A Brief Introduction to Generalized Linear Mixed Models and Generalized Additive Models

ERIC MANN, FCAS

URI KORN, FCAS

Roadmap

- Review of Linear Models and Generalized Linear Models
- Generalized Additive Models
  - Example
- Mixed Effect Models
  - Example
- Questions

Review of Linear Models

Classical Linear Model

- Response: $Y \sim N(X\beta, \sigma^2)$
- $X\beta$ is a linear function that describes how the expected values vary based on characteristics in the data
- Linear: $\beta_0 + \beta_1 X_1^2 + sin(\beta_2 X_2)$
- Non-linear: $\beta_1 X_1 e^{\beta_2 X_2}$
- Constant Variance

Generalized Linear Model

- Response: Poisson, Gamma, Binomial, etc.
- $Y \sim F(\pi, R)$
- Expected Value:

$G(E[Y])^{-1} = X\beta$

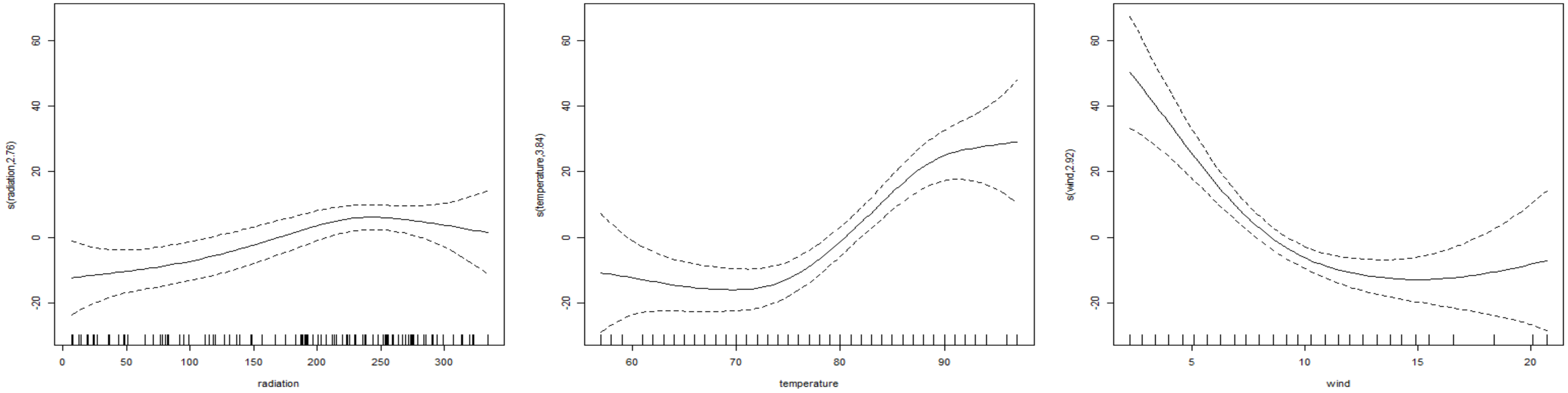
- Variance is a function of expected value
- Responses are independent

Generalized Additive Models

- Linear predictor has a more general form
- $E(Y|X_1, X_2, \cdots X_p) = \alpha + f_1(X_1) + f_2(X_2) + \cdots + f_p(X_p)$

- $f_i(X_i)$ are non-parametric smoother functions
  - Smoothing Splines
  - Kernel Smoothers
  - Local Linear Regression
  - But can also be parametric functions, too

What Does That Mean in Real Life?



- Fit models with less assumptions
  - No ‘nice’ polynomial shapes are necessary
  - No variance assumptions

When Can I Fit a GAM?

- You can fit a GAM with any data where you might try fitting LMs, GLMs, and GLMMs
  - GAMs are more general and with less assumptions
- Common Examples
  - LDF fitting
  - Large data sets with complicated interaction effects
  - Models with many parameters but not a lot of data per parameter
  - Fitting a smoothed trend line that allows the trend to vary by year

GAM Tradeoffs

Advantages

1. Useful for non-parametric and semi-parametric data
2. Useful when data doesn't fit LM/GLM assumptions
3. Can paste splines directly into Excel

Disadvantages

1. Output may be more difficult to interpret to regulators and business side
2. Must be wary of over-fitting

Let Software Do the Hard Work!

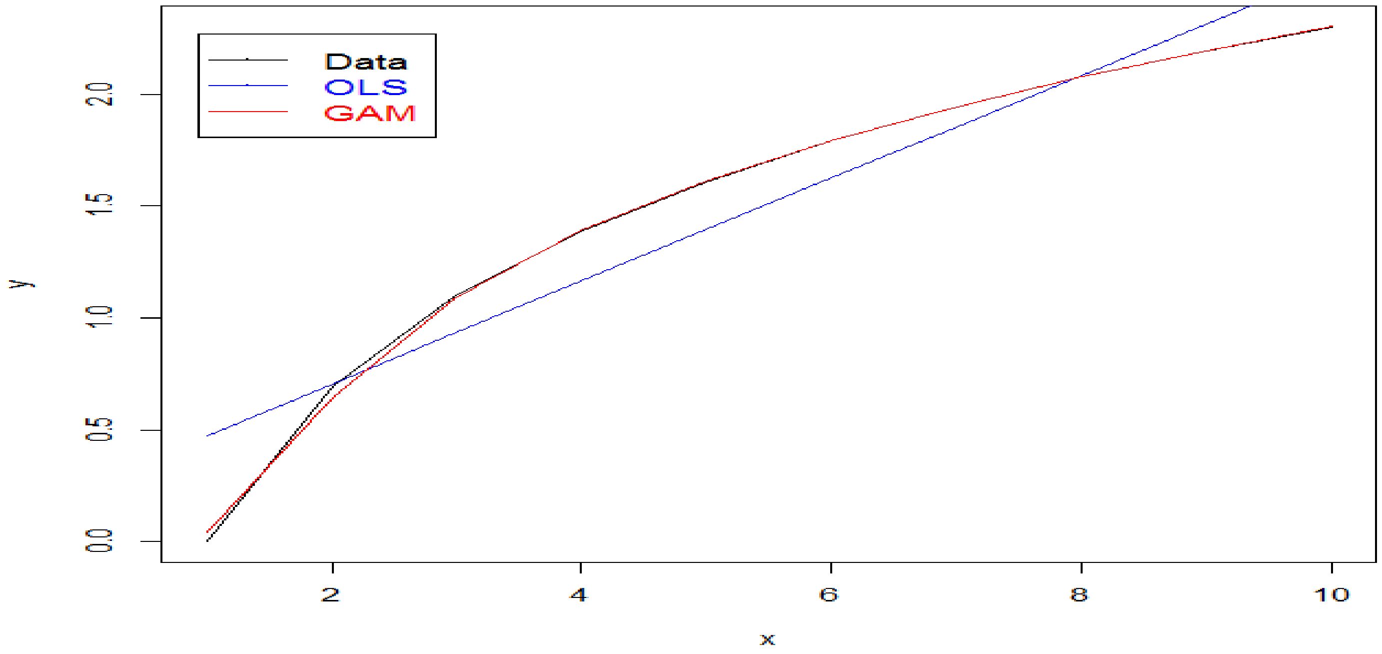
R

- Packages
  - gam
  - mgcv – this package automatically selects smoothing factors

SAS

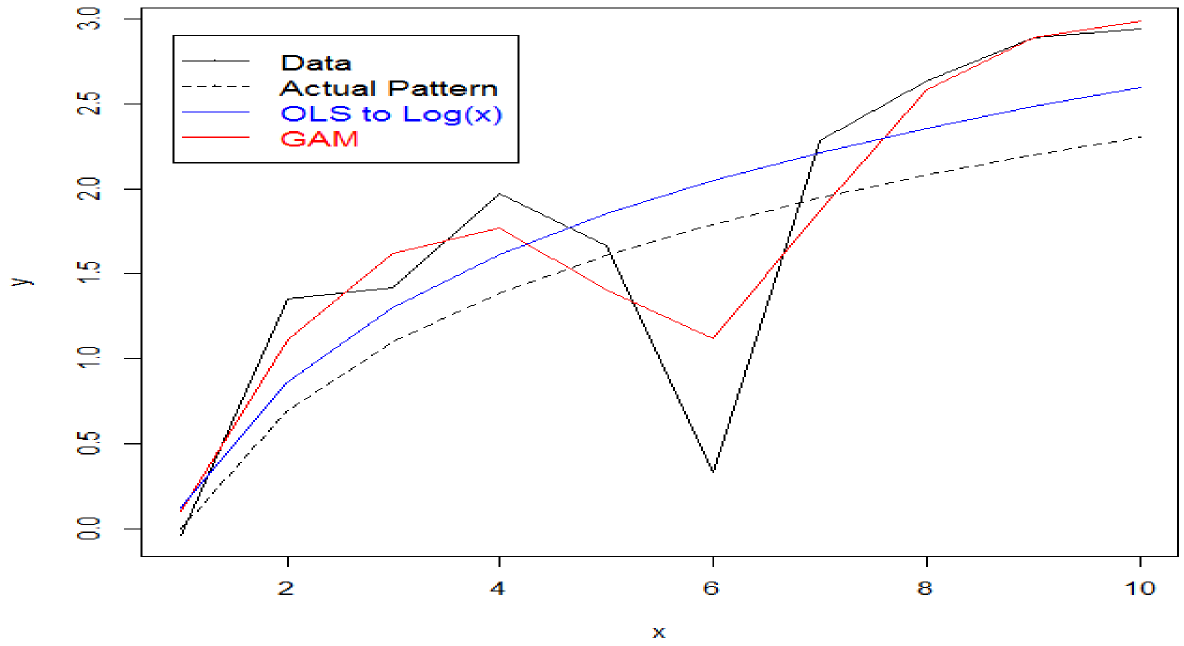
- PROC GAM
  - SAS 9.2

Simple GAM Example



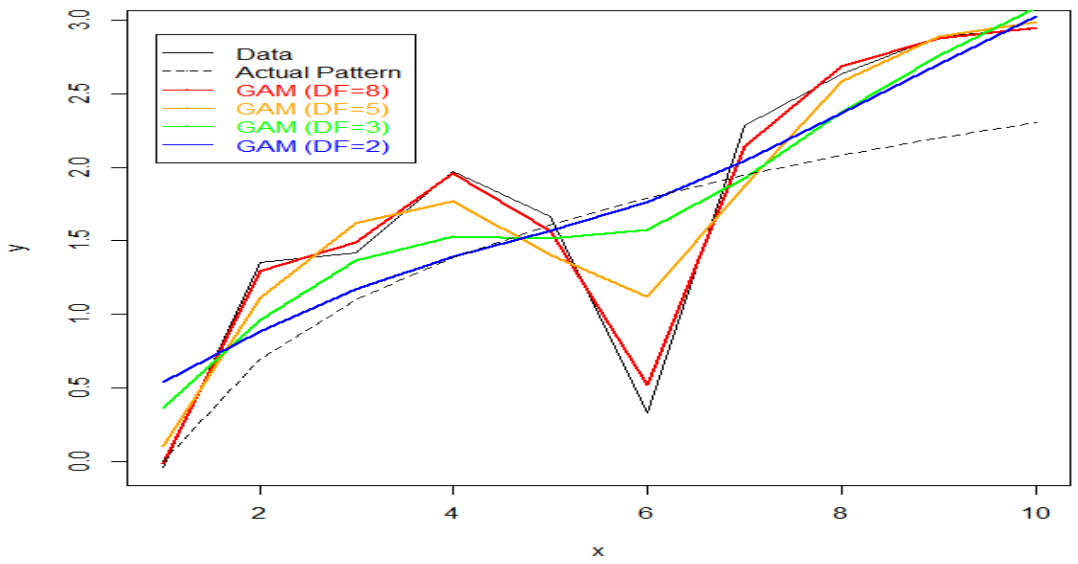
- Smoothing to data can provide a very good fit

GAM Fitting to Noisy Data



- Smoothing to data can sometimes cause over-fitting though
- If a good parametric fit exists, use that instead

Amount of Smoothing Can be Varied



- DF = Degrees of Freedom = the number of parameters we are using to smooth
- A GAM ranges from a linear curve to fitting each point exactly

Some R Code for the Curious

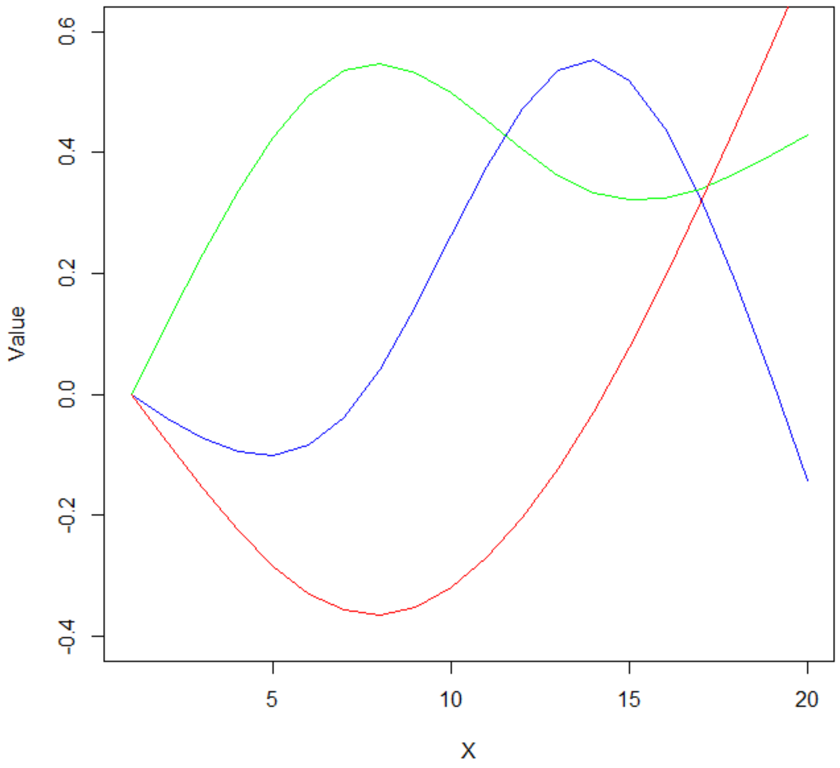
```
x <- 1:10
y <- log(x)

plot( x, y, type='l' )

fit.lm <- lm( y ~ x )
lines( predict( fit.lm ), col='blue' )

library(gam)
fit.gam <- gam( y ~ s(x, df=5) )
lines( predict( fit.gam ), col='red' )
```

Splines



- GAMs work by generating splines
- These can also be copied and pasted into Excel

- In R:
  - library(splines)
  - ns(1:20, df=3)

GAM Practical Application: LDF Fitting

- LDF patterns – Difficult to find a good parametric curve
- A GAM can be used to help smooth the curve to the data
- Will show an approach here that combines best features of two published models: the Inverse Power Curve (Extrapolating, Interpolating, and Smoothing Development Factors, Sherman, 1984) and England and Verrall's GAM model (A Flexible Framework For Stochastic Claims Reserving, 2001)

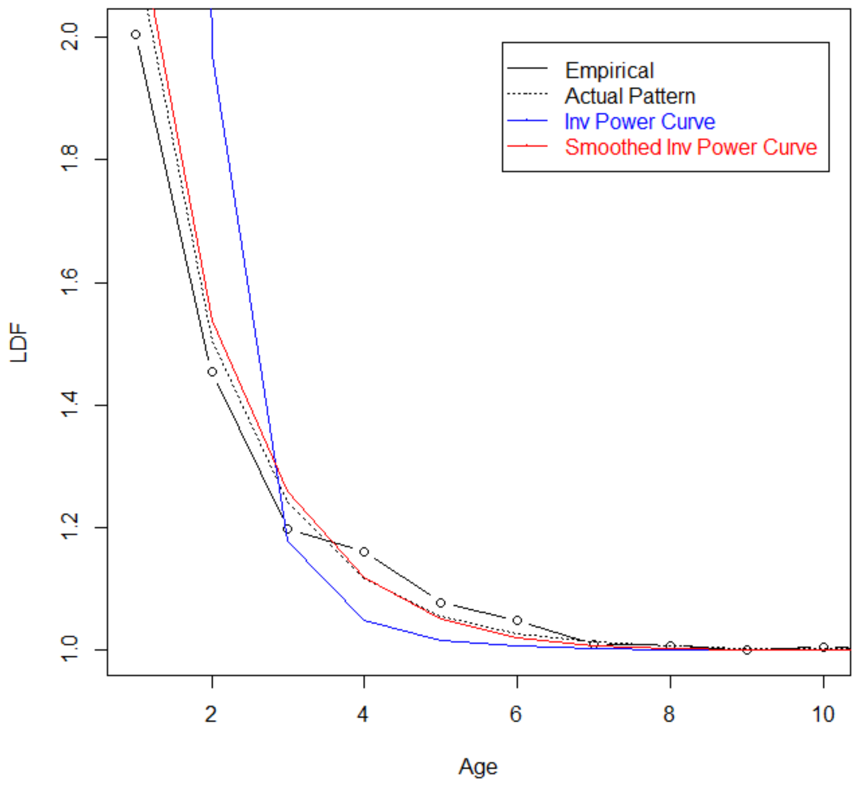
GAM Practical Application: LDF Fitting

- Inverse Power Curve
  - Good: Simple procedure that can fit a portion of the LDFs well
  - Bad: Struggles in many lines to provide a good fit to the entire curve
- England & Verrall's GAM Model
  - Good: Uses a GAM to get a nice fit to the incremental loss pattern (within the common GLM loss development framework)
  - Bad:
    - Negative values difficult to deal with
    - Some of the resulting LDFs can be difficult to interpret when comparing to the empirical LDFs
    - More difficult to implement: Need to find correct Tweedie power
    - Can't implement in Excel
    - Hard to incorporate credibility (teaser)

Proposed Approach

- Smoothed Inverse Power Curve using GAMs (Korn 2015?)
- Inverse Power Curve:
  - log(LDF – 1) = A + B log(t)
- Smoothed Inverse Power Curve:
  - log(LDF – 1) = A + s(log(t))
    - Where "s" means GAM smoothing
    - (Note the smoothing is done on log(t))

Smoothed Inverse Power Curve



- Comparison of two approaches on simulated data
- This same pattern has been observed on actual data, where the inverse power curve has trouble making the "turn"
- The smoothed inverse power curve does a good job of smoothing out the volatility
- (No, I did not fish for a good example)

Still Interested in GAMs?

- Elements of Statistical Learning
  - By Hastie and Tibshirani
  - Free download: http://web.stanford.edu/~hastie/local.ftp/Springer/OLD/ESLII_print4.pdf
- Stochastic Claims Reserving in General Insurance
  - By England and Verrall

Review of Linear Models (Again)

Classical Linear Model

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Correlated Losses

- But in the real world losses may not be independent
  - Why?
- **Hierarchical Data** – Correlation can exist among loss data when the risks come from the same territory or region

- **Repeated Measures** - Unless you have 0% retention, correlation can exist among records as some of them will represent the same risk repeatedly observed over time

Mixed Effect Models to the Rescue!

- Linear Predictor contains fixed effects and random effects
- $X\beta + Zb$
- $Z \sim N(0, G)$
  - G is a covariance matrix that can reflect the extra variability and the correlation within the levels of a territory or across time
  - Flexible enough to specify different G side covariance structures
- Response can still be Normal or from Exponential Class

LM	**LMM**
GLM	**GLMM**

What if I Use a LM/GLM Anyway?

Linear Model

- LM thinks it estimates this:
$$Y \sim N(X\beta, R\ )$$
- But it actually estimates this:
$$Y \sim N(X\beta, G + R\ )$$
- **Result:** correct parameter estimates but incorrect covariance estimate and distorted alpha levels

Generalized Linear Model

- GLM thinks it estimates this:
$$Y \sim F(\pi, R)$$
- But it actually estimates this:
$$Y \sim F(\tilde{\pi}\ , TGT + R\ )$$
- **Result:** Incorrect parameter estimates and incorrect covariance structure and distorted alpha levels

Don't Do Heavy Math by Yourself! Use Software!

R

- Package: lme4
  - Can fit common distributions but **not** the over-dispersed Poisson or Tweedie

SAS

- PROC GLIMMIX
  - SAS 9.1 and later
  - Can fit common distributions **and** over-dispersed Poisson
  - Uncertain about Tweedie

Simulation Example - GLMM

- Random effects:
- Groups Name Variance Std.Dev. Corr
- territory (Intercept) 0.1525 0.3905
- A1 0.3756 0.6129 0.27
- Number of obs: 20000, groups: territory, 100

- Fixed effects:
- Estimate Std. Error z value Pr(>|z|)
- (Intercept) 2.03836 0.03917 52.03 <2e-16 ***
- A1 0.61291 0.06135 9.99 <2e-16 ***

Simulation Example - GLM

- Coefficients:
- Estimate Std. Error z value Pr(>|z|)
- (Intercept) 2.303576 0.002483 927.8 <2e-16 ***
- A1 0.671555 0.002483 270.5 <2e-16 ***

Expected Counts

	Actual	GLMM	GLM
Base Class	7.4	7.6	10.0
A1	12.2	14.1	19.6

Practical Application – Credibility Weighting

- A GLMM with a normal distribution and an identity link will produce identical results as the Buhlmann-Straub method
  - Benefits of GLMM:
    - Easier to automate – no need to manually calculate the within and between variances
    - More flexibility
      - More complicated regression models, such as hierarchal and multi-dimensional
      - Ability to handle different link functions (e.g. log, logit), non-normal errors, and continuous variables
  - A disadvantage is that a GLMM is harder to use from Excel

Credibility Weighting of Expected Loss Ratios/Costs

- Don't just credibility weight the IBNR portion! – This will be credibility weighting only half of the data.

- Don't credibility weight the selected ultimates from a BF (or similar) method! – That would be including what did NOT happen (and lowering the variance = too much credibility).

- Use the observed experience!

Coin Flipping Analogy

- First Time: 20 Flips
  - 15 Heads, 5 Tails = 75% Heads
- Second Time (Same Coin): 5 Flips (out of 20)
  - 0 Heads, 5 Tails
  - IBNR (BF Method): 11.25 Heads, 3.75 Tails
  - Ultimate: 11.25 Heads, 8.75 Tails = 56.25% Heads
- What's the variance??

Credibility Weighting of Expected Loss Ratios/Costs

- Instead, use a Cape-Cod-like method:
  - LR per Year = Reported Losses / Used Premium (= chain ladder)
  - Initial Weight per Year = Used Premium
  - Then apply an off-balance factor so that the total weight for each segment equals the actual premium

Credibility Weighting of Expected Loss Ratios/Costs

- Weights:
  - GLMMs use the same weight for credibility as they do for the regression
  - For calculating the variances, the weight is assumed to be the number of observations
    - Premium Weights = Full Credibility
    - Claim Count Weights = Inconsistencies
  - To reconcile: (Really use weights as above instead of Premium)
    - K = Claim Count / Premium (for all policies)
    - For each policy, Weight(i) = Premium(i) x K
    - Total weight will be consistent with true number of observations and we will still be weighting by premium

Credibility Weighting of Expected Loss Ratios/Costs

- Structure of the GLMM:
  - Normal/Normal (is not the same as assuming that loss ratios are normally distributed)
  - Link function:
    - Log link: Multiplicative (dealing with 0s)
    - Identity link: Additive

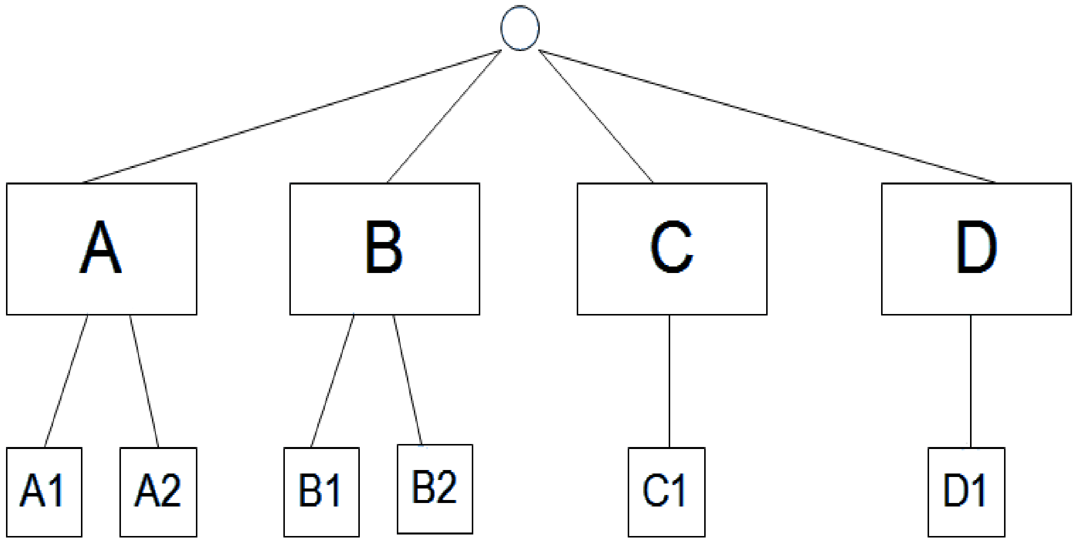
Some R Code for the Curious

- library(lme4)
- fit <- glmer(lr ~ (1 | sic1) + (1 | sic2), weights=w, data=mydata, family=gaussian(link='log'))
- fixef(fit)
- ranef(fit)

Another Practical Use – Credibility Weighted Interaction Terms

- State x Industry Example:
  - No interaction: If New York is running 20% worse overall, it will be 20% worse in every single industry
    - Not Enough Information!
  - With interaction term: Every single state x industry combination will be assigned a loss ratio based on its experience alone
    - Not Enough Data!
  - Credibility weighted interaction term: If New York is running 20% worse overall, this will be the complement of credibility for each industry
    - Makes the most out of limited data
    - In R: "(1 | state:industry)"

Uneven Hierarchies



- A GLM will not produce coefficient values for C1 and D1 since they are redundant
- A GLMM will → Double Credibility!

- To handle, create a dummy variable that is 1 for A & B cells, and 0 for C & D cells
- For the lowest subgroup, create the random effect as a slope parameter on this dummy variable
- This will cause C1 and D1 to not be assigned coefficients
- In R: (1 | group) + (dummy | subgroup)

Other Practical Uses

- Incorporating credibility into pricing or other GLMs
- Credibility weighting of trend (if you have enough data)

Still Interested in GLMMs?

- Further Reading:
- Generalized Linear Mixed Models: Modern Concepts, Methods and Applications
  - By Walter Stroup
  - Examples for SAS
- Mixed-Effect Models in S and S-Plus
  - By Pinheiro and Bates
  - Written for R