A theory of risk for two price market equilibria

Dilip Madan
Department of Finance
Robert H. Smith School of Business

Joint work with Shaun Wang and Phil Heckman

Preview of Results

- A theory of risk for two price economies is overlaid on an underlying one price economy.

- The two price economy is concerned with the failure of markets to converge to the law of one price

- Equations equations for the two prices are developed with a view to ensuring the acceptability of residual unhedgeable risks in incomplete markets.

- The acceptability approach results in nonlinear pricing operators that are concave for bid prices and convex for ask prices.

- Explicit closed forms for the two prices result when the cone of acceptable risks is modeled using parametric concave distortions of distribution functions for the residual risk.

- With assets marked at bid and liabilities valued at ask prices the theory allows a separation of liability valuation from an associated asset pricing theory.

- The static two price theory is then extended to its dynamic counterpart by leveraging recent advances made in the theory of non linear expectations and its association with solutions of backward stochastic difference and differential equations.

- For the hedging of risks we introduce the new criterion of capital minimization defined as the difference between the ask and bid prices.

Broad View of Two Price Economy

- Attention is focused on two prices, the one at which one is guaranteed a purchase or the ask price and the other the one at which one is guaranteed a sale or the bid price.

- In effect we contemplate an economy in which most transactions of interest are for products not traded on any exchange, for which one may be able to observe the ask price and or the bid price, but importantly there is no possibility of trading in both directions at any observed transaction price.

- Every transaction is either near or at the ask or near or at the bid.

Relevance for Insurance

- Products developed for sale to final user have both parties holding position to maturity with little if any trading in a secondary market.

- Products are by design quite specific and therefore lack liquidity.

- Buyers do not buy to sell and sellers do not sell to buy back. Positions are not being reversed and hence there is not much interest in liquidity, but rather in product performance.

- In the absence of two way transactors it is little wonder that two way prices are absent.

- What is needed to mark positions is a theory for the two one way prices that do and must prevail in equilibrium.

- By focusing attention on a two price economy we model liquidity risk not as an anomaly that is absent in the liquid market but as a core risk especially relevant for insurance products even if all financial risks are absent.

Dynamic Models for the Two Prices

- Insurance contracts typically extend over multiple periods and it is important to analyze the two price economy over multiple periods.

- The two prices, bid and ask are known to be nonlinear and we extend these pricing operators to dynamically consistent nonlinear operators by applying the recently developed theory of nonlinear expectations.

- In this regard we follow Madan and Schoutens (2010) and apply these methods to the pricing of insurance claims modeled by increasing compound Poisson processes.

Hedging in Two Price Economies

- The hedging objectives in two price economies turn towards the minimization of ask prices or the maximization of bid prices.

- Equivalently as suggested in Carr, Madan and Vicente Alvarez (2011) one economizes on capital commitments measured by the difference between the ask and the bid price.

- We contrast our capital minimization hedging criteria with other classical criteria like variance minimization and or the maximization of expected utility.

- We also apply these new hedging objectives to illustrate the construction of optimal reinsurance points for contracts insuring losses.

Two Price Economy Pricing Kernels

- Consider a two date one period economy trading state contingent claims paying cash flows at time 1 with prices determined at time 0.

- The claims traded are random variables on a probability space $(\Omega, \mathcal{F}, P)$ and we suppose that there are some zero cost claims with payouts $H \in \mathcal{H}$ that trade in a liquid market with the same zero cost for trading in both directions.

- The class of risk neutral measures is then given by

$$\mathcal{R} = \left\{ Q | Q \sim P \text{ and } E^{Q}[H] = 0, \text{ all } H \in \mathcal{H} \right\}.$$

- We suppose that an equilibrium has selected a base risk neutral measure Q^0 and the set of classically acceptable risks is then given by the set of positive alpha trades or the set of random variables

$$\mathcal{A}_c = \left\{ X | X \in L^{\infty}(\Omega, \mathcal{F}, P), E^{Q^0}[X] \geq 0 \right\}.$$

- The definition of $\mathcal{A}_c$ recognizes that the classical market will accept to buy any amount at a price below the going market price and agree to sell any amount at a price above the price given by the risk neutral expectation.

- We may define by Λ_c the change of measure density

$$\Lambda_c = \frac{dQ^0}{dP}$$

and equivalently write that the return R_X on X with positive risk neutral price $\pi(X) = (1+r)^{-1} E^{Q^0}[X] > 0$ for a periodic interest rate of r, defined by

$$R_X = \frac{X}{\pi(X)} - 1$$

satisfies the condition that

$$E^P[R_X] - r \geq -cov^P\left(\Lambda_c, R_X\right),$$

or we have a positive alpha trade or one that earns in excess of compensation for risk.

- The point of departure for two price economies from the classical model is the recognition that the half space $\mathcal{A}_c$ is too large an acceptance set for realistic economies.

- For two price economies the acceptance set for the market is defined by a smaller convex cone containing the nonnegative random variables.

- It is shown in Artzner, Delbaen, Eber and Heath (1999) that all such cones are defined by requiring a positive expectation under a set of test measures

$Q \in \mathcal{M}$. The set of risks accepted by the market is then

$$\mathcal{A} = \left\{X | X \in L^{\infty}\left(\Omega, \mathcal{F}, Q^0\right), E^Q[X] \geq 0, \text{ all } Q \in \mathcal{M}\right\},$$

where we suppose that our base measure is $Q^0 \in \mathcal{M}$.

- Madan and Schoutens (2011) determine the set $\mathcal{A}$ in equilibrium as the largest set consistent with the aggregate risk held by the market being in a prespecified small cone containing the nonnegative random variables.

- The two prices for a cash flow X of a two price economy are derived from the market's acceptance cone by requiring that the price less the cash flow for a sale by the market or the other way around for a purchase be market acceptable.

- Cherny and Madan (2010) show that the unhedged bid and ask prices, with a periodic interest rate of r, $b(X)$, $a(X)$ respectively are given by

$$\begin{aligned} b(X) &= (1+r)^{-1} \inf_{Q \in \mathcal{M}} E^Q[X] \\ a(X) &= (1+r)^{-1} \sup_{Q \in \mathcal{M}} E^Q[X]. \end{aligned}$$

- Note importantly that the two prices of a two price economy are nonlinear functions on the space of random variables with the bid price being concave while the ask price is convex by virtue of the infimum and supremum operations.

- The hedging price is determined by maximizing the post hedge bid price or minimizing the post hedge ask price. Formally we have (Cherny and Madan (2010)) that

$$\begin{aligned} b(X) &= \sup_{H \in \mathcal{H}} b(X - H) \\ a(X) &= \inf_{H \in \mathcal{H}} a(H - X). \end{aligned}$$

- We now investigate the pricing of risk in our two price economy.

- We may write the bid and ask prices for X as attained at extreme points $Q^{b,X}, Q^{a,X}$ that have densities with respect to the base measure Q^0 of

$$\begin{aligned} \Lambda^{b,X} &= \frac{dQ^{b,X}}{dQ^0} \\ \Lambda^{a,X} &= \frac{dQ^{a,X}}{dQ^0} \end{aligned}$$

and we then have that

$$
\begin{aligned}
b(X) &= (1+r)^{-1} E^P \left[\Lambda^{b,X} \Lambda_c X\right] \\
a(X) &= (1+r)^{-1} E^P \left[\Lambda^{a,X} \Lambda_c X\right]
\end{aligned}
$$

- If we employ a weighted average as a candidate price defining returns $\widetilde{R}_X$ relative to this average by

$$
\begin{aligned}
\widetilde{R}_X &= \frac{X}{m(X)} - 1 \\
m(X) &= \alpha a(X) + (1-\alpha) b(X)
\end{aligned}
$$

then we infer the risk pricing equation

$$
E[\widetilde{R}_X] - r = -cov^P \left(\left(\alpha \Lambda^{a,X} + (1-\alpha)\Lambda^{b,X} \right) \Lambda_c, \widetilde{R}_X \right).
$$

- Note importantly that by virtue of the nonlinearity of the pricing operators of a two price economy the pricing kernels are no longer independent of the risk being priced.

- We build on the classical measure change Λ_c of a one price economy an additional illiquidity based measure change given by $\left(\alpha\Lambda^{a,X} + (1-\alpha)\Lambda^{b,X}\right)$.

- The second measure change is precisely an illiquidity based measure change as it comes into existence with a bid ask spread associated with an absence of a convergence to a law of one price.

Acceptance Cones Modeled by Concave Distortions

- The market primitive of two price economies is the set of zero cost cash flows accepted by the market.

- This set is a convex cone of random variables containing the nonnegative random variables.

- When the acceptance decision for a random variable X is a function solely of its distribution function $F_X(x)$ one may evaluate acceptance as shown in Cherny and Madan (2010) by a positive expectation under a concave distortion of the distribution function.

- Specifically for a concave distribution function $\mathbf{\Psi}(u)$ defined on the unit interval and termed the distortion the random variable X is accepted or belongs to the acceptance cone $\mathcal{A}$,just if

$$\int_{-\infty}^{\infty} x d\mathbf{\Psi}\left(F_X(x)\right) \geq 0.$$

- It is shown in Cherny and Madan (2010) that the set of approving measures $\mathcal{M}$ are all change of measure densities on the unit interval $Z(u)$ with

$$\int_{-\infty}^{\infty} xZ(F_X(x))f_X(x)dx \geq 0$$

for all Z for which $L \leq \mathbf{\Psi}$, where $L' = Z$.

- We mention here two distortions that have been used in earlier work by Cherny and Madan (2010) among other papers and earlier work in the insurance literature Wang (2000).

- These are the transforms $minmaxvar$, $\mathbf{\Psi}^{\gamma}$ and the Wang transform, $\mathbf{\Phi}^{\alpha}$.

- They are defined respectively by

$$
\begin{aligned}
\mathbf{\Psi}^{\gamma}(u) &= 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma} \\
\mathbf{\Phi}^{a}(u) &= N\left(N^{-1}(u) + a\right)
\end{aligned}
$$

- Both these transforms have the desirable property of derivatives tending to infinity as u tends to zero and derivatives that tend to zero as u tends to unity.

- In terms of distortions one has exact expressions for bid and ask prices (Cherny and Madan (2010)). In this case

$$
\begin{aligned}
b(X) &= (1+r)^{-1}\int_{-\infty}^{\infty} x\psi(F_X(x))f_X(x)dx \\
a(X) &= (1+r)^{-1}\int_{-\infty}^{\infty} y\psi\left(1-F_X(y)\right)f_X(y)dy
\end{aligned}
$$

- So

$$
m_\alpha(X) = (1+r)^{-1}E^{Q^0}\left[\begin{array}{c} (\alpha\psi(F_X(X)) \\ +(1-\alpha)\psi\left(1-F_X(X)\right)X \end{array}\right]
$$

Hence we have that

$$
E^P[X]-r = -cov^P(R_X, \left(\begin{array}{c} \alpha\psi(F_X(X)) \\ +(1-\alpha)\psi\left(1-F_X(X)\right)\Lambda_c \end{array}\right)
$$

The kernel is then $U-shaped$ and we graph the quantile pricing kernel for $\gamma = .5$ and $\alpha = .5$.

The kernel cannot be uncorrelated with X as it is a deterministic function of X.

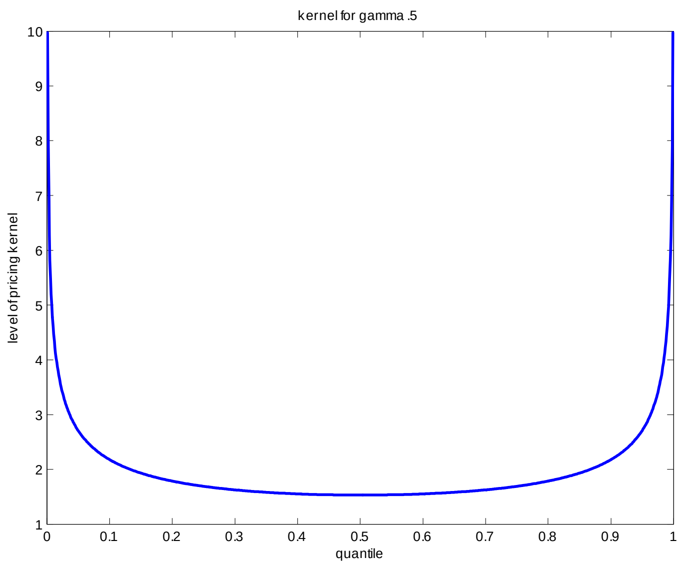

Figure 1: Quantile Risk Pricing Kernel for gamma equal to 0.5

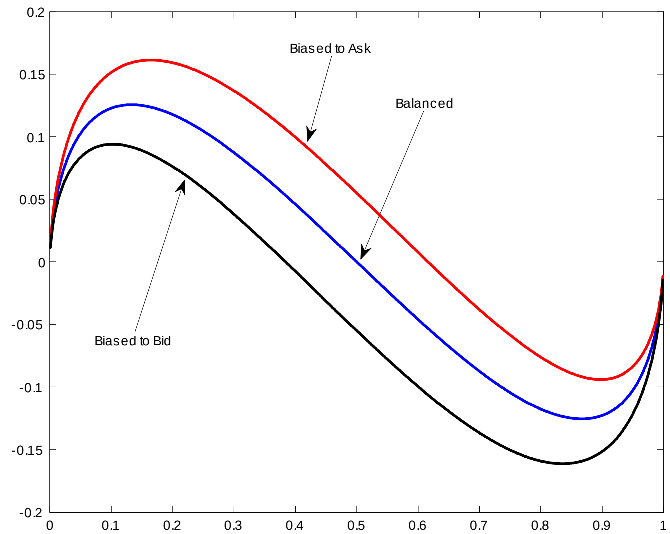

Figure 2: Mid quote base expectation gap at quantiles for digitals

- For most insurance contracts we have sensitivity in the lower quantiles and so we expect the mid quote to rise above the base expectation as may be observed on noting directly that the gap $g(a)$ for a digital at quantile a is

$$g(a) = \Psi(a) + 1 - \Psi(1-a) - 2a.$$

We graph in Figure 2 this digital gap.

- The gap is positive at quantiles below a half and negative for quantiles above a half.

Dynamic Two Price Economies

- We now consider the dynamic valuation of a discrete time stochastic claims or receipts process $X = (X_t, t = 1, \cdots, T)$.

- The valuation is as at time t and is denoted $V_t^B(X)$, $V_t^A(X)$ depending on whether we are constructing a bid price or an ask price.

- We suppose that the length of the interperiod time interval is h.

- We suppose the existence of a base risk neutral measure selected by an equilibrium under which one may construct the risk neutral valuation process V_t^R by

$$\begin{aligned} V_t^R &= \sum_{j \le t} \frac{B(j)}{B(t)} X_j \\ &\quad + E^{Q_0}\left[\sum_{j > t} \frac{B(j)}{B(t)} X_j\right] \\ &=_{def} \sum_{j \le t} \frac{B(j)}{B(t)} X_j + W_t^R \end{aligned}$$

where $B(t)$ is the time zero discount curve supposed fixed in this exercise.

- Risk neutral valuation is a well understood linear pricing operator and as in the static case it constitutes our starting point.

- What we shall present are the nonlinear pricing operators for the bid and ask prices.

- We note in this regard the partitioning of total value into the part of that has been realized and the part that is yet to be realized by defining

$$\begin{aligned} V_t^A(X) &= {}_{def} \sum_{j \le t} \frac{B(j)}{B(t)} X_j \\ &\quad + W_t^A(X) \\ V_t^B(X) &= {}_{def} \sum_{j \le t} \frac{B(j)}{B(t)} X_j \\ &\quad + W_t^B(X) \end{aligned}$$

- Such nonlinear pricing operators are given by nonlinear expectations that are related to solutions of backward stochastic difference equations.

- We define risk charges directly for the risk defined for example as the zero mean random variable

$$\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)$$
$$-E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right].$$

- We therefore apply the recursions

$$\begin{aligned}
& W_t^A(X) \\
= \;& E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right] \\
& +h \sup_{Q\in\mathcal{M}}\left(\begin{array}{c} \frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right) \\ -E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^A(t+1)\right)\right] \end{array}\right) \\
& W_t^B(X) \\
= \;& E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)\right] \\
& +h \inf_{Q\in\mathcal{M}}\left(\begin{array}{c} \frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right) \\ -E^{Q^0}\left[\frac{B(t+1)}{B(t)}\left(X_{t+1}+W^B(t+1)\right)\right] \end{array}\right)
\end{aligned}$$

Drivers for nonlinear expectations based on distortions

- The driver for a translation invariant nonlinear expectation is basically a positive risk charge for the ask price and a positive risk shave for a bid price applied to a zero mean risk exposure to be held over an interim.

- We are then given as input the risk exposure ideally spanned by some martingale differences as $Z_u M_{u+1}$ or alternatively a zero risk neutral mean random variable X with a distribution function $F(x)$.

- We consider in the rest of the paper drivers based on the distortion $minmaxvar$. In this case

$$
\begin{aligned}
F^B(Z_u M_{u+1}) &= \int_{-\infty}^{\infty} x d\Psi^{\gamma}(\Theta^B(x)) \\
F^A(Z_u M_{u+1}) &= -\int_{-\infty}^{\infty} x d\Psi^{\gamma}\left(1 - \Theta^A(-x)\right)
\end{aligned}
$$

and in particular

$$\Theta^B(x) = Q^0 \left(\begin{array}{c} \frac{B(t+1)}{B(t)} \Big(X_{t+1} + W^B(t+1) \Big) \\ -E^{Q^0} \left[\frac{B(t+1)}{B(t)} \Big(X_{t+1} + W^B(t+1) \Big) \right] \\ \leq x \end{array} \right)$$

$$\Theta^A(x) = Q^0 \left(\begin{array}{c} \frac{B(t+1)}{B(t)} \Big(X_{t+1} + W^A(t+1) \Big) \\ -E^{Q^0} \left[\frac{B(t+1)}{B(t)} \Big(X_{t+1} + W^A(t+1) \Big) \right] \\ \leq x \end{array} \right)$$

Hedging in Two Price Economies

- We note that hedge instruments should have zero means under the base probability measure for otherwise these instruments would become vehicles for investment or speculation instead of being used as hedges.

- With hedges having zero means one may take target cash flows to be hedged to also have a zero mean and hence the hedging criterion should be receptive of negative as well as positive cash flows.

- A classical criterion often used in studies related to hedging in incomplete markets is variance minimization or quadratic hedging

- This criterion has no parameter with which to reflect some degree of aggressiveness or otherwise in hedge design.

- An often studied alternative criterion is the maximization of expected utility.

- In the context of two price markets Carr, Madan and Vicente Alvarez (2011) and Madan and Schoutens (2011) suggest capital minimization defined as the difference between ask and bid prices.

- Given that bid and ask prices reflect stress parameters embedded in distortions capital minimization becomes a hedging criterion with a parameter allowing for the expression of different levels of aggressiveness in hedge design.

- We present a graph of hedge functions in the context of an inhomogeneous Poisson with compound gamma losses.

- We observe that certainty equivalents are quite asymmetric in their effects on the hedging criterion. Variance as already noted is symmetric but lacks a parameter.

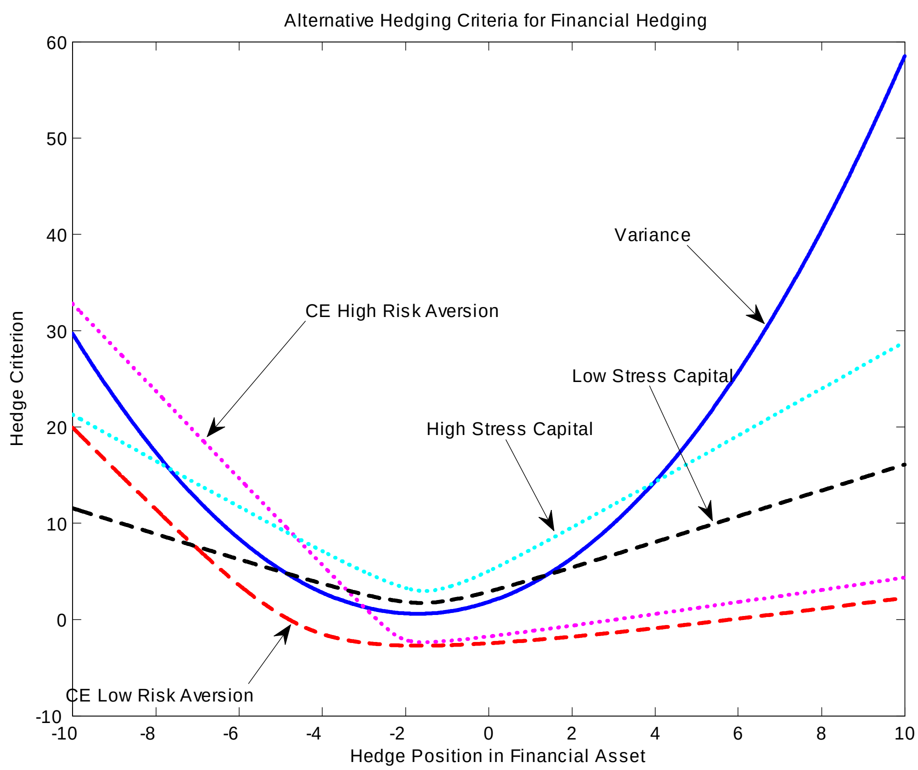

Figure 3: Alternative Hedging Criteria for Inhomogeneous Poisson Compound Gamma Losses hedged by security tracking cumulated exceedances

Conclusion

- A theory of risk for two price economies is overlaid on an underlying one price economy.

- The two price economy is concerned with the failure of markets to converge to the law of one price and goes on to develop explicit equations for bid and ask prices with a view to ensuring the acceptability of residual unhedgeable risks in incomplete markets.

- The acceptability approach results in nonlinear pricing operators that are concave for bid prices and convex for ask prices.

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