Univariate and Copula Regression Models of Insurance Claims: A Personal Auto Case Study

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CAS RPM Seminar
April 1, 2014

Outline

Background

- Ratemaking is a classic actuarial problem
- Unique features of insurance data require advanced statistical methods
  - Heavy tailed and skewed data
  - Multivariate nature of bundling products
- Some background (promoting the book?)
  - Predictive modeling book edited by Frees, Meyers and Derrig
  - This case study contributes a chapter in Volume II
  - Data and code will be available on book website
- We discuss different modeling strategy, and we emphasize that model selection depends on the data format

Some Notations

- For each policy i, an analyst could observe
  - N_i - the number of claims
  - K_i - the type of claims
  - Y_{ink} - the amount of each claim by type
  - $Y_{in} = \sum_k Y_{ink}$, $n = 1, \cdots, N_i$ - amount of each claim
  - $S_{ik} = Y_{i1k} + \cdots + Y_{iN_ik}$ - aggregate claim amount by type
  - $S_i = \sum_k S_{ik}$ - aggregate claim amount for policyholder i

Personal Auto Dataset

- Massachusetts automobile claims dataset from CAR
  - Made public by Massachusetts Executive Office of Energy and Environmental Affairs
  - Contain experience in year 2006 for about 3.25 million policies
  - Two types of claims: liability and PIP
- We draw a random sample of 100,000 policyholders
  - Claim frequency

Count	0	1	2	3	4	4+
Frequency	95,443	4,324	219	12	2	0

  - Percentiles of claim size

	5%	10%	25%	50%	75%	90%	95%
Liability	237.00	350.00	675.50	1,464.00	3,465.00	10,596.90	19,958.75
PIP	2.00	5.00	84.00	1,371.50	3,300.00	7,548.50	8,232.00

Covariates

	Mean			Average Loss		
	Overall	No claim	$\geq$ 1 claim	Liability	PIP	Total
Rating Group						
1 - adult	0.747	0.749	0.703	155.20	18.45	173.65
2 - business	0.014	0.014	0.014	199.65	16.48	216.13
3 - <3 yrs exp	0.043	0.042	0.078	332.38	26.24	358.63
4 - 3-6 yrs exp	0.044	0.043	0.067	283.92	22.32	306.24
5 - senior	0.152	0.153	0.138	119.15	12.29	131.44
Territory Group						
1 - least risky	0.185	0.188	0.132	92.53	8.76	101.29
2	0.193	0.194	0.167	135.00	9.82	144.81
3	0.113	0.114	0.091	137.21	7.47	144.68
4	0.201	0.201	0.194	154.69	16.39	171.08
5	0.189	0.187	0.227	203.39	24.58	227.97
6 - most risky	0.120	0.117	0.189	296.94	47.58	344.52

Tweedie

- A Poisson sum of gamma random variables
  - $S_i = (Y_{i1} + \cdots + Y_{iN_i})/\omega_i$
  - $N_i \sim Poisson(\omega_i \lambda_i)$
  - Y_{ij} $(j = 1, \cdots, N_i) \sim gamma(\alpha, \gamma_i)$
- The Tweedie belongs to the exponential familiy with the reparameterizations:

$$\lambda_i = \frac{\mu_i^{2-p}}{\phi(2-p)}, \quad \alpha = \frac{2-p}{p-1}, \quad \gamma_i = \phi(p-1)\mu_i^{p-1}$$

- Location μ, dispersion ϕ, and power p, denoted by $Tweedie(\mu, \phi, p)$

$$\mathrm{E}(Y_i) = \mu_i \quad \text{and} \quad \mathrm{Var}(Y_i) = \frac{\phi}{\omega_i}\mu_i^p$$

Tweedie

- Data availability
  - Both S_i and N_i are observed

$$f_i(n, s) = a(n, s; \phi/\omega_i, p) \exp\left\{\frac{\omega_i}{\phi} b(s; \mu_i, p)\right\}$$

  - Only S_i are recorded

$$f_i(y) = \exp\left[\frac{\omega_i}{\phi} b(s; \mu_i, p) + c(s; \phi/\omega_i)\right]$$

- Dispersion modeling?
  - Tweed GLM: $g_\mu(\mu_i) = \mathbf{x}_i'\beta$
  - Dispersion model: $g_\phi(\phi_i) = \mathbf{z}_i'\eta$

Tweedie

	Cost and Claim Counts				Cost Only				
	Mean Model		Dispersion Model			Mean Model		Dispersion Model	
Parameter	Est	S.E.	Est	S.E.	Parameter	Est	S.E.	Est	S.E.
int	5.634	0.087	5.647	0.083	int	5.634	0.088	5.646	0.084
A	0.267	0.070	0.263	0.071	A	0.267	0.071	0.263	0.072
B	0.499	0.206	0.504	0.211	B	0.500	0.209	0.506	0.213
I	1.040	0.120	1.054	0.106	I	1.040	0.121	1.054	0.108
M	0.811	0.122	0.835	0.113	M	0.811	0.123	0.834	0.114
t1	-1.209	0.086	-1.226	0.086	t1	-1.210	0.087	-1.226	0.087
t2	-0.830	0.083	-0.850	0.080	t2	-0.831	0.084	-0.850	0.081
t3	-0.845	0.095	-0.863	0.097	t3	-0.845	0.097	-0.862	0.098
t4	-0.641	0.081	-0.652	0.077	t4	-0.641	0.082	-0.652	0.078
t5	-0.359	0.080	-0.368	0.074	t5	-0.360	0.081	-0.368	0.075
p	1.631	0.004	1.637	0.004	*p*	1.629	0.004	1.634	0.004
dispersion									
int	5.932	0.015	5.670	0.041	int	5.968	0.016	5.721	0.043
A			0.072	0.034	A			0.064	0.035
B			0.006	0.101	B			0.010	0.105
I			-0.365	0.051	I			-0.356	0.054
M			-0.206	0.054	M			-0.209	0.056
t1			0.401	0.042	t1			0.374	0.043
t2			0.323	0.039	t2			0.301	0.040
t3			0.377	0.047	t3			0.365	0.048
t4			0.266	0.037	t4			0.260	0.039
t5			0.141	0.036	t5			0.132	0.037
loglik	-61121.090		-60988.180			-60142.140		-60030.140	

Frequency-Severity Models

- Suppose one can observe data at claim level, i.e. both N_i and Y_{in} are available
- Two-part model follows

$$f(N, Y) = f(N) \times f(Y|N)$$

- Based on conditional decomposition and does not require independence between Y and N like Tweedie
- Use count regression for the frequency component $f(N)$
  - Poisson, NB, Zero-inflated, Hurdle ... (see *Volume I*)
- Use fat-tailed regression for the severity component $f(Y|N)$
  - GLM, parametric (GG,GB2 etc.), quantile regression ... (see *Volume I*)
- The above formulation allows us to estimate the two parts separately

Frequency-Severity Models

- Suppose one can observe data only at policy level, i.e. S_i or $\{N_i, S_i\}$ are available
- Strategy:
  - Model the mass probability at zero, i.e. $Pr(S = 0)$, using a binary regression, such as logit or probit.
  - Model the positive claim amount, i.e. $f_S(s|S > 0)$, using a fat-tailed regression.
- Likelihood

$$f_S(s) = \begin{cases} Pr(S = 0) & s = 0 \\ f_S(s|S > 0) \times Pr(S > 0) & s > 0 \end{cases}$$

- Estimation

$$loglik = \sum_{\{i:S_i=0\}} Pr(S_i = 0) + \sum_{\{i:S_i>0\}} Pr(S_i > 0) \qquad \leftarrow frequency$$

$$+ \sum_{\{i:S_i>0\}} \ln f_S(s_i|S_i > 0) \qquad \leftarrow severity$$

Frequency-Severity Models

	Frequency				Severity				
	NegBin		ZINB		Gamma		GG		
Parameter	Est	S.E.	Est	S.E.	Parameter	Est	S.E.	Est	S.E.
int	-2.559	0.051	-2.185	0.865	int	8.179	0.066	7.601	0.079
A	0.039	0.044	-0.133	0.678	A	0.235	0.056	0.207	0.064
B	0.186	0.130	-0.025	0.835	B	0.382	0.167	0.306	0.190
I	0.793	0.067	0.551	0.873	I	0.257	0.084	0.259	0.096
M	0.550	0.070	0.398	0.683	M	0.284	0.089	0.208	0.102
t1	-0.866	0.053	-1.068	0.121	t1	-0.376	0.068	-0.245	0.079
t2	-0.647	0.050	-0.867	0.128	t2	-0.223	0.064	-0.166	0.073
t3	-0.703	0.060	-0.777	0.111	t3	-0.168	0.077	-0.115	0.088
t4	-0.517	0.048	-0.655	0.091	t4	-0.175	0.061	-0.119	0.070
t5	-0.283	0.046	-0.451	0.112	t5	-0.117	0.059	-0.053	0.067
zero model					loglik	-43748.500		-43504.510	
int			-0.104	1.709					
A			-1.507	0.818					
B			-2.916	3.411					
I			-5.079	5.649					
M			-1.260	1.388					
t1			-2.577	11.455					
t2			-3.894	52.505					
t3			-0.509	0.965					
t4			-1.145	2.264					
t5			-1.583	4.123					
loglik	-19147.500		-19139.000						

Copula

- Copula (linguistics): a word used to link subject and predicate
- Copula (music), a type of polyphonic texture similar to organum
- Copula linguae, an embryonic structure of the tongue
- Copula (probability theory), a function linking marginal variables into a multivariate distribution

Copula

- A *copula* is a multivariate distribution function with uniform marginals. Let $U_1, \ldots, U_T$ be T uniform random variables on (0,1). Their distribution function

$$H(u_1, \ldots, u_T) = Pr(U_1 \leq u_1, \ldots, U_T \leq u_T)$$

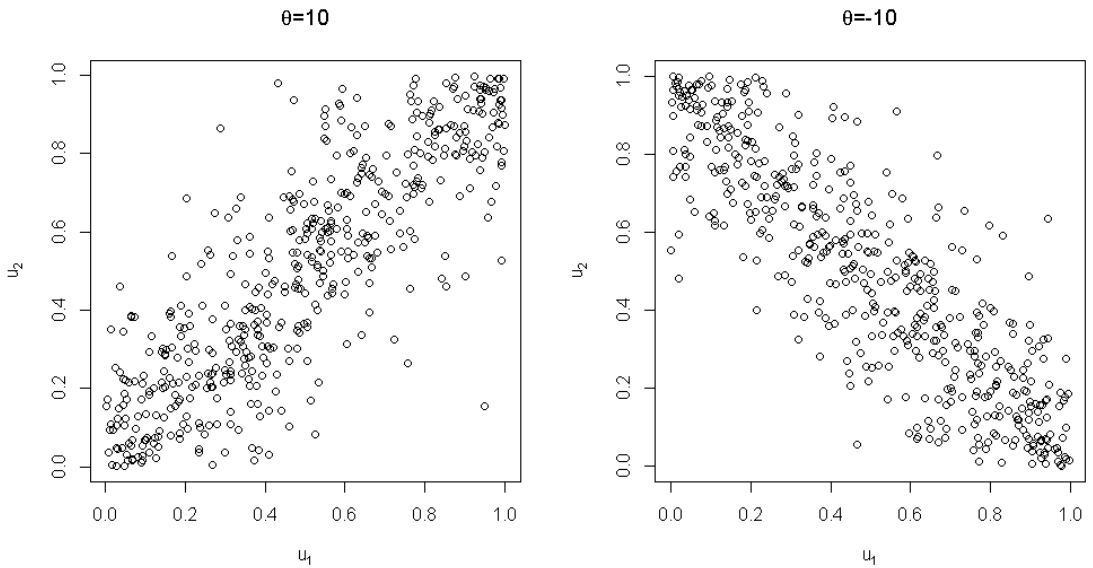
- For general applications, consider arbitrary marginal distributions $F_1(y_1), \ldots, F_T(y_T)$. Define a multivariate distribution function using the copula such that

$$\begin{aligned} F(y_1, \ldots, y_T) &= Pr(Y_1 \leq y_1, \cdots, Y_T \leq y_T) \\ &= Pr(F_1(Y_1) \leq F_1(y_1), \cdots, F_T(Y_T) \leq F_T(y_T)) \\ &= Pr(U_1 \leq F_1(y_1), \cdots, U_T \leq F_T(y_T)) \\ &= H(F_1(y_1), \ldots, F_T(y_T)) \end{aligned}$$

- Sklar(1959) established the converse: any multivariate distribution function F can be written in the form of a copula
- Density function: $h(u_1, \cdots, u_T) = \partial^T H(u_1, \cdots, u_T)/\partial u_1 \cdots \partial u_T$

Bivariate Examples

- Examples of Copula:
  - Frank: $H(u_1, u_2) = -\frac{1}{\theta}\log\left(1 + \frac{(\exp(-u_1\theta - 1))(\exp(-u_2\theta - 1))}{\exp(-\theta) - 1}\right)$
  - Gumbel: $H(u_1, u_2) = \exp\left(-((-\log(u_1))^\theta + (-\log(u_2))^\theta)^{1/\theta}\right)$



Bivariate Distributions

- Continuous variables (Y_1, Y_2)

$$f(y_1, y_2) = h(F_1(y_1), F_2(y_2)) f_1(y_1) f_2(y_2)$$

- Discrete variables (Y_1, Y_2)

$$\begin{aligned} f(y_1, y_2) =& H(F_1(y_1), F_2(y_2)) - H(F_1(y_1 - 1), F_2(y_2)) \\ & - H(F_1(y_1), F_2(y_2 - 1)) + H(F_1(y_1 - 1), F_2(y_2 - 1)) \end{aligned}$$

- Y_1 - Continuous and Y_2 - Discrete

$$f(y_1, y_2) = f_1(y_1)(h_1(F_1(y_1), F_2(y_2)) - h_1(F_1(y_1), F_2(y_2 - 1)))$$

where $h_1(u_1, u_2) = \partial H(u_1, u_2)/\partial u_1$

Bivariate Tweedie

- Two types of coverage: S_1-Liability, S_2-PIP
- Use Tweedie for S_1 and another Tweedie for S_2
- Use a parametric copula H to construct the joint distribution of S_1 and S_2

$$f(s_1, s_2) = \begin{cases} H(F_1(0), F_2(0)) & \text{if } s_1 = 0 \text{ and } s_2 = 0 \\ f_1(s_1) h_1(F_1(s_1), F_2(0)) & \text{if } s_1 > 0 \text{ and } s_2 = 0 \\ f_2(s_2) h_2(F_1(0), F_2(s_2)) & \text{if } s_1 = 0 \text{ and } s_2 > 0 \\ f_1(s_1) f_2(s_2) h(F_1(s_1), F_2(s_2)) & \text{if } s_1 > 0 \text{ and } s_2 > 0 \end{cases}$$

Bivariate Tweedie

Tweedie		
	Marginal	Frank Copula
θ		4.659
		(0.332)
Loglik	65930.30	65520.92
$\chi^2(1)$		818.76
Double GLM		
	Marginal	Frank Copula
θ		5.580
		(0.384)
Loglik	65771.470	65308.59
$\chi^2(1)$		925.76
$\chi^2(18)$	317.66	424.66

Bivariate Two-Part Model

- Two semi-continuous claim outcomes
- Consider four scenarios: $\{S_1 = 0, S_2 = 0\}$, $\{S_1 > 0, S_2 = 0\}$, $\{S_1 = 0, S_2 > 0\}$, $\{S_1 > 0, S_2 > 0\}$
- The joint distribution can be expressed as

$$
\begin{aligned}
& f(s_1, s_2) \\
&= \begin{cases}
\Pr(S_1 = 0, S_2 = 0) & \text{if } s_1 = 0, s_2 = 0 \\
\Pr(S_1 > 0, S_2 = 0) \times f_1(s_1 | s_1 > 0) & \text{if } s_1 > 0, s_2 = 0 \\
\Pr(S_1 = 0, S_2 > 0) \times f_2(s_2 | s_2 > 0) & \text{if } s_1 = 0, s_2 > 0 \\
\Pr(S_1 > 0, S_2 > 0) \times f(s_1, s_2 | s_1 > 0, s_2 > 0) & \text{if } s_1 > 0, s_2 > 0
\end{cases}
\end{aligned}
$$

- Define $R_1 = I(S_1 > 0)$ and $R_2 = I(S_2 > 0)$

Bivariate Two-Part Model

- Bivariate frequency (R_1, R_2)
  - Copula

$$\begin{cases} \Pr(R_1 = 1, R_2 = 1) = 1 - F_1(0) - F_2(0) - H(F_1(0), F_2(0)) \\ \Pr(R_1 = 1, R_2 = 0) = F_2(0) - H(F_1(0), F_2(0)) \\ \Pr(R_1 = 0, R_2 = 1) = F_1(0) - H(F_1(0), F_2(0)) \\ \Pr(R_1 = 0, R_2 = 0) = H(F_1(0), F_2(0)) \end{cases}$$

  - Dependence ratio (see Chapter)
  - Odds ratio (see Chapter)
- Bivariate severity (S_1, S_2)
  - Use another copula for the joint distribution of (S_1, S_2)

$$f(s_1, s_2 | s_1 > 0, s_2 > 0)$$

$$= h(F_1(s_1|s_1 > 0), F_2(s_{i2}|s_2 > 0)) \prod_{j=1}^{2} f_j(s_j|y_j > 0)$$

Bivariate Two-Part Model - Frequency

	Dependence Ratio		Odds Ratio		Frank Copula	
Parameter	Estimate	StdErr	Estimate	StdErr	Estimate	StdErr
Liability						
rating group = A	-0.008	0.046	-0.003	0.046	-0.006	0.095
rating group = B	0.210	0.137	0.202	0.137	0.206	0.094
rating group = I	0.680	0.068	0.795	0.072	0.781	0.022
rating group = M	0.415	0.075	0.471	0.077	0.455	0.019
territory group = 1	-0.739	0.057	-0.795	0.058	-0.788	0.023
territory group = 2	-0.502	0.054	-0.565	0.054	-0.555	0.043
territory group = 3	-0.585	0.064	-0.643	0.065	-0.635	0.054
territory group = 4	-0.397	0.052	-0.458	0.053	-0.448	0.037
territory group = 5	-0.184	0.050	-0.231	0.051	-0.226	0.038
PIP						
rating group = A	0.356	0.124	0.363	0.124	0.362	0.099
rating group = B	0.223	0.373	0.217	0.372	0.224	0.598
rating group = I	0.872	0.179	0.968	0.180	0.961	0.137
rating group = M	1.039	0.170	1.094	0.170	1.083	0.130
territory group = 1	-1.466	0.137	-1.502	0.137	-1.498	0.124
territory group = 2	-1.182	0.123	-1.224	0.123	-1.218	0.118
territory group = 3	-1.298	0.156	-1.336	0.156	-1.331	0.144
territory group = 4	-0.874	0.110	-0.915	0.110	-0.909	0.110
territory group = 5	-0.650	0.105	-0.679	0.105	-0.677	0.080
dependence	6.893	0.309	13.847	1.094	10.182	1.084
loglik	-20698.810		-20669.230		-20676.890	
Chi-square	799.420		858.580		843.260	

Bivariate Two-Part Model - Severity

	Liability		PIP	
Parameter	Estimate	StdErr	Estimate	StdErr
intercept	7.437	0.081	7.955	0.220
rating group = A	0.269	0.065	0.121	0.185
rating group = B	0.272	0.190	-0.156	0.523
rating group = I	0.417	0.098	-0.033	0.275
rating group = M	0.428	0.106	-0.448	0.263
territory group = 1	-0.233	0.081	-0.049	0.226
territory group = 2	-0.196	0.075	-0.519	0.190
territory group = 3	-0.090	0.090	-0.427	0.249
territory group = 4	-0.105	0.073	-0.178	0.171
territory group = 5	-0.073	0.070	-0.100	0.164
σ	1.428	0.016	1.673	0.062
κ	0.210	0.029	1.655	0.105
θ	0.326	0.047		
df	11.258	4.633		
loglik	-44041.970			
$\chi^2(1)$	7.480			
$\chi^2(2)$	48.200			

Three-Part Model

- Examine data at claim level
- Three-part model follows

$$f(N, T, Y) = f(N) \times f(T|N) \times f(Y|N, T)$$

  - N - number of claims
  - T - the type of claim: liability, PIP, or both
  - Y - amount of claims: (Y_1), (Y_2), or (Y_1, Y_2)
- Strategy:
  - Use a count regression for $f(N)$
  - Given an accident, use a multinomial logit regression for claim type $f(T|N)$
  - Given the type of an accident, use a copula regression for the amount $f(Y|N, T)$

Three-Part Model

- Part I: Poisson/NB2 ...
- Part II:

$$\Pr(T = \textit{Liability}) = \frac{\exp(\mathbf{x}_{i1}'\beta_1)}{1 + \exp(\mathbf{x}_{i1}'\beta_1) + \exp(\mathbf{x}_{i2}'\beta_2)}$$

$$\Pr(T = \textit{PIP}) = \frac{\exp(\mathbf{x}_{i2}'\beta_2)}{1 + \exp(\mathbf{x}_{i1}'\beta_1) + \exp(\mathbf{x}_{i2}'\beta_2)}$$

- Part III:
  - If T=Liability, $f_1(y_1) \sim \textit{Gamma/GG/GB2}$...
  - If T=PIP, $f_2(y_2) \sim \textit{Gamma/GG/GB2}$...
  - If T=Both, $f(y_1, y_2) = h(F_1(y_1), F_2(y_2)) f_1(y_1) f_2(y_2)$

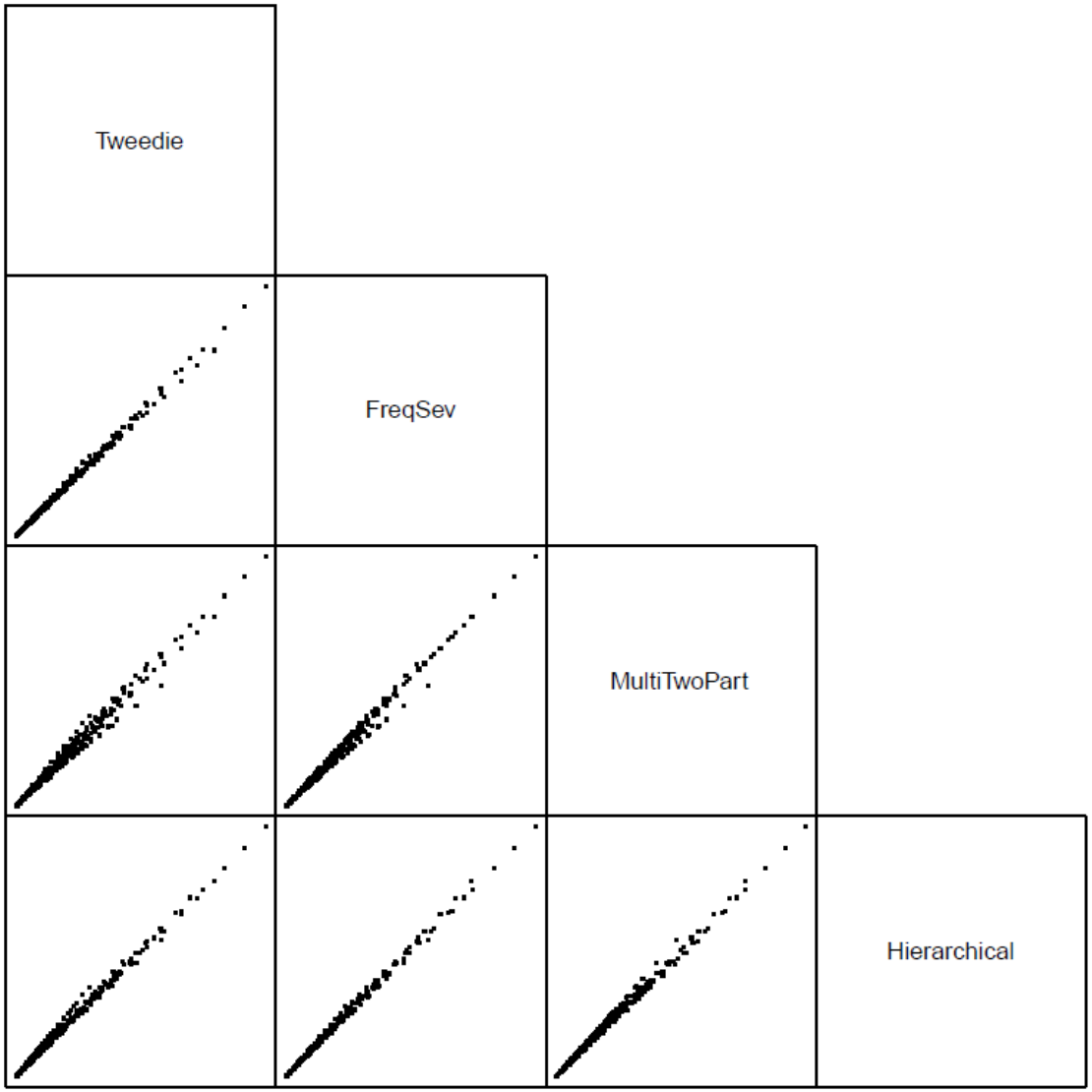
Three-Part Model

	Liability		PIP	
Parameter	Estimate	StdErr	Estimate	StdErr
intercept	2.799	0.126	0.390	0.178
cgroup_A	0.091	0.135	0.403	0.188
cgroup_B	-0.225	0.381	-0.851	0.592
cgroup_I	-0.021	0.204	-0.170	0.276
cgroup_M	0.027	0.229	0.731	0.278
tgroup_1	0.429	0.200	0.287	0.232
tgroup_2	0.028	0.155	-0.210	0.191
tgroup_3	0.299	0.221	0.088	0.261
tgroup_4	-0.226	0.135	-0.254	0.166
tgroup_5	0.003	0.138	0.070	0.163

Maximum Likelihood Analysis of Variance

Source	DF	Chi-Square	*p*-value
Intercept	2	766.340	$<$0.0001
Rating group	8	26.480	0.001
Territory group	10	54.750	$<$0.0001
Likelihood Ratio	40	55.890	0.049

Out-of-Sample Comparison



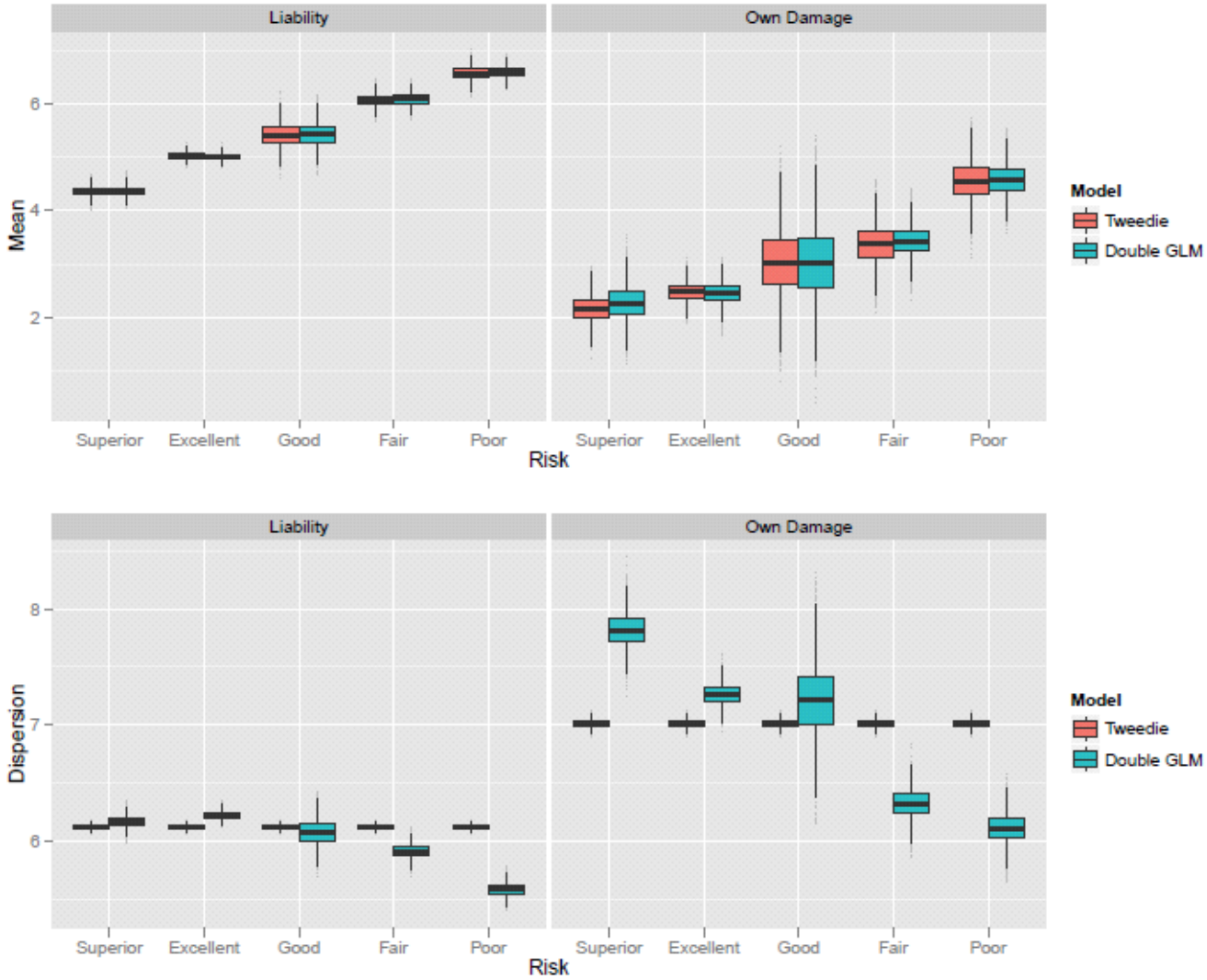
Prediction - Risk Classification

- Risk class profile

Risk Class	Rating Group					Territory Group					
	=1	=2	=3	=4	=5	=1	=2	=3	=4	=5	=6
Superior	0	0	0	0	1	1	0	0	0	0	0
Excellent	1	0	0	0	0	0	1	0	0	0	0
Good	0	1	0	0	0	0	0	0	1	0	0
Fair	0	0	0	1	0	0	0	0	0	1	0
Poor	0	0	1	0	0	0	0	0	0	0	1

- We calculate expected cost of claims for each risk class
- We quantify the variability of prediction

Prediction - Mean and Dispersion



Prediction - Frequency

- Joint distribution for high risk

	Poor			
	Tweedie		Double GLM	
	Product	Frank	Product	Frank
$\Pr(Y_1 = 0, Y_2 = 0)$	0.9215	0.9238	0.8634	0.8727
$\Pr(Y_1 > 0, Y_2 = 0)$	0.0671	0.0649	0.1088	0.0994
$\Pr(Y_1 = 0, Y_2 > 0)$	0.0106	0.0083	0.0247	0.0154
$\Pr(Y_1 > 0, Y_2 > 0)$	0.0008	0.0030	0.0031	0.0124

- For intermediate risk, predictions from the two models are similar
- For low risk, predictions are opposite of high risk

Conclusion

- We discussed alternative strategies for modeling insurance claims
- Each approach has its own strength and limitations
- Model selection also relies on the format of data