GLM I
An Introduction to Generalized Linear Models

CAS Ratemaking and Product Management Seminar
March 2012

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Outline

- Overview of Statistical Modeling
- Linear Models
  - ANOVA
  - Simple Linear Regression
  - Multiple Linear Regression
  - Categorical Variables
  - Transformations
- Generalized Linear Models
  - Why GLM?
  - From Linear to GLM
  - Basic Components of GLM's
  - Common GLM structures
- References

Generic Modeling Schematic

Predictor Vars
- Driver Age
- Region
- Relative Equity
- Credit Score

Weights
- Claims
- Exposures
- Premium

Response Vars
- Losses
- Default
- Persistency

Statistical Model

Model Results
- Parameters
- Validation Statistics

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Basic Linear Model Structures - Overview

- Simple ANOVA :
  - $Y_{ij} = \mu + e_{ij}$ or more generally $Y_{ij} = \mu + \psi_i + e_{ij}$
  - In Words: Y is equal to the mean for the group with random variation and possibly fixed variation
  - Traditional Classification Rating – Group Means
  - Assumptions: errors independent & follow $N(0,\sigma_e^2)$
  - $\sum \psi_i = 0 \quad i = 1,\ldots,k$ (fixed effects model)
  - $\psi_i \sim N(0,\sigma_\psi^2)$ (random effects model)

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Basic Linear Model Structures - Overview

- Simple Linear Regression : $y_i = b_o + b_1 x_i + e_i$
  - Assumptions:
    - linear relationship
    - errors independent and follow $N(0,\sigma_e^2)$
- Multiple Regression : $y_i = b_o + b_1 x_{1i} + \ldots + b_n x_{ni} + e_i$
  - Assumptions: same, but with n independent random variables (RV's)
- Transformed Regression : transform x, y, or both; maintain errors are $N(0,\sigma_e^2)$

$$y_i = \exp(x_i) \Rightarrow \log(y_i) = x_i$$

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Simple Regression *(special case of multiple regression)*

- Model: $Y_i = b_o + b_1 X_i + e_i$
  - Y is the dependent variable explained by X, the independent variable
  - Y could be Pure Premium, Default Frequency, etc
  - Want to estimate relationship of how Y depends on X using observed data
  - Prediction: $Y = b_o + b_1 x^*$ for some new x* (usually with some confidence interval)

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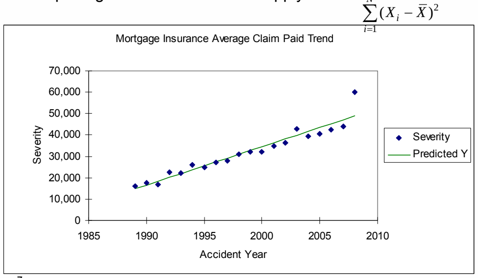
Simple Regression

- A formalization of best fitting a line through data with a ruler and a pencil
- Correlative relationship
- Simple e.g. determine a trend to apply $\beta = \frac{\sum_{i=1}^{N}(Y_i - \overline{Y})(X_i - \overline{X})}{\sum_{i=1}^{N}(X_i - \overline{X})^2}, \quad a = \overline{Y} - \beta\overline{X}$



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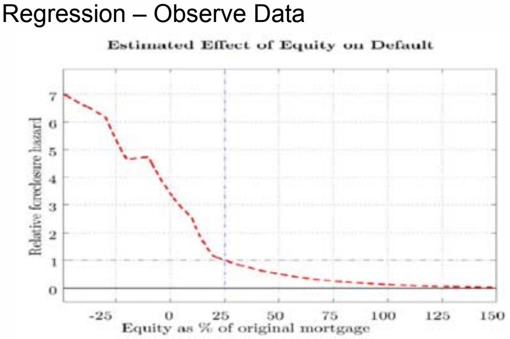
Note: All data in this presentation are for illustrative purposes only

Regression – Observe Data



Source: Foote et al., "Negative Equity and Foreclosure: Theory and Evidence."[32]

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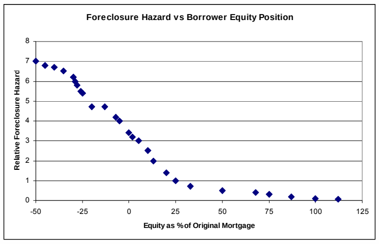
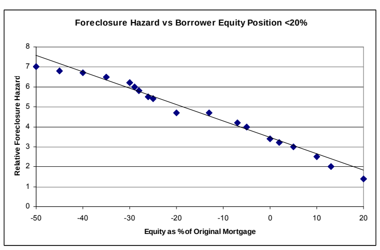
Regression – Observe Data



Regression – Observe Data



Simple Regression

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	52.7482	52.7482	848.2740	<0.0001
Residual	17	1.0571	0.0622		
Total	18	53.8053			

- How much of the sum of squares is explained by the regression?

  SS = Sum Squared Errors

  SSTotal = SSRegression + SSResidual (Residual also called Error)

  SSTotal = $\sum (y_i - \bar{y})^2 = 53.8053$

  SSRegression = $b_{1\ est} * [\sum x_i y_i - 1/n(\sum x_i)(\sum y_i)] = 52.7482$

  SSResidual = $\sum (y_i - y_{i\ est})^2$

  = SSTotal – SSRegression

  1.0571 = 53.8053 – 52.742

Simple Regression

ANOVA

	df	SS	MS	F	Significance F
Regression	1	52.7482	52.7482	848.2740	<0.0001
Residual	17	1.0571	0.0622		
Total	18	53.8053			

Regression Statistics	
Multiple R	0.9901
R Square	0.9804
Adjusted R Square	0.9792

- MS = SS divided by df
- R^2: (SS Regression/SS Total)
  0.9804 = 52.7482 / 53.8053
  - percent of variance explained
- F statistic: (MS Regression/MS Residual)
- significance of regression:
  - F tests H_o: b_1=0 v. H_A: $b_1 \neq 0$

Simple Regression

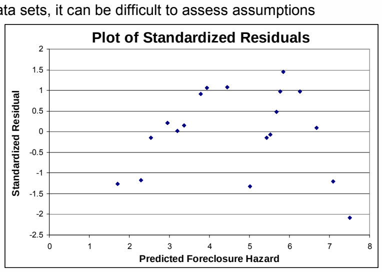
	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%
Intercept	3.3630	0.0730	46.0615	0.0000	3.2090	3.5170	3.2090	3.5170
X	-0.0828	0.0028	-29.1251	0.0000	-0.0888	-0.0768	-0.0888	-0.0768

T statistics: $(b_{i\,est} - H_o(b_i)) / s.e.(b_{i\,est})$

- significance of individual coefficients
- T^2 = F for b_1 in simple regression
- $(-29.1251)^2 = 848.2740$
- F in multiple regression tests that at least one coefficient is nonzero. For the simple case, at least one is the same as the entire model. F stat tests the global null model.

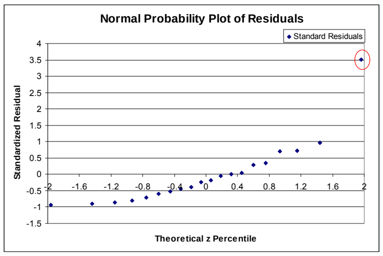
Residuals Plot

- Looks at $(y_{obs} - y_{pred})$ vs. y_{pred}
- Can assess linearity assumption, constant variance of errors, and look for outliers
- Standardized Residuals (raw residual scaled by standard error) should be random scatter around 0, standard residuals should lie between -2 and 2
- With small data sets, it can be difficult to assess assumptions



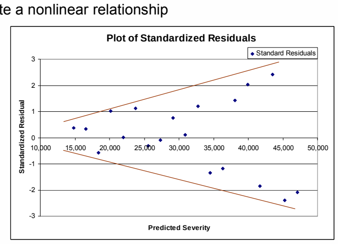
Normal Probability Plot

- Can evaluate assumption $e_i \sim N(0,\sigma_e^2)$
  - Plot should be a straight line with intercept μ and slope σ_e^2
  - Can be difficult to assess with small sample sizes



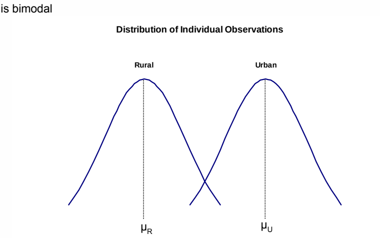
Residuals

- If absolute size of residuals increases as predicted value increases, may indicate nonconstant variance
- May indicate need to transform dependent variable
- May need to use weighted regression
- May indicate a nonlinear relationship



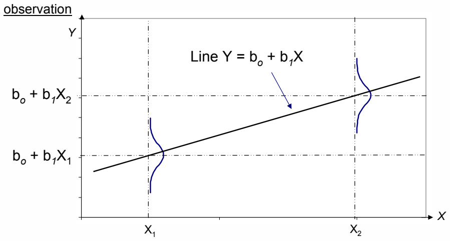
Distribution of Observations

- Average claim amounts for Rural drivers are normally distributed as are average claim amounts for Urban drivers
- Mean for Urban drivers is twice that of Rural drivers
- The variance of the observations is equal for Rural and Urban
- The total distribution of average claim amounts across Rural and Urban is not Normal
  - here it is bimodal



Distribution of Observations

- The basic form of the regression model is $Y = b_o + b_1X + e$
- $\mu_i = E[Y_i] = E[b_o + b_1X_i + e_i] = b_o + b_1X_i + E[e_i] = b_o + b_1X_i$
- The mean value of Y, rather than Y itself, is a linear function of X
- The observations Y_i are normally distributed about their mean μ_i $Y_i \sim N(\mu_i, \sigma_e^2)$
- Each Y_i can have a different mean μ_i but the variance σ_e^2 is the same for each observation



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Multiple Regression *(special case of a GLM)*

- $Y = \beta_0 + \beta_1X_1 + \beta_2X_2 + \ldots + \beta_nX_n + \varepsilon$
- $E[\underline{Y}] = \underline{\beta}\,\mathbf{X}$
  - $\underline{\beta}$ is a vector of the parameter coefficients
  - $\underline{Y}$ is a vector of the dependent variable
  - $\mathbf{X}$ is a matrix of the independent variables
    - Each column is a variable
    - Each row is an observation
- Same assumptions as simple regression
  1) model is correct (there exists a linear relationship)
  2) errors are independent
  3) variance of e_i constant
  4) $e_i \sim N(0,\sigma_e^2)$
- Added assumption the n variables are independent

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Multiple Regression

- Uses more than one variable in regression model
  - R-sq always goes up as add variables
  - Adjusted R-Square puts models on more equal footing
  - Many variables may be insignificant
- Approaches to model building
  - Forward Selection - Add in variables, keep if “significant”
  - Backward Elimination - Start with all variables, remove if not “significant”
  - Fully Stepwise Procedures – Combination of Forward and Backward

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Multiple Regression

- **Goal** : Find a simple model that explains things well with assumptions reasonably satisfied
- **Cautions**:
  - All predictor variables assumed independent
    - as add more, they may not be
    - multicollinearity— linear relationships among the X's
  - Tradeoff:
    - Increase # of parameters (1 for each variable in regression) ➔ lose degrees of freedom (df)
    - keep df as high as possible for general predictive power ➔ problem of over-fitting

Multiple Regression

- ***Model***: Claim Rate = *ƒ* (Loan-to-Value (LTV), Delinquency Status, Home Price Appreciation (HPA))
- Degrees of freedom ~ # observations - # parameters
- Any parameter with a t-stat with absolute value less than 2 is not significant

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.97
R Square	0.94
Adjusted R Square	0.94
Standard Error	0.05
Observations	586

ANOVA

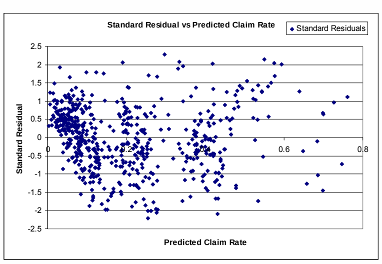
	df	*SS*	*MS*	*F*	*Significance F*
Regression	10	17.716	1.772	849.031	< 0.00001
Residual	575	1.200	0.002		
Total	585	18.916			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	1.30	0.03	41.4	0.00	1.24	1.36
ltv85	-0.10	0.01	-12.9	0.00	-0.11	-0.09
ltv90	-0.07	0.01	-9.1	0.00	-0.08	-0.06
ltv95	-0.04	0.01	-9.1	0.00	-0.05	-0.03
ltv97	-0.02	0.01	-6.0	0.00	-0.03	-0.01
ss30	-0.75	0.01	-55.3	0.00	-0.77	-0.73
ss60	-0.61	0.01	-56.0	0.00	-0.63	-0.59
ss90	-0.45	0.01	-53.5	0.00	-0.47	-0.43
ss120	-0.35	0.01	-40.1	0.00	-0.37	-0.33
ssFCL	-0.24	0.01	-22.8	0.00	-0.26	-0.22
HPA	-0.48	0.03	-18.0	0.00	-0.53	-0.43

- *T-stats are also used for evaluating significance of coefficients in GLM's*

Multiple Regression

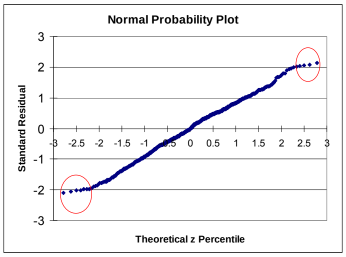
- Residuals Plot



- *Residual Plots are also used to evaluate fits of GLM's*

Multiple Regression

- Normal Probability Plot



- *Percentile or Quantile Plots are also used to evaluate fits of GLM's*

Categorical Variables *(used in LM's and GLM's)*

- Explanatory variables can be discrete or continuous
- Discrete variables generally referred to as "factors"
- Values each factor takes on referred to as "levels"
- Discrete variables also called Categorical variables
- In the multiple regression example given, all variables were categorical except HPA

Categorical Variables

- Assign each level a "Dummy" variable
  - A binary valued variable
  - X=1 means member of category and 0 otherwise
  - Always a reference category
    - defined by being 0 for all other levels
  - If only one factor in model, then reference level will be intercept of regression
  - If a category is not omitted, there will be linear dependency
    - "Intrinsic Aliasing"

Categorical Variables

- Example: Loan – To – Value (LTV)
  - Grouped for premium – 5 Levels
    - <=85%, LTV85
    - 85.01% - 90%, LTV90
    - 90.01% - 95%, LTV95
    - 95.01% - 97%, LTV97
    - >97% Reference
  - Generally positively correlated with claim frequency
  - Allowing each level it's own dummy variable allows for the possibility of non-monotonic relationship
  - Each modeled coefficient will be relative to reference level

		X1	X2	X3	X4
Loan #	LTV	LTV85	LTV90	LTV95	LTV97
1	97	0	0	0	1
2	93	0	0	1	0
3	95	0	0	1	0
4	85	1	0	0	0
5	100	0	0	0	0

Design Matrix

Transformations

- A possible solution to nonlinear relationship or unequal variance of errors
- Transform predictor variables, response variable, or both
- Examples:
  - $Y' = \log(Y)$
  - $X' = \log(X)$
  - $X' = 1/X$
  - $Y' = \sqrt{Y}$
- Substitute transformed variable into regression equation
- Maintain assumption that errors are $N(0,\sigma_e^2)$

Why GLM?

- What if the variance of the errors increases with predicted values?
  - More variability associated with larger claim sizes
- What if the values for the response variable are strictly positive?
  - assumption of normality violates this restriction
- If the response variable is strictly non-negative, intuitively the variance of Y tends to zero as the mean of X tends to zero
  - Variance is a function of the mean (poisson, gamma)
- What if predictor variables do not enter additively?
  - Many insurance risks tend to vary multiplicatively with rating factors

Classic Linear Model to Generalized Linear Model

- LM:
  - **X** is a matrix of the independent variables
    - Each column is a variable
    - Each row is an observation
  - $\underline{\beta}$ is a vector of parameter coefficients
  - $\underline{\varepsilon}$ is a vector of residuals

LM

$\underline{Y} = \underline{\beta}\,\mathbf{X} + \underline{\varepsilon}$

$E[\underline{Y}] = \underline{\beta}\,\mathbf{X}$

$E[\underline{Y}] = \underline{\mu} = \underline{\eta}$

$\varepsilon \sim N(0, \sigma_e^2)$

- GLM:
  - **X**, $\underline{\beta}$ same as in LM
  - $\underline{\varepsilon}$ is still vector of residuals
  - *g* is called the "link function"

GLM

$g\,(\underline{\mu}) = \underline{\eta} = \underline{\beta}\,\mathbf{X}$

$E[\underline{Y}] = \underline{\mu} = g^{-1}(\underline{\eta})$

$\underline{Y} = g^{-1}(\underline{\eta}) + \underline{\varepsilon}$

ε ~ exponential family

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Classic Linear Model to Generalized Linear Model

- LM:
  1) *Random Component* : Each component of $\underline{Y}$ is independent and normally distributed. The mean μ_i allowed to differ, but all Y_i have common variance σ_e^2
  2) *Systematic Component* : The n covariates combine to give the "linear predictor"

     $\underline{\eta} = \underline{\beta}\,\mathbf{X}$
  3) *Link Function* : The relationship between the random and systematic components is specified via a link function. In linear model, link function is identity fnc.

     $E[\underline{Y}] = \underline{\mu} = \underline{\eta}$
- GLM:
  1) *Random Component* : Each component of $\underline{Y}$ is independent and from one of the exponential family of distributions
  2) *Systematic Component* : The n covariates are combined to give the "linear predictor"

     $\underline{\eta} = \underline{\beta}\,\mathbf{X}$
  3) *Link Function* : The relationship between the random and systematic components is specified via a link function g, that is differentiable and monotonic

     $E[\underline{Y}] = \underline{\mu} = g^{-1}(\underline{\eta})$

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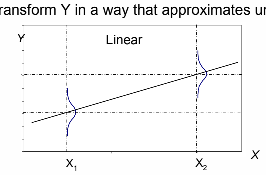
Linear Transformation versus a GLM

- Linear transformation uses transformed variables
  - GLM transforms the mean
  - GLM not trying to transform Y in a way that approximates uniform variability



- The error structure
  - Linear transformation retains assumption $Y_i \sim N(\mu_i\,,\,\sigma_e^2)$
  - GLM relaxes normality
  - GLM allows for non-uniform variance
  - Variance of each observation Y_i is a function of the mean $E[Y_i] = \mu_i$

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The Link Function

- Example: the log link function $g(x) = ln\ (x)$; $g^{-1}(x) = e^x$
- Suppose Premium (Y) is a multiplicative function of Policyholder Age (X_1) and Rating Area (X_2) with estimated parameters β_1 , β_2
  - $\eta_i = \beta_1 X_1 + \beta_2 X_2$
  - $g(\mu_i) = \eta_i$
  - $E[Y_i] = \mu_i = g^{-1}(\eta_i)$
  - $E[Y_i] = \exp(\beta_1 X_1 + \beta_2 X_2)$
  - $E[\underline{Y}] = g^{-1}(\underline{\beta}\ \mathbf{X})$
  - $E[Y_i] = \exp(\beta_1 X_1) \bullet \exp(\beta_2 X_2) = \mu_i$
  - $g(\mu_i) = \ln[\exp(\beta_1 X_1) \bullet \exp(\beta_2 X_2)] = \eta_i = \beta_1 X_1 + \beta_2 X_2$
  - The GLM here estimates logs of multiplicative effects

Examples of Link Functions

- Identity
  - $g(x) = x$ — $g^{-1}(x) = x$ — additive rating plan
- Reciprocal
  - $g(x) = 1/x$ — $g^{-1}(x) = 1/x$
- Log
  - $g(x) = \ln(x)$ — $g^{-1}(x) = e^x$ — multiplicative rating plan
- Logistic
  - $g(x) = \ln(x/(1-x))$ — $g^{-1}(x) = e^x/(1+e^x)$

Error Structure

- Exponential Family
  - Distribution completely specified in terms of its mean and variance
  - The variance of Y_i is a function of its mean $E[Y_i] = \mu_i$
  - $Var(Y_i) = \varphi V(\mu_i) / \omega_i$
  - $V(\mu)$ structure specifies the *distribution* of Y, but
  - $V(\mu)$, the variance function, is not the variance of *Y*
  - φ is a parameter that scales the variance
  - ω_i is a constant that assigns a weight, or credibility, to observation i

Error Structure

- Members of the Exponential Family
  - Normal (Gaussian) -- used in classic regression
  - Poisson (common for frequency)
  - Binomial
  - Negative Binomial
  - Gamma (common for severity)
  - Inverse Gaussian
  - Tweedie (common for pure premium)
    - *aka* Compound Gamma-Poisson Process
      - Claim count is Poisson distributed
      - Size-of-Loss is Gamma distributed

General Examples of Error/Link Combinations

- Traditional Linear Model
  - response variable: a continuous variable
  - error distribution: normal
  - link function: identity
- Logistic Regression
  - response variable: a proportion
  - error distribution: binomial
  - link function: logit
- Poisson Regression in Log Linear Model
  - response variable: a count
  - error distribution: Poisson
  - link function: log
- Gamma Model with Log Link
  - response variable: a positive, continuous variable
  - error distribution: gamma
  - link function: log

Specific Examples of Error/Link Combinations

Observed Response	Link Fnc	Error Structure	Variance Fnc
Claim Frequency	Log	Poisson	μ
Claim Severity	Log	Gamma	μ^2
Pure Premium	Log	Tweedie	μ^p $(1<p<2)$
Retention Rate	Logit	Binomial	$\mu(1-\mu)$

References

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