Utility Theoretic Decision-Making

Leigh J. Halliwell

Economic agent with current wealth *W* bids for the amount θ of *X* at price *q* that maximizes its resultant expected utility:

$$f(\theta, q) = E[u(W + \theta X - \theta q)]$$

Equivalently, the agent will chose amount-price combinations that set to zero the derivative of this function with respect to θ.

$$\frac{\partial f(\theta, q)}{\partial \theta} = E[u'(W + \theta X - \theta q)(X - q)]$$

Again equivalently, one may normalize the function and set it to zero:

$$\begin{aligned} h(\theta, q) &= \frac{E[u'(W + \theta X - \theta q)(X - q)]}{E[u'(W + \theta X - \theta q)]} \\ &= E[\Psi(X - q)] \\ &= E[\Psi X] - q \end{aligned}$$

$$E[\Psi] = 1$$

Therefore, $h(\theta, q) = 0 \Leftrightarrow q = E[\Psi X]$.

The agent's utility is exponential with risk-aversion parameter $a \geq 0$:

$$u(x) = \begin{cases} \frac{1 - e^{-ax}}{a} & a \neq 0 \\ x & a = 0 \end{cases}$$

$$u'(x) = e^{-ax}$$

Therefore,

$$
\begin{aligned}
q &= E[\Psi X] \\
&= \frac{E[u'(W+\theta X-\theta q)X]}{E[u'(W+\theta X-\theta q)\quad]} \\
&= \frac{E\left[e^{-a(W+\theta X-\theta q)}X\right]}{E\left[e^{-a(W+\theta X-\theta q)}\right]} \\
&= \frac{E\left[e^{-a(W+\theta X)}X\right]}{E\left[e^{-a(W+\theta X)}\right]} \\
&= -\frac{1}{a}\frac{\partial}{\partial\theta}\ln E\left[e^{-a(W+\theta X)}\right] \\
&= -\frac{1}{a}\frac{\partial\psi_{W,X}(-a,-a\theta)}{\partial\theta}
\end{aligned}
$$

The derivative of q with repect to θ is:

$$
\begin{aligned}
\frac{dq}{d\theta} &= \frac{d}{d\theta}\frac{E\left[e^{-a(W+\theta X)}X\right]}{E\left[e^{-a(W+\theta X)}\right]} \\
&= \frac{E\left[e^{-a(W+\theta X)}\right]E\left[e^{-a(W+\theta X)}(-aX)X\right]-E\left[e^{-a(W+\theta X)}(-aX)\right]E\left[e^{-a(W+\theta X)}X\right]}{E\left[e^{-a(W+\theta X)}\right]^2} \\
&= (-a)\frac{E\left[e^{-a(W+\theta X)}\right]E\left[e^{-a(W+\theta X)}X^2\right]-E\left[e^{-a(W+\theta X)}X\right]E\left[e^{-a(W+\theta X)}X\right]}{E\left[e^{-a(W+\theta X)}\right]^2} \\
&= (-a)\left\{\frac{E\left[e^{-a(W+\theta X)}X^2\right]}{E\left[e^{-a(W+\theta X)}\right]}-\left(\frac{E\left[e^{-a(W+\theta X)}X\right]}{E\left[e^{-a(W+\theta X)}\right]}\right)^2\right\} \\
&= -a\left(E\left[\Psi X^2\right]-E[\Psi X]^2\right) \\
&= -aE\left[\Psi(X-q)^2\right] \\
&= -aVar_{Q=P\Psi}[X]
\end{aligned}
$$

If (*W*, *X*) is bivariate-normal, then:

$$\begin{aligned} q &= -\frac{1}{a}\frac{\partial\psi_{W,X}(-a,-a\theta)}{\partial\theta} \\ &= -\frac{1}{a}\frac{\partial\left\{-a\mu_W - a\theta\mu_X + a^2\left(\sigma_{WW} + 2\theta\sigma_{WX} + \theta^2\sigma_{XX}\right)/2\right\}}{\partial\theta} \\ &= -\frac{1}{a}\left\{-a\mu_X + a^2\left(\sigma_{WX} + \theta\sigma_{XX}\right)\right\} \\ &= \mu_X - a\left(\sigma_{WX} + \theta\sigma_{XX}\right) \end{aligned}$$

$$\frac{dq}{d\theta} = -a\sigma_{XX}$$

These formulas hold for $a = 0$ also.