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GLM I: Introduction to Generalized Linear Models

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Liberty Mutual Insurance

Casualty Actuarial Society
Ratemaking and Product Management Seminar
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Overview

Standard Linear Model Specification

$$y = \beta_0 + x_1\beta_1 + \cdots + x_k\beta_k + \epsilon \qquad \text{with } \epsilon \in N(0, \sigma^2)$$

Standard Linear Model Specification

$$y = \beta_0 + x_1\beta_1 + \cdots + x_k\beta_k + \epsilon \qquad \text{with } \epsilon \in N(0, \sigma^2)$$

A better way to think about this would be

$$\mathbb{E}[y] = \beta_0 + x_1\beta_1 + \cdots + x_k\beta_k$$

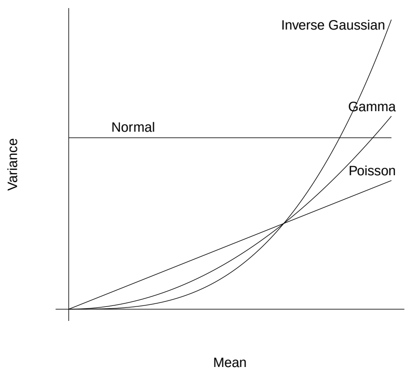
where $y \in N(\mu, \sigma^2)$ and $\mu = \beta_0 + x_1\beta_1 + \cdots + x_k\beta_k$ is the linear predictor.

Generalized Linear Model Specification

$$g(\mathbb{E}[y]) = \beta_0 + x_1\beta_1 + \cdots + x_k\beta_k + \text{offset}$$

1. The link function is g
2. The distribution of y is a member of the exponential family
3. The explanatory variables x_i may be continuous or discrete
4. Offset terms have a known coefficient of 1 in the linear predictor

Mean–Variance Relationship



Personal Injury Dataset

The dataset contains 22,036 settled personal injury claims. These claims arose from accidents occurring from July 1989 through January 1999. This is the `persinj.xls` dataset featured in the book by de Jong & Heller [2].

I have taken a random sample of 200 claims.
The variables are:

1. Settled Amount
2. Injury codes
3. Legal representation
4. Accident month
5. Report month
6. Finalization month
7. Operational time

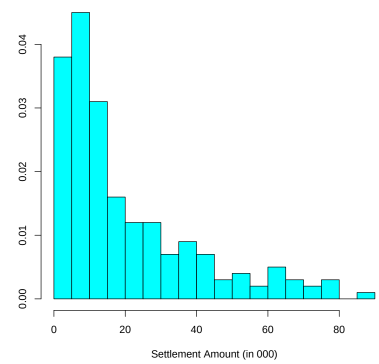
Derived variables:

1. Injured count
2. Accident injury code
3. Report delay
4. Settlement delay

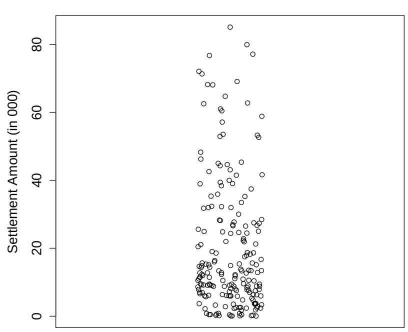
Variable Descriptions

Variable	Type	Comments
Settled Amount	Cont	range: \$40 to \$85,000
Injury Codes	Cat	Injury level: $1, 2, \ldots, 6$ = death, 9 = missing
Legal Rep.	Bin	Attorney involved? 1 = Yes, 0 = No
Accident Month	Coded	1 = July 1989, 120 = June 1999
Report Month	Coded	same as accident month
Fin. Month	Coded	same as accident month
Injured Count	Count	Number of persons injured: $1, 2, \ldots, 5$
Acc. Injury	Cat	Highest injury code among those injured
Report Delay	Cont	# months between accident and report
Settle. Delay	Cont	# months between report and settlement

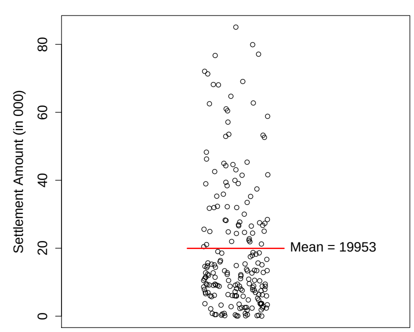
Histogram of Settlement Amount



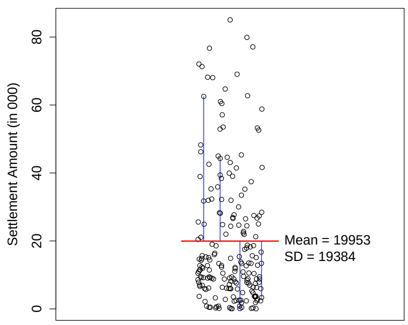
Distribution of Settlement Amount



Settlement Amount: mean



Settlement Amount: mean & standard deviation



Linear Model—Intercept only

```
Call:
lm(formula = total ~ 1, data = spinj)

Residuals:
   Min     1Q Median     3Q    Max
-19913 -13570  -7199   7591  65110

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    19953       1371   14.56   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 19380 on 199 degrees of freedom
```


Generalized Linear Model—Normal Id—Intercept only

```
Call: glm(formula = total ~ 1,
          family  = gaussian(link = identity), data = spinj)
Deviance Residuals:
   Min      1Q  Median      3Q     Max
-19913  -13570   -7199    7591   65110
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    19953       1371   14.56   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for gaussian family taken to be 375744867)

    Null deviance: 7.4773e+10  on 199  degrees of freedom
Residual deviance: 7.4773e+10  on 199  degrees of freedom
AIC: 4519.5

Number of Fisher Scoring iterations: 2
```

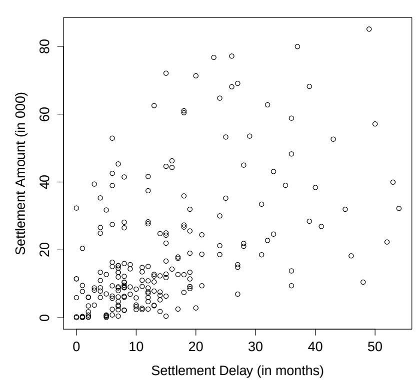

Generalized Linear Model—Gamma Id—Intercept only

```
Call: glm(formula = total ~ 1,
          family  = Gamma(link = identity), data = spinj)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-3.2293  -0.9588  -0.4165   0.3407   1.9043
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    19953       1371   14.56   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for Gamma family taken to be 0.9438079)

    Null deviance: 252.05  on 199  degrees of freedom
Residual deviance: 252.05  on 199  degrees of freedom
AIC: 4366.6

Number of Fisher Scoring iterations: 3
```


Generalized Linear Model—Gamma Log—Intercept only

```
Call: glm(formula = total ~ 1,
          family  = Gamma(link = "log"), data = spinj)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-3.2293  -0.9588  -0.4165   0.3407   1.9043
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   9.9011     0.0687   144.1   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for Gamma family taken to be 0.9438079)

    Null deviance: 252.05  on 199  degrees of freedom
Residual deviance: 252.05  on 199  degrees of freedom
AIC: 4366.6

Number of Fisher Scoring iterations: 6
```

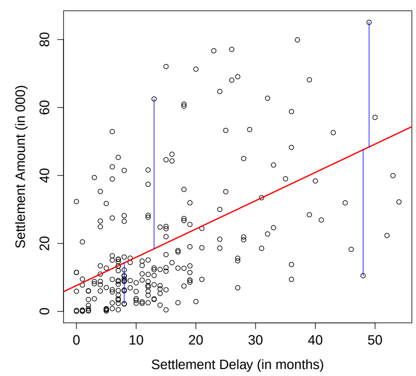

Settlement Amount vs. Settlement Delay



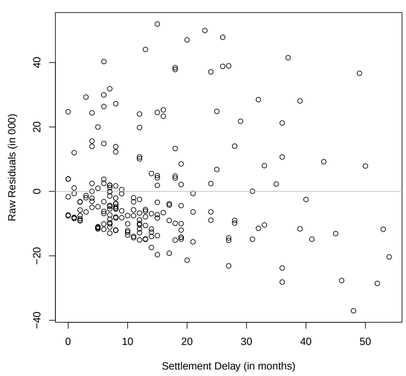
Linear Model–Intercept and Slope

```
Call:
lm(formula = total ~ settle.delay, data = spinj)

Residuals:
   Min     1Q Median     3Q    Max
-37059 -10395  -5085   4366  51957

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   7614.05    1861.85   4.089 6.28e-05 ***
settle.delay   832.30      97.44   8.542 3.50e-15 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

Residual standard error: 16610 on 198 degrees of freedom
Multiple R-squared: 0.2693, Adjusted R-squared: 0.2656
F-statistic: 72.96 on 1 and 198 DF,  p-value: 3.504e-15
```

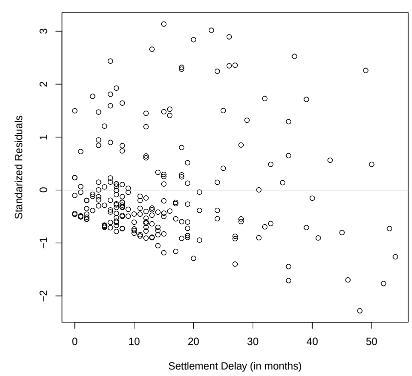

Settlement Amount vs. Delay: Least Squares Line



Raw Residuals vs. Settlement Delay



Standarized Residuals vs. Settlement Delay



Many Flavors of Residuals

Raw	$y - \hat{y}$ or $y - \mu$ or $y - \mathbb{E}[y]$
Pearson	$(y - \mu)/\sqrt{V}$
Deviance	$\text{sgn}(y - \mu)\sqrt{\text{deviance}}$

Standarized	Divide residual by $\sqrt{1-h}$, which aims to make its variance constant; where h are the diagonal elements of the projection (‘hat’) matrix, $H = X(X^tX)^{-1}X^t$, which maps y into $\hat{y}$
Studentized	Divide residual by $\sqrt{\phi}$; where ϕ is the scale parameter
Stan & Stud	Divide residual by both standarized and studentized adjustments

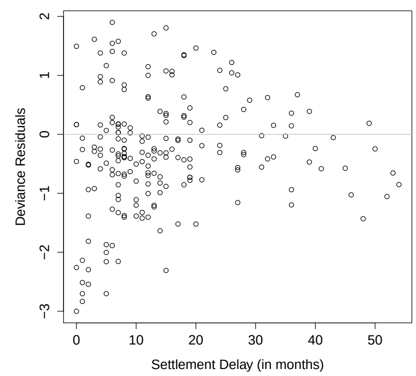
Deviance

Distribution	Contribution to Squared Deviance
Normal	$(y_i - \mu_i)^2$
Poisson	$2\{y_i \log(y_i/\mu_i) - y_i + \mu_i\}$
Gamma	$2\{-\log(y_i/\mu_i) + (y_i - \mu_i)/\mu_i\}$
Inverse Gaussian	$(y_i - \mu_i)^2/(\mu_i^2 y_i)$

Gamma Log GLM–Intercept and Slope

```
Call: glm(formula = total ~ settle.delay,
          family  = Gamma(link = "log"), data = spinj)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-3.0008  -0.8017  -0.3145   0.1991   1.8982
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  9.187173   0.102174  89.917  < 2e-16 ***
settle.delay 0.040473   0.005347   7.569 1.39e-12 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

(Dispersion parameter for Gamma family taken to be 0.8310652)

    Null deviance: 252.05  on 199  degrees of freedom
Residual deviance: 206.47  on 198  degrees of freedom
AIC: 4321.8

Number of Fisher Scoring iterations: 7
```

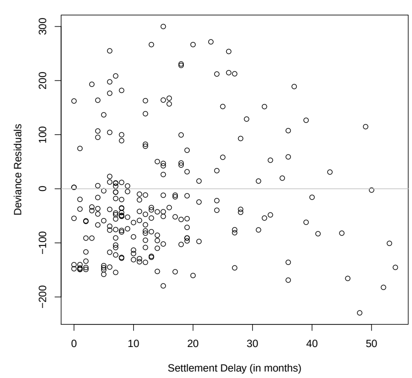

Gamma Model: Deviance Residuals vs. Settlement Delay



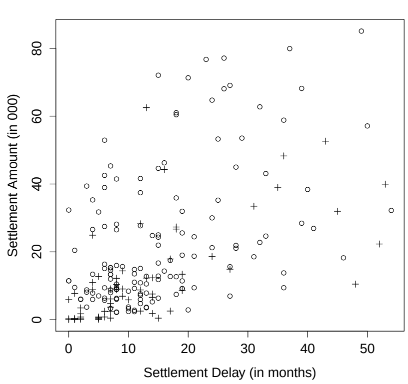
Poisson Log GLM–Intercept and Slope

```
Call: glm(formula = tot.amt ~ settle.delay,
           family = poisson(link = "log"), data = spinj)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-229.41   -92.18   -42.51    35.74   299.99
Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  9.323e+00  8.583e-04 10862.1   <2e-16 ***
settle.delay 3.280e-02  3.338e-05   982.7   <2e-16 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 3366902  on 199  degrees of freedom
Residual deviance: 2515703  on 198  degrees of freedom
AIC: 2517928

Number of Fisher Scoring iterations: 5
```


Poisson Model: Deviance Residuals vs. Settlement Delay



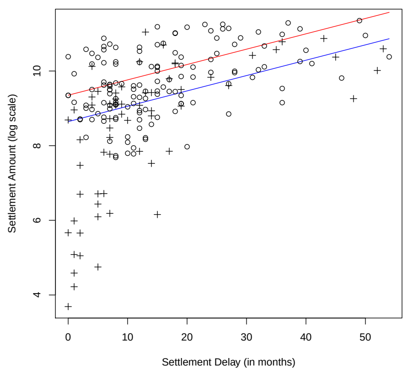
Legal Representation?



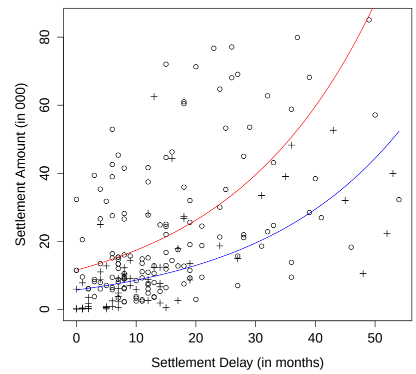
Gamma Log GLM–Legal Representation?

```
Call: glm(formula = total ~ settle.delay + legrep,
          family  = Gamma(link = "log"), data = spinj)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.8152  -0.8183  -0.3115   0.2864   2.6778
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   8.64459    0.13476  64.148  < 2e-16 ***
settle.delay  0.04112    0.00539   7.628 9.96e-13 ***
legrep1       0.70702    0.13989   5.054 9.85e-07 ***
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1

(Dispersion parameter for Gamma family taken to be 0.8354751)

    Null deviance: 252.05  on 199  degrees of freedom
Residual deviance: 186.98  on 197  degrees of freedom
AIC: 4300.9

Number of Fisher Scoring iterations: 8
```

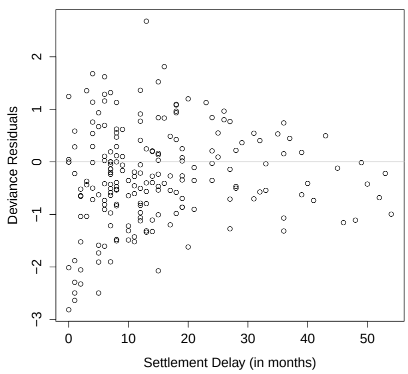
Legal Representation: Linear Predictor



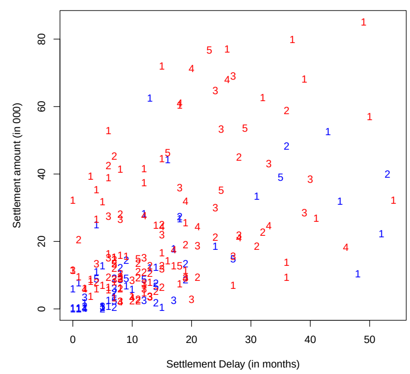
Legal Representation: Fitted Values



Legal Representation: Deviance Residuals



Number of Injured Persons



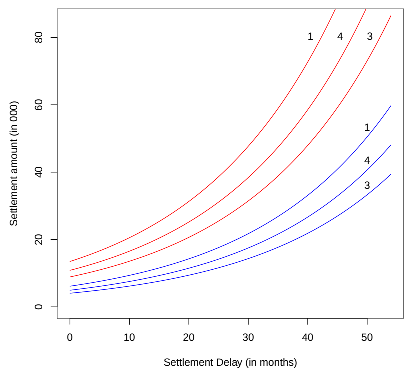
Gamma Log GLM–Many Predictors

```
Call: glm(formula = total ~ settle.delay + legrep + inj.count,
          family  = Gamma(link = "log"), data = spinj)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)   8.722358   0.141721  61.546  < 2e-16 ***
settle.delay  0.042138   0.005222   8.069 7.38e-14 ***
legrep1       0.786161   0.139411   5.639 6.01e-08 ***
inj.count2   -0.300230   0.160788  -1.867   0.0634 .
inj.count3   -0.416338   0.177247  -2.349   0.0198 *
inj.count4   -0.216891   0.244640  -0.887   0.3764
inj.count5    0.005267   0.254395   0.021   0.9835

    Null deviance: 252.05  on 199  degrees of freedom
Residual deviance: 181.44  on 193  degrees of freedom
AIC: 4302

Number of Fisher Scoring iterations: 9
```


Predicted Values

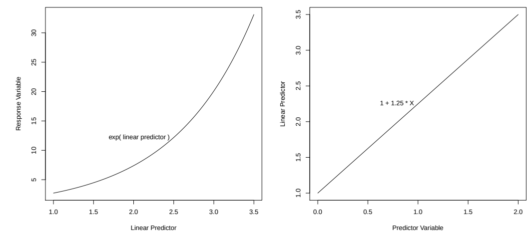
Settle Delay	Legal Rep?	Injured Count	Linear Predictor	Fitted Value
0	No	1	$8.7 + 0 \cdot 0.042 = 8.7$	$e^{8.7} = 6003$
0	Yes	1	$8.7 + 0 \cdot 0.042 + 0.79 = 9.5$	$e^{9.5} = 13360$
10	No	4	$8.7 + 10 \cdot 0.042 - 0.22 = 8.5$	$e^{8.9} = 7332$

Many Predictors: Fitted Values



Summary Key Concepts: Link Function

The link function is the bridge between the space of the linear predictor and the space of the response.



Summary Key Concepts: Deviance

The deviance tells us how to measure the distance between an observation and its fitted value.

Distribution	Contribution to Squared Deviance
Normal	$(y_i - \mu_i)^2$
Poisson	$2\{y_i \log(y_i/\mu_i) - (y_i - \mu_i)\}$
Gamma	$2\{-\log(y_i/\mu_i) + (y_i - \mu_i)/\mu_i\}$
Inverse Gaussian	$(y_i - \mu_i)^2/(\mu_i^2 y_i)$

References

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