Agenda

1. The Problem
2. The Approach
3. Results and Conclusions

Swiss Re 2

The Problem

- Very often in actuarial practice we need to estimate the distribution of the aggregate losses
- This is especially important for QS Reinsurance treaties with aggregate features (Loss Ratio Cap, Annual Aggregate Deductible, Loss Corridor, etc.)
- However, in practice, there is little data available to construct a separate frequency / severity model, and only the first two moments of the historical loss distributions might be available
- So: **what shape of the Aggregate Loss Distribution should one assume to achieve the best results of the approximation?**
- **Does the answer to the prior question depend on the size of the book or line of business?**

Swiss Re 3

The Approach − General Idea

1. Create a (very) large sample of plausible annual aggregate losses
2. Fit different probability distributions to the sample
3. Test the goodness-of-fit and compare

The Approach - Details

1. Choose frequency and severity distributions
2. Simulate the number of claims (N) and individual claim amounts (X_i), put the individual loss amounts into per-occurrence layers ($X_1^l, \dots, X_N^l$), and calculate the corresponding aggregate loss ($S^l = \sum_{i=1}^{N} X_i^l$) in each layer l
3. Repeat many times (50,000) to obtain a sample of aggregate loss in each layer l
4. Estimate the parameters of different (candidate) probability distributions for each layer l
5. Test the goodness of fit of the distributions and compare results

The Approach

1. **Choose frequency and severity distributions**
2. Simulate the number of claims (N) and individual claim amounts (X_i), put the individual loss amounts into per-occurrence layers ($X_1^l, \dots, X_N^l$), and calculate the corresponding aggregate loss ($S^l = \sum_{i=1}^{N} X_i^l$) in each layer l
3. Repeat many times (50,000) to obtain a sample of aggregate loss in each layer l
4. Estimate the parameters of different candidate probability distributions for each layer l
5. Test the goodness of fit of the distributions and compare results

Severity Distributions – Casualty

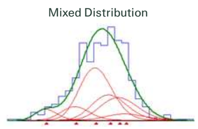
Source: ISO's Size of Loss Curves

1. Mixed Exponential Distributions:
- Prem Ops – e.g. Table 1 Section Group CA
- Products – e.g. Table C
- Commercial Auto – e.g. Extra Heavy: Section Group 7
- Different means, different weights

• Mixed distributions provide better fit to data than parametric distributions

Source: Swiss Re Pricing System

2. Lognormal Distributions:
- E&O – e.g., Medium Lawyers
- D&O – e.g., Public – Non F500



Data
Fitted Mixed Dist.
Individual Dists.

Severity Distributions – Commercial Property (All Perils)

Source: ISO's Size of Loss Curves

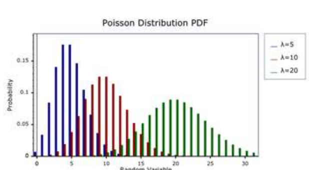
1. Mixed Exponential Distributions:
- $5M-$6M AOI (Small)
- $25M-$30M AOI (Middle)
- $100M-$125M AOI (High)
- Same means, different weights

Source: Company Data

2. Loss Submissions.

Frequency Distributions

- Poisson
- $\lambda_{standard\ lines} = 100, 500, \&\ 1{,}000$
- $\lambda_{professional\ lines} = 50\ \&\ 500$
- $\lambda_{property} = 100\ \&\ 500$



The Approach

1. Choose frequency and severity distributions
2. **Simulate the number of claims (N) and individual claim amounts (X_i), put the individual loss amounts into per-occurrence layers ($X_1^l, \ldots, X_N^l$), and calculate the corresponding aggregate loss ($S^l = \sum_{i=1}^{N} X_i^l$) in each layer l**
3. **Repeat many times (50,000) to obtain a sample of aggregate loss in each layer l**
4. Estimate the parameters of different candidate probability distributions for each layer l
5. Test the goodness of fit of the distributions and compare results

Simulation Methods

1. Latin Hypercube Sampling for Poisson frequency

2. Latin Hypercube Sampling, or

- For Mixed Exponential and Lognormal severity

Bootstrapping

- Used for simulation of severity from the property loss submissions
- Without replacement

Separating Individual Losses into Layers

- Amount of penetration of each simulated severity of loss within a layer =

  Min (Max (LOSS - RETENTION, 0), LIMIT)

  - For instance, for the layer $750K xs of $250K, RETENTION would be $250,000 and LIMIT would be $750,000
- The layers we used in Prem Ops, Products, and Auto are listed below:
  1. $250K Limit (Retention = $0)
  2. $500K Limit
  3. $1M Limit
  4. $750K xs of $250K
  5. $500K xs of $500K
  6. $4M xs of $1M

The Approach

1. Choose frequency and severity distributions
2. Simulate the number of claims (N) and individual claim amounts (X_i), put the individual loss amounts into per-occurrence layers ($X_1^l, \ldots, X_N^l$), and calculate the corresponding aggregate loss ($S^l = \sum_{i=1}^{N} X_i^l$) in each layer l
3. Repeat many times (50,000) to obtain a sample of aggregate loss in each layer l
4. **Estimate the parameters of different candidate probability distributions for each layer l**
5. Test the goodness of fit of the distributions and compare results

Swiss Re 13

Candidate Aggregate Loss Distributions

- Two-parameter distributions, as observed data is often too sparse to reliably estimate more than two parameters:
  - Normal
  - Logistic
  - Gamma
  - Inverse Gauss
  - Lognormal

Swiss Re 14

Candidate Aggregate Loss Distributions

Distribution	Parameters	Probability Density Function	Mean	Variance
Normal	μ - location $\sigma > 0$ - scale	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2
Logistic	μ - location $s > 0$ - scale	$\frac{e^{-(x-\mu)/s}}{s\left(1+e^{-(x-\mu)/s}\right)^2}$	μ	$\frac{s^2\pi^2}{3}$
Gamma	$\alpha > 0$ - shape $\beta > 0$ - rate	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$
Inverse Gauss	$\mu > 0$ - location $\lambda > 0$ - shape	$\left[\frac{\lambda}{2\pi x^3}\right]^{1/2} \exp\left\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right\}$	μ	$\frac{\mu^3}{\lambda}$
Lognormal	μ - scale $\sigma > 0$ - shape	$\frac{1}{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$	$e^{(\mu+\sigma^2/2)}$	$e^{(2\mu+\sigma^2)}(e^{\sigma^2}-1)$

Swiss Re 15

Candidate Aggregate Loss Distributions

Distribution	CV	Skewness	Ex. Kurtosis
Normal	c	0	0
Logistic	c	0	1.2
Gamma	c	$2c$	$6c^2$
Inverse Gauss	c	$3c$	$15c^2$
Lognormal	c	$c + c^3$	$16c^2 + 15c^4 + 6c^6 + c^8$

Swiss Re 16

Parameter Estimation

- Method of Moments

Swiss Re 17

The Approach

1. Choose frequency and severity distributions
2. Simulate the number of claims (N) and individual claim amounts (X_i), put the individual loss amounts into per-occurrence layers ($X_1^l, \ldots, X_N^l$), and calculate the corresponding aggregate loss ($S^l = \sum_{i=1}^{N} X_i^l$) in each layer l
3. Repeat many times (50,000) to obtain a sample of aggregate loss in each layer l
4. Estimate the parameters of different candidate probability distributions for each layer l
5. **Test the goodness of fit of the distributions and compare results**

Swiss Re 18

Percentile Matching Test

- Compares the survival functions $Prob\{X > x\}$ of the simulated aggregate loss distribution with fitted probability distributions
- Allows us to compare distributions in their tails

Swiss Re 19

Excess Expected Loss Cost Test

- Compares the conditional means of distributions in excess of different amounts, $E[X - x|X > x] * Prob\{X > x\}$
- Important for Aggregate Stop Loss Coverage and Aggregate Deductible Coverage (AAD)

Swiss Re 20

Results

Swiss Re 21

Results (Comm Auto, 100, $500K Limit)

LoB: 'Commercial Auto', Size: 100, Layer: 500x0
Mean Aggregate Loss: $2,038,218

Percentile matching LoB: 'Commercial Auto', Size: 100, Layer: 500x0

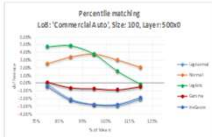
% of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	70%	-0.18%	2.52%	4.74%	0.09%	-0.47%
90%	58%	-2.07%	3.40%	4.87%	-0.62%	-2.26%
100%	46%	-2.77%	3.79%	3.79%	-0.73%	-2.80%
110%	35%	-2.90%	3.02%	1.55%	-0.86%	-2.79%
120%	26%	-2.09%	2.04%	-0.16%	-0.46%	-1.89%

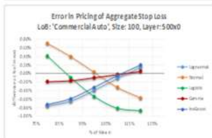
Error in Pricing of Aggregate Stop Loss LoB: 'Commercial Auto', Size: 100, Layer: 500x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	Inf	512,046	-0.76%	0.71%	0.41%	-0.19%	-0.73%
90%	Inf	381,743	-0.65%	0.40%	-0.09%	-0.17%	-0.59%
100%	Inf	275,393	-0.39%	0.03%	-0.54%	-0.10%	-0.33%
110%	Inf	192,640	-0.11%	-0.32%	-0.82%	-0.02%	-0.05%
120%	Inf	130,463	0.15%	-0.57%	-0.87%	0.05%	0.19%

Error in Corridor Pricing LoB: 'Commercial Auto', Size: 100, Layer: 500x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
75%	80%	73,880	-0.01%	-0.11%	-0.22%	-0.01%	0.00%
80%	90%	130,303	0.11%	-0.31%	-0.50%	0.02%	0.14%
90%	100%	106,350	0.25%	-0.37%	-0.44%	0.07%	0.26%
100%	110%	82,753	0.28%	-0.36%	-0.28%	0.07%	0.28%
110%	120%	62,177	0.26%	-0.25%	-0.06%	0.07%	0.24%





X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

Swiss Re 22

Results (GL Products, 500, $1M Limit)

LoB: 'GL: Products', Size: 500, Layer: 1000x0
Mean Aggregate Loss: $38,737,178

Percentile matching LoB: 'GL: Products', Size: 500, Layer: 1000x0

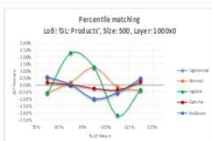
% of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	97%	0.56%	-0.55%	-0.60%	0.18%	0.57%
90%	81%	0.05%	0.20%	2.28%	0.03%	0.02%
100%	49%	-0.98%	1.26%	1.26%	-0.24%	-0.98%
110%	19%	-0.57%	-0.11%	-2.19%	-0.36%	-0.54%
120%	4%	0.47%	-0.39%	-0.34%	0.22%	0.47%

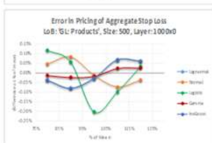
Error in Pricing of Aggregate Stop Loss LoB: 'GL: Products', Size: 500, Layer: 1000x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	Inf	7,795,920	-0.04%	0.05%	0.12%	-0.01%	-0.04%
90%	Inf	4,289,446	-0.08%	0.08%	0.06%	-0.02%	-0.08%
100%	Inf	1,744,726	-0.03%	-0.01%	-0.20%	-0.02%	-0.03%
110%	Inf	475,856	0.07%	-0.07%	-0.10%	0.02%	0.07%
120%	Inf	80,948	0.06%	-0.04%	0.03%	0.03%	0.06%

Error in Corridor Pricing LoB: 'GL: Products', Size: 500, Layer: 1000x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
75%	80%	1,900,012	-0.02%	0.02%	0.04%	-0.01%	-0.02%
80%	90%	3,506,474	-0.04%	0.04%	-0.06%	-0.01%	-0.04%
90%	100%	2,544,720	0.05%	-0.09%	-0.26%	0.01%	0.05%
100%	110%	1,268,870	0.10%	-0.06%	0.10%	0.04%	0.09%
110%	120%	394,908	-0.01%	0.03%	0.13%	0.00%	-0.01%





X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

Swiss Re 23

Results (GL PremOps, 1000, $250K Limit)

LoB: 'GL: PremOps', Size: 1000, Layer: 250x0
Mean Aggregate Loss: $15,888,810

Percentile matching LoB: 'GL: PremOps', Size: 1000, Layer: 250x0

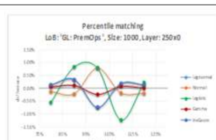
% of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	100%	0.13%	-0.13%	-0.55%	0.06%	0.13%
90%	91%	0.34%	-0.22%	0.83%	0.12%	0.33%
100%	49%	-0.74%	0.79%	0.79%	-0.23%	-0.74%
110%	10%	0.17%	-0.18%	-1.23%	0.08%	0.18%
120%	1%	0.13%	-0.20%	0.22%	0.01%	0.13%

Error in Pricing of Aggregate Stop Loss LoB: 'GL: PremOps', Size: 1000, Layer: 250x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
80%	Inf	3,978,997	0.00%	0.01%	0.03%	0.00%	0.00%
90%	Inf	2,051,967	-0.03%	0.04%	0.07%	-0.01%	-0.03%
100%	Inf	610,646	0.00%	0.00%	-0.13%	0.00%	0.00%
110%	Inf	76,850	0.03%	-0.03%	0.00%	0.01%	0.03%
120%	Inf	3,840	0.01%	-0.01%	0.02%	0.00%	0.01%

Error in Corridor Pricing LoB: 'GL: PremOps', Size: 1000, Layer: 250x0 (difference as % of mean)

From % of Mean	To % of Mean	Empirical	Lognormal	Normal	Logistic	Gamma	InvGauss
75%	80%	993,216	0.00%	0.00%	0.02%	0.00%	0.00%
80%	90%	1,927,030	-0.03%	0.03%	0.04%	-0.01%	-0.03%
90%	100%	1,441,321	0.03%	-0.03%	-0.19%	0.01%	0.03%
100%	110%	533,796	0.03%	-0.04%	0.12%	0.01%	0.03%
110%	120%	73,010	-0.03%	0.03%	0.02%	-0.01%	-0.03%



X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

Swiss Re 24

Results (Commercial Auto, 100, $750K xs of $250K)

LoB: 'Commercial Auto', Size: 100, Layer: 750x250
Mean Aggregate Loss: $1,015,045

Percentile matching LoB: 'Commercial Auto', Size: 100, Layer: 750x250

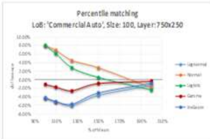
% of Mean	Empirical P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
		difference				
100%	42%	-4.16%	7.96%	7.96%	-1.09%	-4.41%
110%	37%	-5.14%	6.85%	6.07%	-1.72%	-5.12%
125%	31%	-6.04%	4.49%	2.76%	-2.58%	-5.71%
150%	20%	-3.65%	2.78%	0.52%	-0.86%	-3.08%
200%	9%	-1.33%	-1.84%	-2.43%	-0.33%	-0.85%

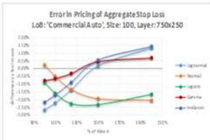
Error in Pricing of Aggregate Stop Loss LoB: 'Commercial Auto', Size: 100, Layer: 750x250

From % of Mean	To % of Mean	Empirical E[X-x|X>x]P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
100%	Inf	273,255	-2.68%	0.24%	-0.90%	-0.78%	-2.18%
110%	Inf	233,028	-2.22%	-0.50%	-1.61%	-0.64%	-1.70%
125%	Inf	181,017	-1.37%	-1.36%	-2.27%	-0.31%	-0.88%
150%	Inf	112,924	0.18%	-1.95%	-2.33%	0.44%	0.55%
200%	Inf	42,599	1.35%	-2.05%	-1.67%	0.68%	1.44%

Error in Corridor Pricing LoB: 'Commercial Auto', Size: 100, Layer: 750x250

From % of Mean	To % of Mean	Empirical E[X^]	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
90%	100%	45,231	0.34%	-0.84%	-0.88%	0.07%	0.39%
100%	110%	40,227	0.47%	-0.74%	-0.70%	0.14%	0.48%
110%	125%	52,011	0.85%	-0.86%	-0.66%	0.33%	0.82%
125%	150%	68,093	1.55%	-0.59%	-0.06%	0.76%	1.43%
150%	200%	70,325	1.16%	-0.10%	0.66%	0.24%	0.89%





X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

Swiss Re

Results (GL Products, 500, $ 4M xs $ 1M)

LoB: 'GL: Products', Size: 500, Layer: 4000x1000
Mean Aggregate Loss: $15,097,244

Percentile matching LoB: 'GL: Products', Size: 500, Layer: 4000x1000

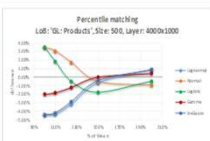
% of Mean	Empirical P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
		difference				
100%	47%	-4.37%	3.47%	3.47%	-2.01%	-4.43%
110%	37%	-4.30%	3.06%	1.82%	-1.84%	-4.18%
125%	25%	-3.14%	1.73%	-0.51%	-1.20%	-2.88%
150%	12%	-0.65%	-0.49%	-1.77%	0.00%	-0.39%
200%	2%	0.88%	-0.92%	-0.47%	0.43%	0.90%

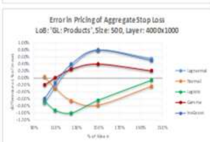
Error in Pricing of Aggregate Stop Loss LoB: 'GL: Products', Size: 500, Layer: 4000x1000

From % of Mean	To % of Mean	Empirical E[X-x|X>x]P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
100%	Inf	2,474,087	-0.71%	0.04%	-0.66%	-0.20%	-0.59%
110%	Inf	1,841,240	-0.26%	-0.29%	-0.92%	0.00%	-0.15%
125%	Inf	1,133,486	0.32%	-0.65%	-1.00%	0.25%	0.40%
150%	Inf	455,191	0.79%	-0.77%	-0.64%	0.39%	0.80%
200%	Inf	50,799	0.55%	-0.23%	-0.06%	0.20%	0.50%

Error in Corridor Pricing LoB: 'GL: Products', Size: 500, Layer: 4000x1000

From % of Mean	To % of Mean	Empirical E[X^]	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
90%	100%	776,522	0.41%	-0.34%	-0.40%	0.20%	0.43%
100%	110%	632,847	0.44%	-0.33%	-0.26%	0.20%	0.44%
110%	125%	707,752	0.58%	-0.36%	-0.08%	0.24%	0.55%
125%	150%	678,297	0.47%	-0.13%	0.36%	0.15%	0.40%
150%	200%	404,392	-0.24%	0.54%	0.58%	-0.19%	-0.30%





X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

Swiss Re

Results (GL PremOps, 1000, $500K xs of $500K)

LoB: 'GL: PremOps', Size: 1000, Layer: 500x500
Mean Aggregate Loss: $3,273,075

Percentile matching LoB: 'GL: PremOps', Size: 1000, Layer: 500x500

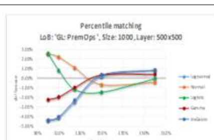
% of Mean	Empirical P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
		difference				
100%	47%	-4.39%	2.55%	2.55%	-2.25%	-4.43%
110%	37%	-4.15%	2.19%	0.80%	-1.99%	-4.04%
125%	23%	-2.53%	1.06%	-1.20%	-1.01%	-2.31%
150%	9%	0.14%	-0.69%	-1.48%	0.34%	0.31%
200%	1%	0.83%	-0.44%	-0.07%	0.39%	0.81%

Error in Pricing of Aggregate Stop Loss LoB: 'GL: PremOps', Size: 1000, Layer: 500x500

From % of Mean	To % of Mean	Empirical E[X-x|X>x]P(X>x)	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
100%	Inf	470,620	-0.50%	0.01%	-0.60%	-0.15%	-0.42%
110%	Inf	332,831	-0.07%	-0.23%	-0.77%	0.07%	0.01%
125%	Inf	186,092	0.45%	-0.48%	-0.71%	0.30%	0.50%
150%	Inf	60,980	0.72%	-0.50%	-0.32%	0.36%	0.71%
200%	Inf	3,629	0.32%	-0.08%	0.02%	0.12%	0.29%

Error in Corridor Pricing LoB: 'GL: PremOps', Size: 1000, Layer: 500x500

From % of Mean	To % of Mean	Empirical E[X^]	Lognormal	Normal	Logistic	Gamma	InvGauss
			difference as % of mean				
90%	100%	173,326	0.40%	-0.26%	-0.33%	0.21%	0.41%
100%	110%	137,789	0.43%	-0.24%	-0.17%	0.22%	0.43%
110%	125%	146,739	0.52%	-0.24%	0.06%	0.24%	0.49%
125%	150%	125,112	0.26%	-0.03%	0.39%	0.06%	0.21%
150%	200%	57,351	-0.39%	0.42%	0.34%	-0.24%	-0.42%



X is aggregate loss
X^ is loss in an aggregate layer, namely max(min(X-x_1, 0), x_2-x_1)

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Conclusions

Gamma distribution provides a fit that is almost always the best for both ground up and excess layers (out of the five candidate distributions considered)

Gamma distribution provides a uniformly reasonable approximation to the aggregate loss on the interval from the mean to at least two means of the aggregate distribution

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Thank you!

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