Trial	Model Uniform	Model Selected	Event Count	Loss
1	0.599	AIR	1	100
2	0.041	RMS	0	-
3	0.401	RMS	2	168
3	-	-	-	268
4	0.5	AIR	1	100

- It produces a blended set of results that can be used to model dependencies with other portfolio results under certain constraints

OEP Blending

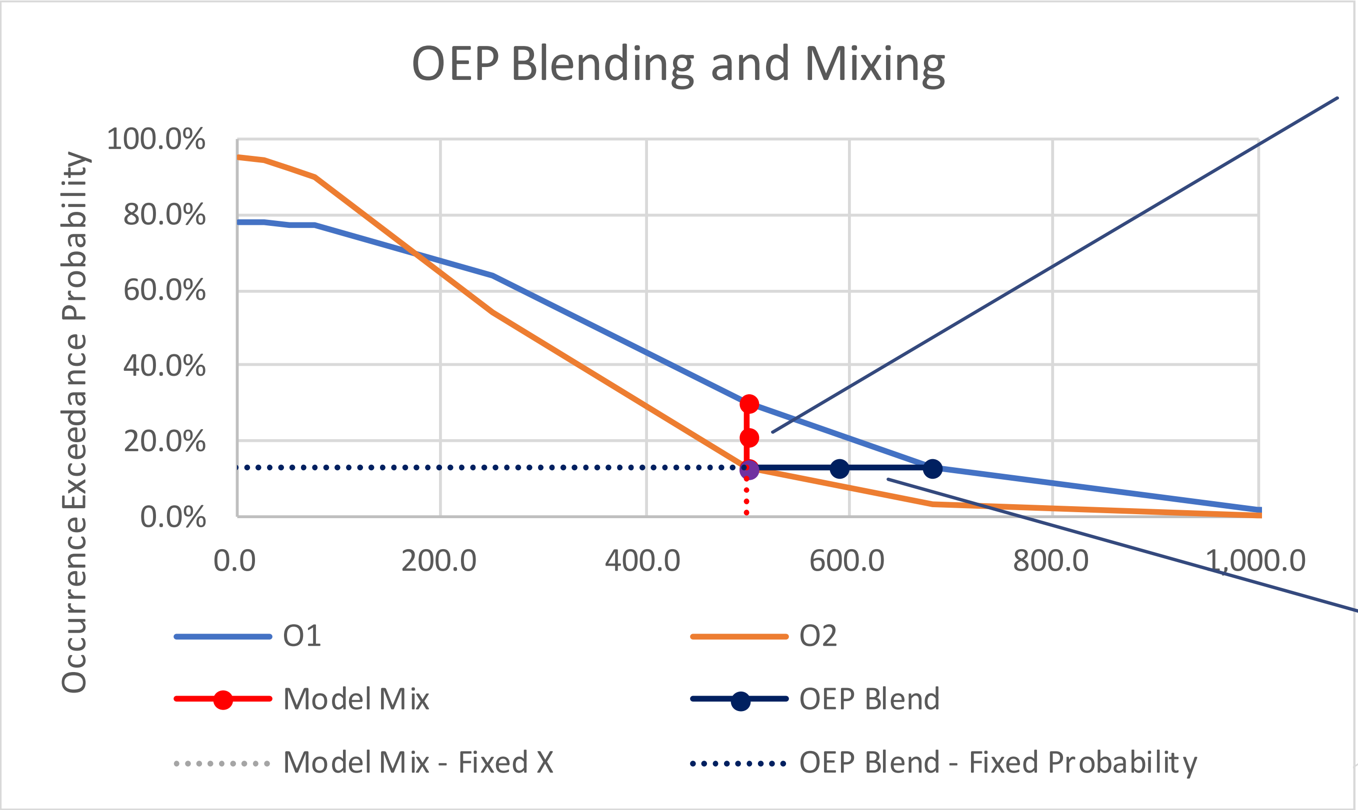
Return Period	AIR PML	RMS PML	50/50 PML
1.33	0	0	0
2	50	204	127
4	100	268	186
∞	500	272	384

- It is intuitive and it has become a common practice to present the blended results at a high level

ELT/YELT Blending vs. OEP Blending

- ELT/YELT blending is essentially a blend of probabilities
- OEP blending is essentially a blend of losses
- The two approaches produce different results and the differences vary by return periods
- The choice of which approach to use depends on business context and application of blended results

Model Mixing weights probabilities, OEP Blending weights PML's or Event Sizes



Model Mix -randomly draw from different models or weight the probabilities for fixed event sizes(PML's)

OEP Blend – weight the event sizes (PML's) for fixed probabilities

When is the AEP like the OEP?

AEP vs OEP

$$Z = X_1 + ... + X_N$$

$$F_Z(x) = \sum_n P_N(n) F_X^{(n)}(x)$$

$$A(x) = 1 - F_Z(x)$$

$$M = \max(X_1, ..., X_N)$$

$$F_M(x) = \sum_n P_N(n)(F_X(x))^n$$

$$O(x) = 1 - F_M(x)$$

$$A(x) - O(x) = \sum_{n=2}^{\infty} P_N(n)(F_X^{(n)}(x) - (F_X(x))^n)$$

“The next step is magic in actuarial science.”

$$\mathcal{F}(F_Z(x)) = \mathrm{PGF}(\mathcal{F}(F_X(x)))$$
$$F_M(x) = \mathrm{PGF}(F_X(x))$$

Mildenhall, Stephen J., The Report of the Research Working Party on Correlations and Dependencies Among All Risk Sources (Part 1): Correlation and Aggregate Loss Distributions With An Empahasis On The Iman-Conover Method, pg.136. https://www.casact.org/pubs/forum/06wforum/06w107.pdf