GLM I: Introduction to Generalized Linear Models

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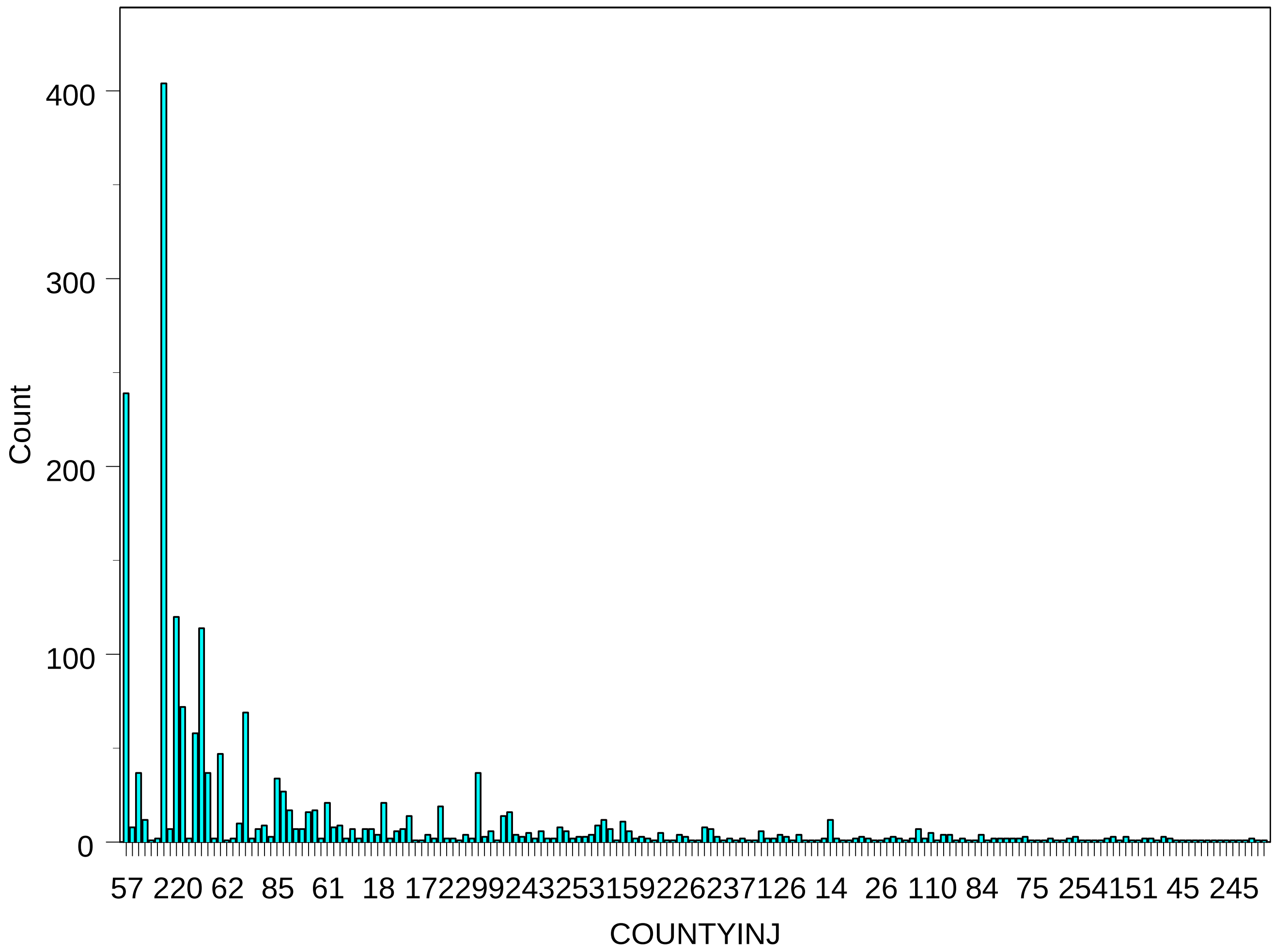
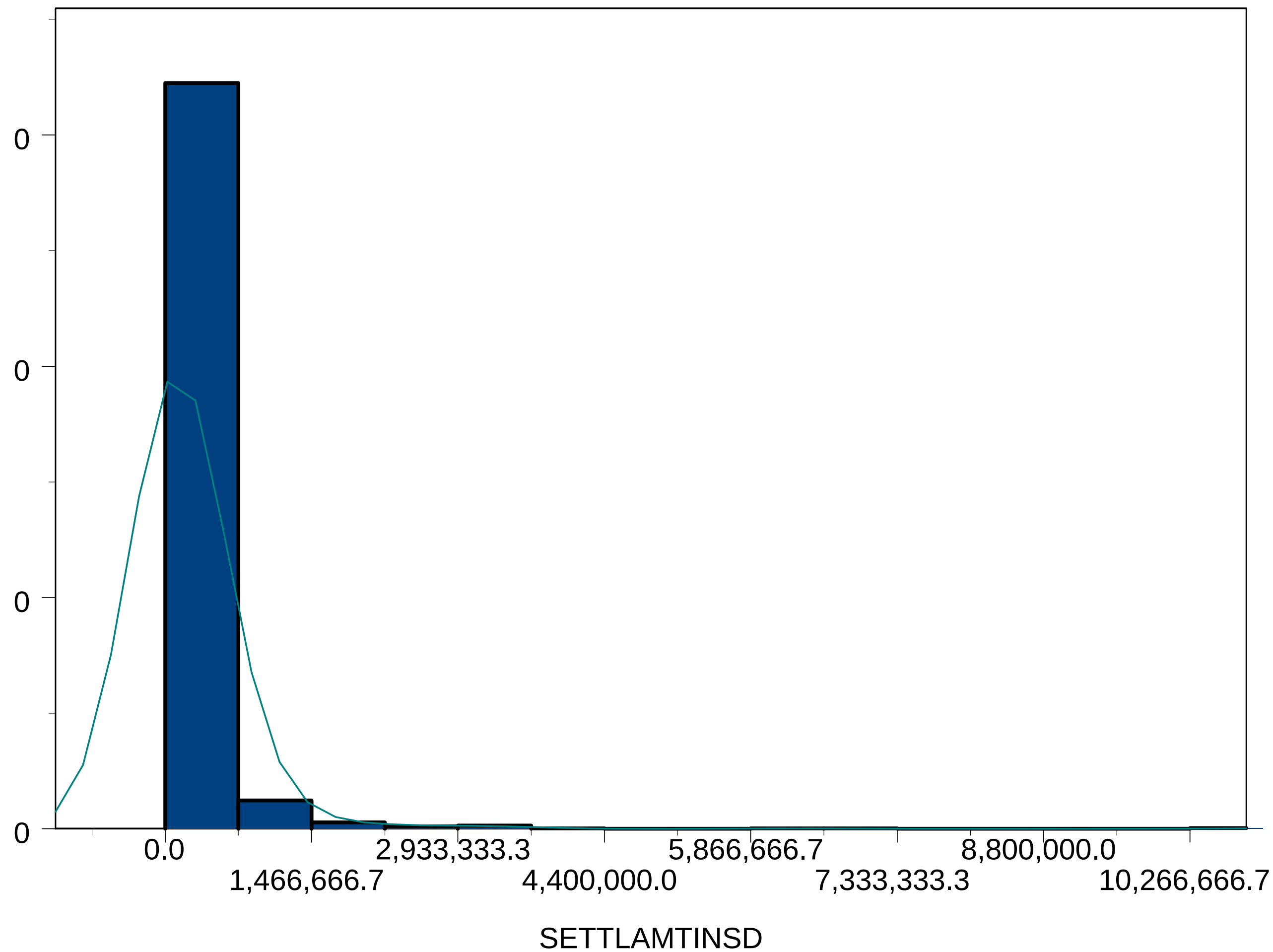
Modeling Number of Claims

Policy	Sex	Territory	Number of Claims in 5 Years
1	M	02	0
2	F	01	0
3	F	01	0
4	F	02	1
5	F	01	0
6	F	02	1
7	M	02	2
8	M	02	2
9	M	02	1
10	F	01	1
:	:	:	:

Number of Claims by County (Discrete Distribution)



Histogram of Paid(Insurer) from Texas Data



Problems with Regresssion Model

- Number of claims is discrete
- Claim sizes are skewed to the right
- Probability of an event is in [0,1]
- Variance is not constant across data points *i*
- Nonlinear relationship between *X*'s and *Y*'s

Generalized Linear Models - GLMs

- Fewer restrictions
- *Y* can model number of claims, probability of renewing, loss severity, loss ratio, etc.
- Large and small policies can be put into one model
- Y can be nonlinear function of X's
  - Only some nonlinear relationships can be modeled
- Classical linear regression model is a special case

Classical Multiple Linear Regression

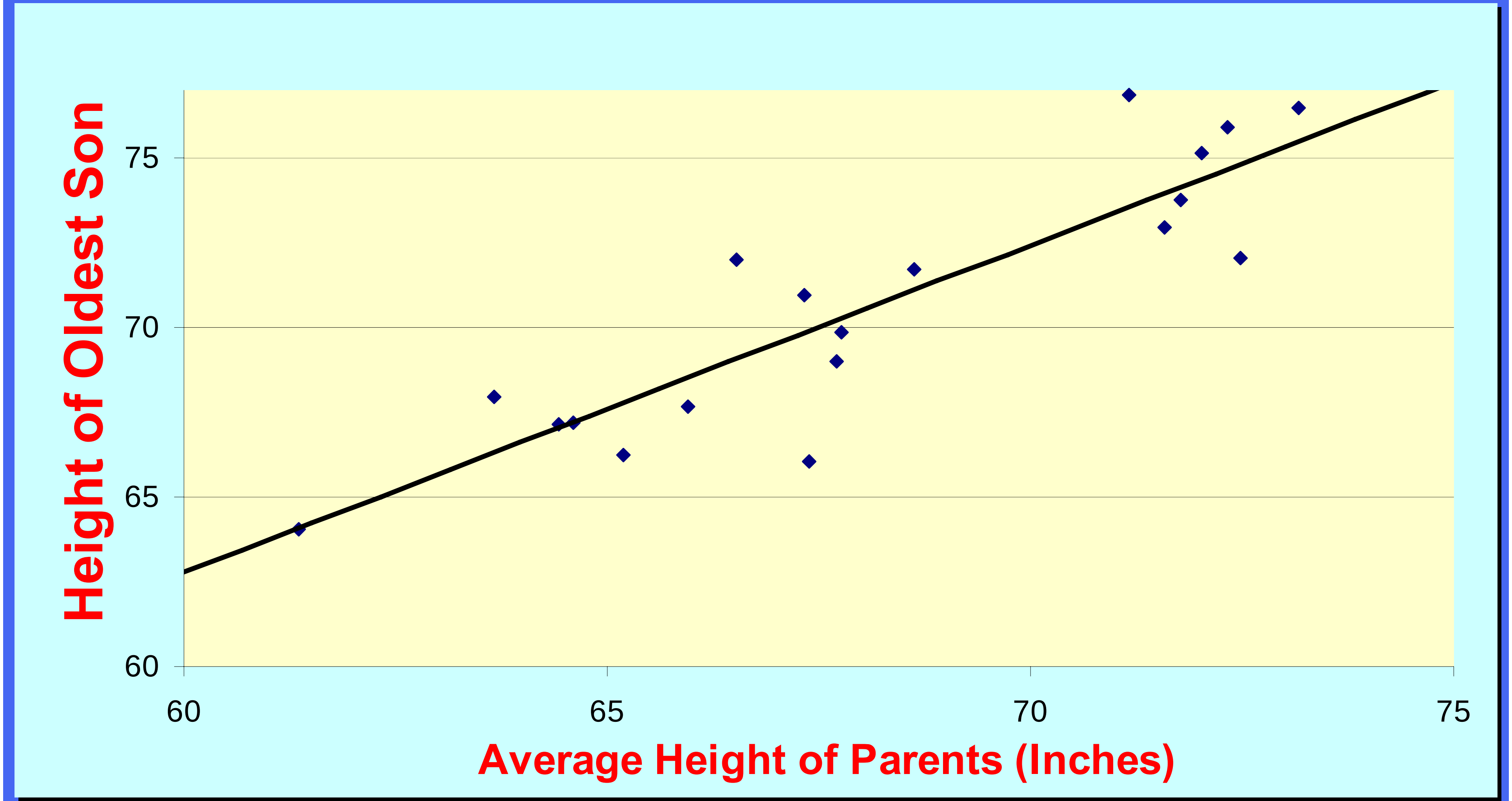
- $Y_i = a_0 + a_1 X_{i1} + a_2 X_{i2} \ldots + a_m X_{im} + e_i$

- Y_i are the response variables
- X_{ij} are predictors

- *i* subscript denotes i^{th} observation

- *j* subscript identifies j^{th} predictor

One Predictor: $Y_i = a_0 + a_1X_i$



Classical Multiple Linear Regression: Solving for a_i

$$\text{Minimize} \quad G(a_0, a_1, ..a_m) =$$

$$\sum_{i=1}^{n} (Y_i - a_0 - a_1 X_{i1} - \ldots - a_m X_{im})^2$$

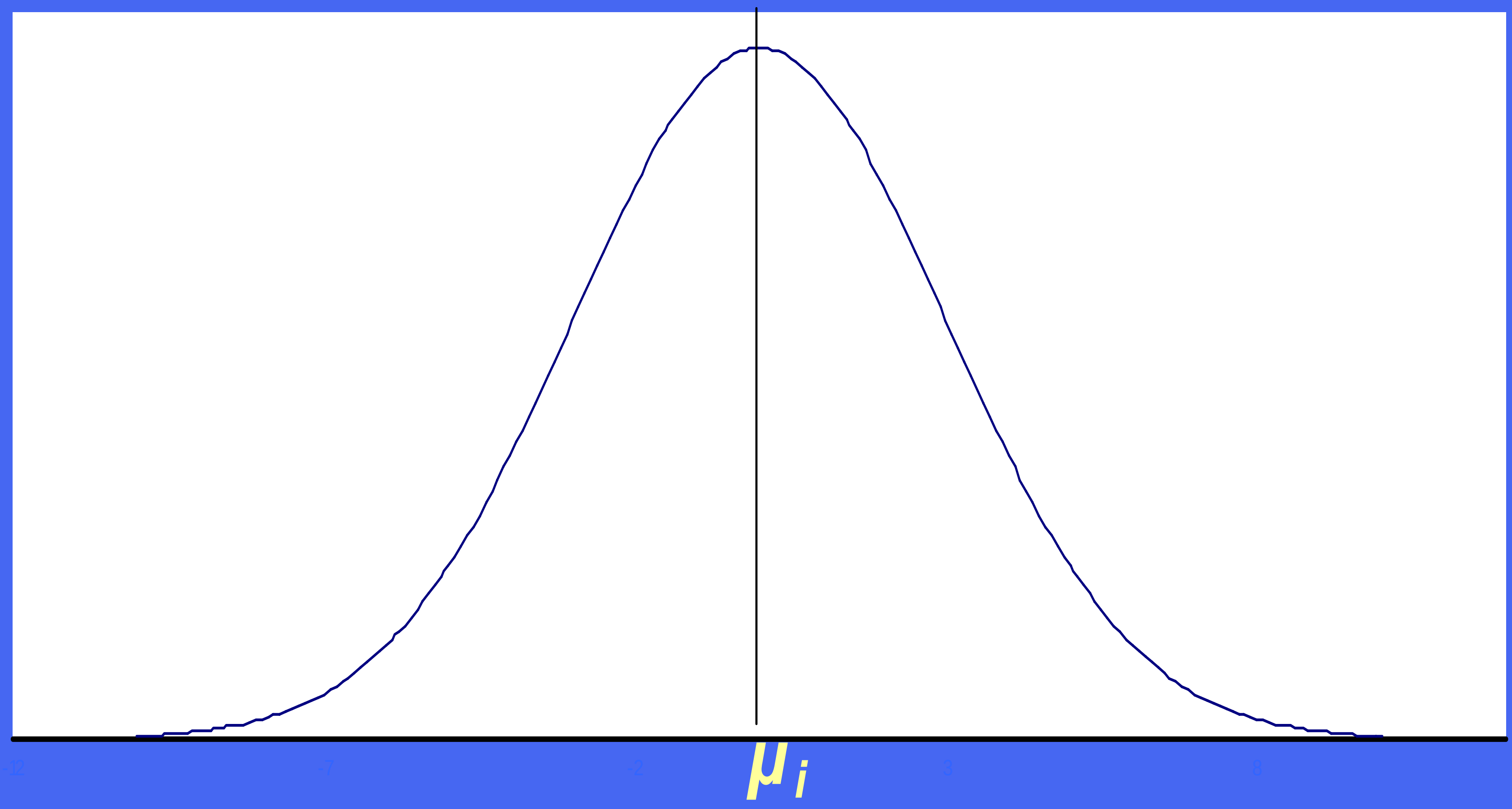
Classical Multiple Linear Regression

- $\mu_i = \mathrm{E}[Y_i] = a_0 + a_1 X_{i1} + \ldots + a_m X_{im}$

- Y_i is Normally distributed random variable with constant variance σ^2

- Want to estimate $\mu_i = \mathrm{E}[Y_i]$ for each i

Response Y_i has Normal Distribution



Generalized Linear Models - GLMs

- **Same goal as Linear Model**

$$\text{Predict}: \quad \mu_i = E[Y_i]$$

Generalized Linear Models - GLMs

- $g(\mu_i) = a_0 + a_1X_{i1} + \ldots + a_mX_{im}$
- $g(\,)$ is a function of the dependent variable
  - Referred to as the link function
  - *A transformation such as log*
- $E[Y_i] = \mu_i = g^{-1}(a_0 + a_1X_{i1} + \ldots + a_mX_{im})$
  - Must reverse the transformation to get original dependent variable back
- Y_i can be Normal, Poisson, Gamma, Binomial, Compound Poisson, …
- Variance can be modeled

GLMs Extend Classical Linear Regression

- If link function is identity: $g(\mu_i) = \mu_i$
- And Y_i has Normal distribution

→ GLM gives same answer as Classical Linear Regression*

* Least squares and MLE equivalent for Normal dist.

Exponential Family of Distributions – Canonical Form

$$f\left(y;\theta,\phi\right)=\exp\left[\frac{\{\theta\cdot y-b(\theta)\}}{a(\phi)}+c\left(y,\phi\right)\right]$$

$$E[Y] \quad = \quad b'(\theta)$$

$$Var[Y] \quad = \quad b''(\theta)a(\phi)$$

θ is the parameter of interest !

ϕ is often called a nuisance parameter.

Some Math Rules: Refresher

$$(1) \quad \exp(x) = e^{x}$$

$$(2) \quad x = \exp[\ln x] = \ln[\exp x]$$

$$(3) \quad \ln(xy) = \ln x + \ln y$$

$$(4) \quad \ln(x^{r}) = r \ln x$$

$$(5) \quad \ln(1/x) = \ln(x^{-1}) = -\ln x$$

$$(6) \quad \ln(x/y) = \ln x - \ln y$$

Normal Distribution in Exponential Family

$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$$

$$= \exp\left(\ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)\right) \exp\left(-\frac{y^2 - 2\mu y + \mu^2}{2\sigma^2}\right)$$

$$= \exp\left(\frac{\mu y - \mu^2/2}{\sigma^2} - \frac{y^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}\right)$$

Normal Distribution in Exponential Family

θ $\qquad b(\theta)$

$$f(y;\mu,\sigma^2) = \exp\left(\frac{\mu y - \mu^2/2}{\sigma^2} - \frac{y^2}{2\sigma^2} - \ln\sqrt{2\pi\sigma^2}\right)$$

Let $\theta = \mu$ and $a(\phi) = \sigma^2$,

then $b(\theta) = \theta^2/2 \rightarrow b'(\theta) = \theta = \mu$

and $Var[Y] = b''(\theta)a(\phi) = 1\cdot\sigma^2 = \sigma^2$

Poisson Distribution in Exponential Family

$$\Pr[Y = y] = \frac{\mu^y e^{-\mu}}{y!}$$

$$\Pr[Y = y] = \exp\left\{ \ln\left(\frac{\mu^y e^{-\mu}}{y!} \right) \right\}$$

θ

$$\Pr[Y = y] = \exp\left\{ \frac{(\ln \mu) \cdot y - \mu}{1} - \ln(y!) \right\}$$

Poisson Distribution in Exponential Family

$$\theta = \ln \mu \rightarrow \mu = e^{\theta}$$

$$b(\theta) = \mu = e^{\theta} \quad \text{and} \quad a(\phi) = 1$$

$$E[Y] = b'(\theta) = \frac{d}{d\theta} e^{\theta} = e^{\theta} = \mu$$

$$Var[Y] = b''(\theta) a(\phi) = \frac{d^2}{d\theta^2} e^{\theta} = \mu$$

Compound Poisson Distribution

- $Y = C_1 + C_2 + . . . + C_N$
- N is Poisson random variable
- C_i are i.i.d. with Gamma distribution
- This is an example of a Tweedie distribution
- Y is member of Exponential Family

Members of the Exponential Family

- Normal
- Poisson
- Binomial
- Gamma
- Inverse Gaussian
- Compound Poisson (Tweedie)

Variance Structure

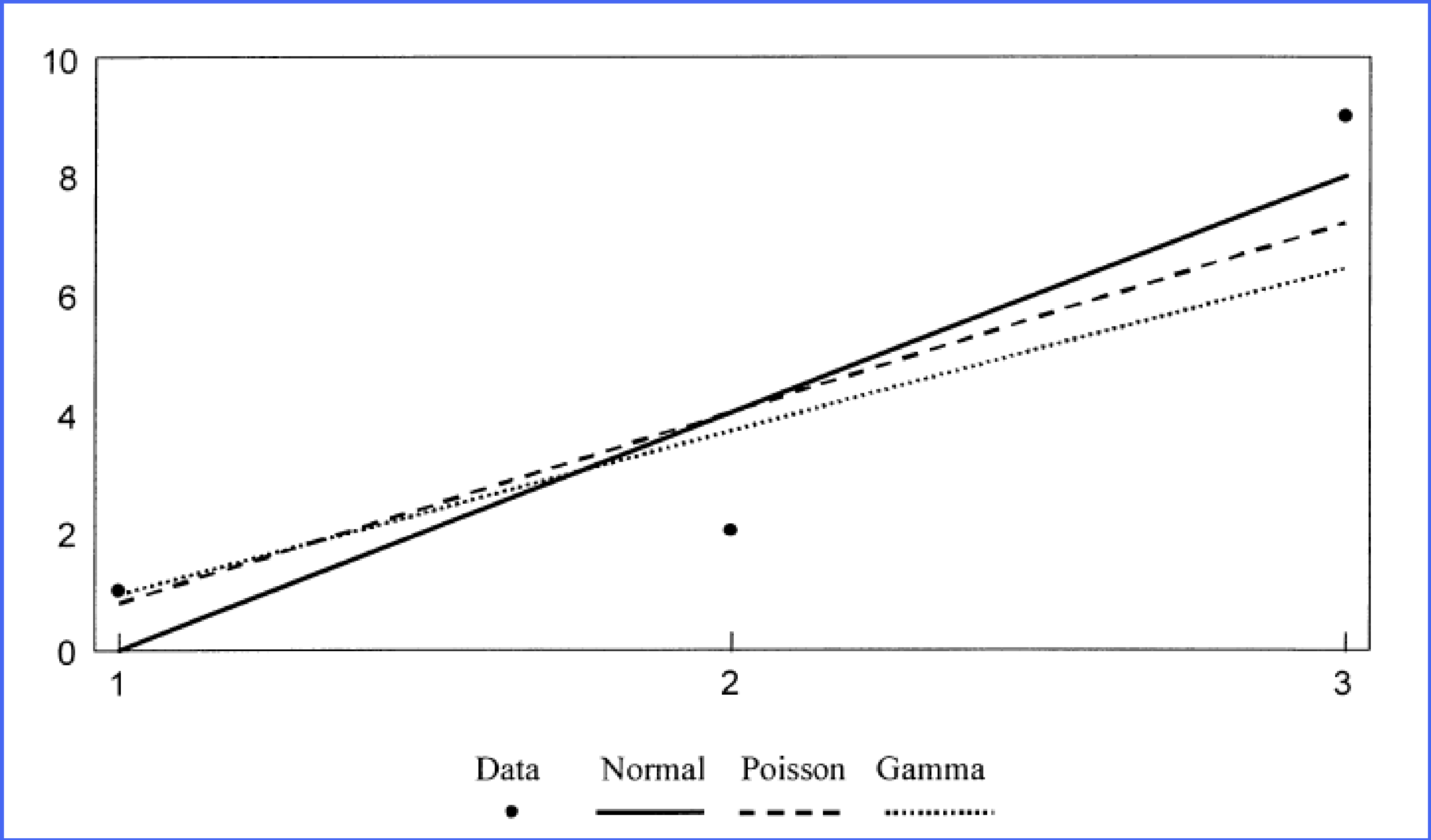
- $E[Y_i] = \mu_i = b'(\theta_i) \rightarrow \theta_i = b'^{(-1)}(\mu_i)$
- $Var[Y_i] = a(\Phi_i)\, b''(\theta_i) = a(\Phi_i)\, V(\mu_i)$
- Common form: $Var[Y_i] = \Phi\, V(\mu_i)/w_i$
  - ✓ Φ is constant across data but weights applied to data points

Variance Functions $V(\mu)$

	$V(\mu)$
❖ **Normal**	μ^0
❖ **Poisson**	μ
❖ **Binomial**	$\mu(1-\mu)$
❖ **Tweedie**	μ^p, $1<p<2$
❖ **Gamma**	μ^2
❖ **Inverse Gaussian**	μ^3

- **Recall:** $\mathrm{Var}[Y_i] = \Phi\, V(\mu_i)/w_i$

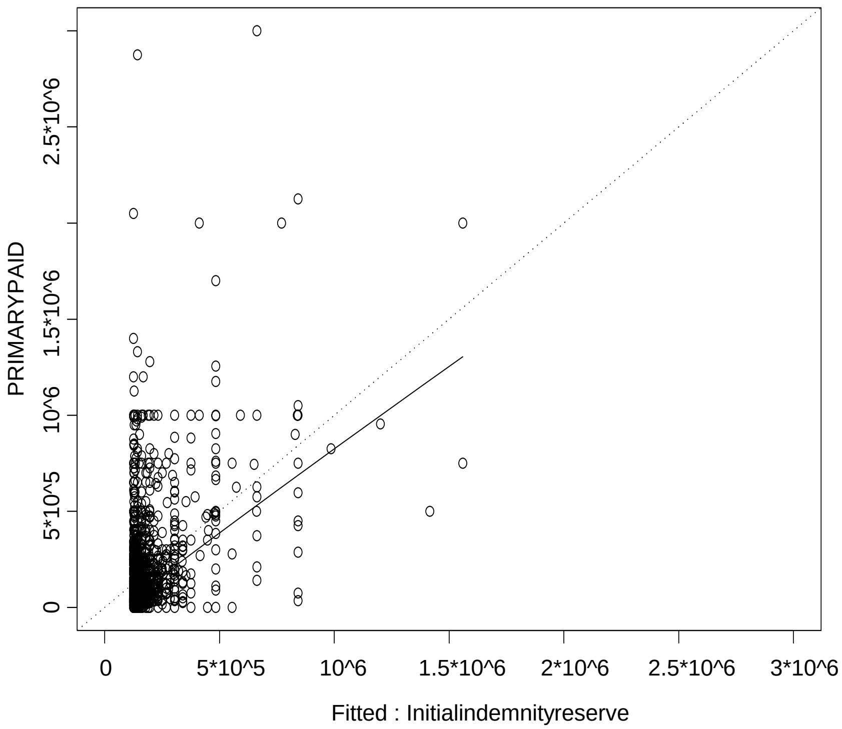
Variance at Point and Fit

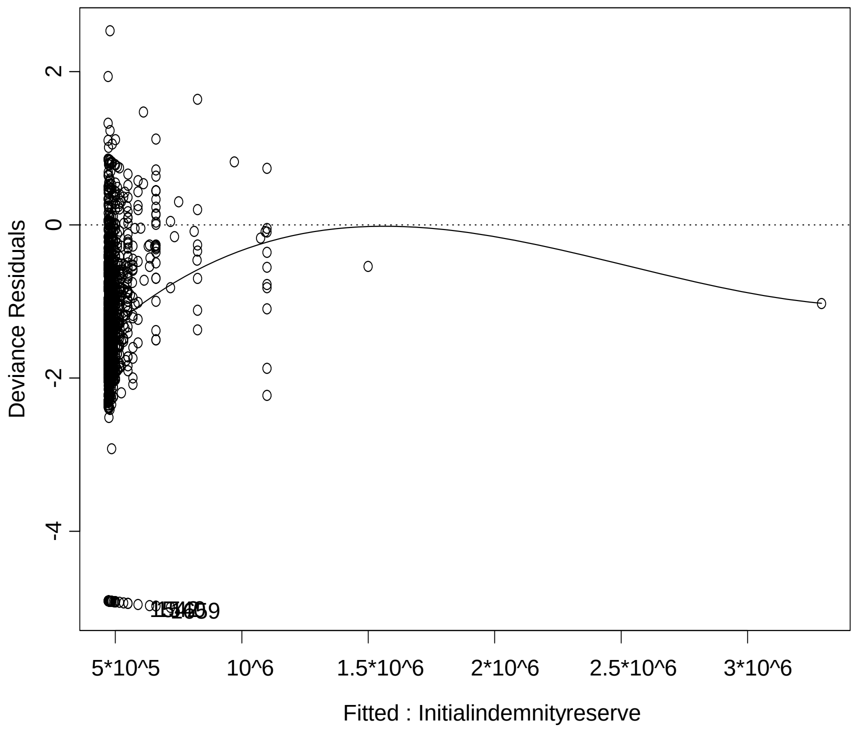


A Practioner's Guide to Generalized Linear Models: A CAS Study Note

Normal
vs Gamma (Inverse Link)

Primary Paid = f(Initial ndemnity Reserve)





Variance of Y_i and Fit at Data Point *i*

- $\text{Var}(Y_i)$ is big $\rightarrow$ looser fit at data point *i*
- $\text{Var}(Y_i)$ is small $\rightarrow$ tighter fit at data point *i*

$$\text{Tightness of fit} \propto \frac{1}{\text{Var}(Y_i)}$$

Why Exponential Family?

- **Distributions in Exponential Family can model a variety of problems**

- **Standard algorithm for finding coefficients $a_0, a_1, \ldots, a_m$**

Modeling Number of Claims

Policy	Sex	Territory	Number of Claims in 5 Years
1	M	02	0
2	F	01	0
3	F	01	0
4	F	02	1
5	F	01	0
6	F	02	1
7	M	02	2
8	M	02	2
9	M	02	1
10	F	01	1
:	:	:	:

Assume a Multiplicative Model

- μ_i = expected number of claims in five years
- $\mu_i = B_{F,01} \times C_{Sex(i)} \times C_{Terr(i)}$
- If *i* is Female and Terr 01

  $\rightarrow \mu_i = B_{F,01} \times 1.00 \times 1.00$

Multiplicative Model

- $\mu_i = \exp(a_0 + a_S X_{S(i)} + a_T X_{T(i)})$

- $\mu_i = \exp(a_0) \text{ x } \exp(a_S X_{S(i)}) \text{ x } \exp(a_T X_{T(i)})$

- i is Female → $X_{S(i)} = 0$; Male → $X_{S(i)} = 1$
- i is Terr 01 → $X_{T(i)} = 0$; Terr 02 → $X_{T(i)} = 1$

Values of Predictor Variables

Policy	Sex	$X_{S(i)}$	Territory	$X_{T(i)}$
1	M	**1**	02	**1**
2	F	**0**	01	**0**
3	F	**0**	01	**0**
4	F	**0**	02	**1**
5	F	**0**	01	**0**
6	F	**0**	02	**1**
7	M	**1**	02	**1**
8	M	**1**	02	**1**
9	M	**1**	02	**1**
10	F	**0**	01	**0**

Natural Log Link Function

- $\ln(\mu_i) = a_0 + a_S X_{S(i)} + a_T X_{T(i)}$
- μ_i is in $(0, \infty)$
- $\ln(\mu_i)$ is in $(-\infty, \infty)$

Poisson Distribution in Exponential Family

$$\Pr[Y = y] = \exp\left\{\frac{\ln\mu \cdot y - \mu}{1} - \ln(y!)\right\}$$

$$\theta = \ln\mu$$

$$b(\theta) = e^{\theta}$$

Natural Log is Canonical Link for Poisson

- $\theta_i = \ln(\mu_i)$

- $\theta_i = a_0 + a_S X_{S(i)} + a_T X_{T(i)}$

Estimating Coefficients a_1, a_2, .., a_m

- Classical linear regression uses least squares
- GLMs use Maximum Likelihood Method
- Solution will exist for distributions in exponential family

Likelihood and Log Likelihood

$$L(y_1,..;\theta_1,..) = \prod_{i=1}^{n} f(y_i;\theta_i)$$

$$\lambda(y_1,..;\theta_1,..) = \ln[L(y_1,..;\theta_1,..)]$$

$$\lambda(y_1,..;\theta_1,..) = \sum_{i=1}^{n} \ln f(y_i;\theta_i)$$

Find a_0, a_S, and a_T for Poisson

Maximize:

$$\lambda(y_1,..;\theta_1,...) = \sum_{i=1}^{n} \theta_i y_i - e^{\theta_i} - \ln y_i!$$

with $\theta_i = a_0 + a_S x_{S(i)} + a_T x_{T(i)}$

Iterative Numerical Procedure to Find a_i's

- Use statistical package or actuarial software
- Specify link function and distribution type
- "Iterative weighted least squares" is the numerical method used

Solution to Our Example

- $a_0 = -.288 \rightarrow \exp(-.288) = .75$
- $a_S = .262 \rightarrow \exp(.262) = 1.3$
- $a_T = .095 \rightarrow \exp(.095) = 1.1$

- $\mu_i = \exp(a_0) \text{ x } \exp(a_S X_{S(i)}) \text{ x } \exp(a_T X_{T(i)})$

- $\mu_i = .75 \text{ x } 1.3^{X_{S(i)}} \text{ x } 1.1^{X_{T(i)}}$

- *i* is Male,Terr 01 $\rightarrow \mu_i = .75 \text{ x } 1.3^1 \text{ x } 1.1^0$

Testing New Drug Treatment

X_1	X_2		Y
Dosage	**Age**	**Cure**	**Value**
1.0	30	Yes	1
1.0	43	No	0
1.0	82	No	0
1.5	45	No	0
1.5	67	No	0
1.5	26	Yes	1
2.0	33	Yes	1
2.0	50	Yes	1
2.0	72	No	0
2.5	31	Yes	1
2.5	45	Yes	1
2.5	75	Yes	1

Multiple Linear Regression

	Dependent Variable: Y	
Cure	**Actual**	**Predicted Probability**
Yes	1	0.5179
No	0	0.3298
No	0	-0.2345
No	0	0.5366
No	0	0.2183
Yes	1	0.8115
Yes	1	0.9460
Yes	1	0.7000
No	0	0.3817
Yes	1	1.2107
Yes	1	1.0081
Yes	1	0.5740

Logistic Regression Model

- p = probability of cure, p in [0,1]
- odds ratio: p/(1-p) in [0, + ∞]
- ln[p/(1-p)] in [- ∞ , + ∞]
- $\ln[p/(1-p)] = a + b_1X_1 + b_2X_2$

Link function

Logistic Regression Model

X_1	X_2		Dependent Variable Y	
Dosage	Age	Cure	Value	Predicted Probability
1.0	30	Yes	1	0.568
1.0	43	No	0	0.000
1.0	82	No	0	0.000
1.5	45	No	0	0.648
1.5	67	No	0	0.000
1.5	26	Yes	1	1.000
2.0	33	Yes	1	1.000
2.0	50	Yes	1	1.000
2.0	72	No	0	0.000
2.5	31	Yes	1	1.000
2.5	45	Yes	1	1.000
2.5	75	Yes	1	0.784

Which Exponential Family Distribution?

- **Frequency:** Poisson, {Negative Binomial}
- **Severity:** Gamma, sometimes Inverse Gaussian
  - Real data is frequently heavier tailed that any of these
- **Loss ratio:** Compound Poisson
- **Pure Premium:** Compound Poisson
- **How many policies will renew:** Binomial

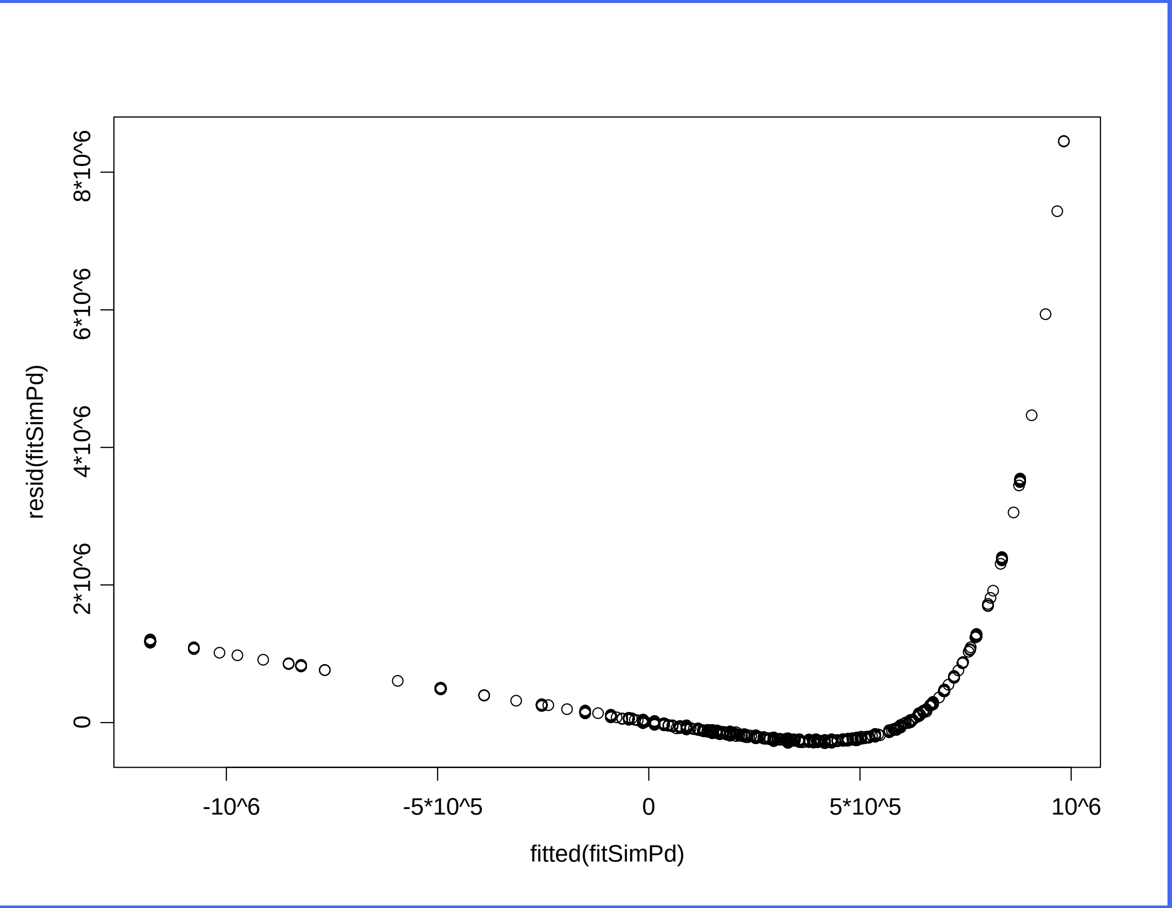
What link function?

- Additive model: identity
- Multiplicative model: natural log
- Modeling probability of event: logistic
- Form of nonlinear relationship (i.e., inverse or other)

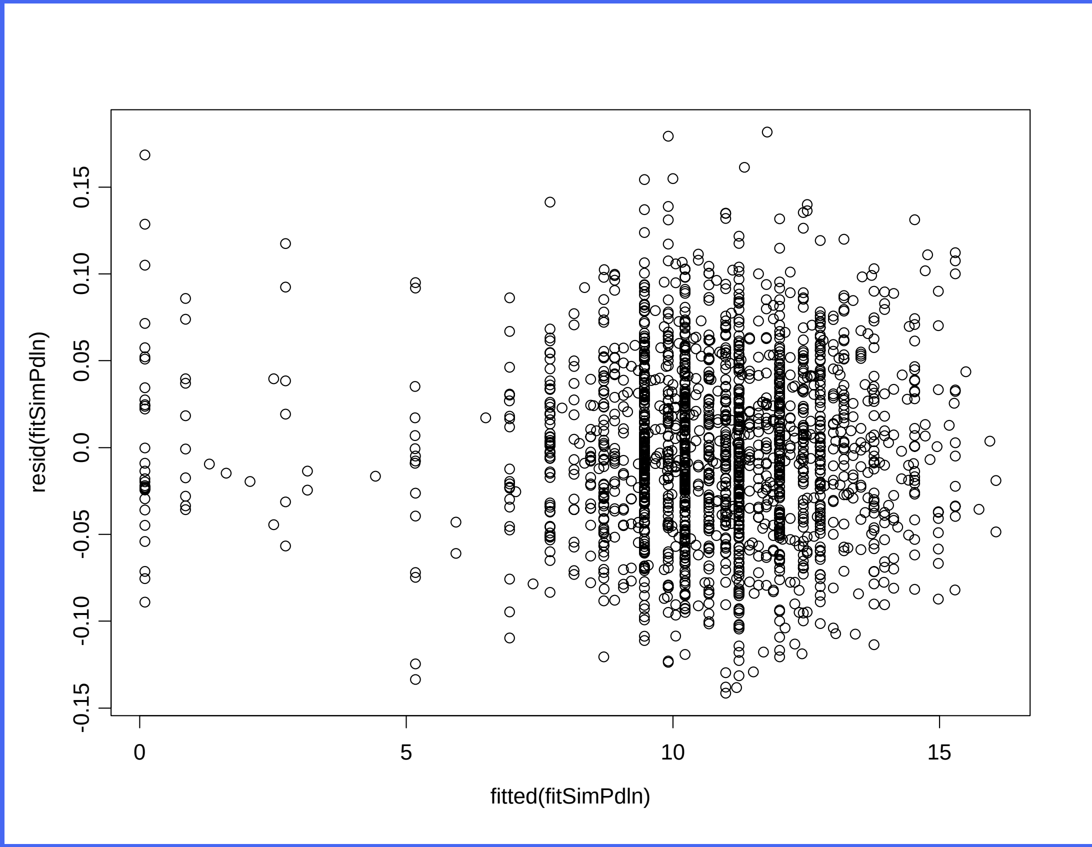
Pearson Residual

- Residual= (Actual-Fitted)/Var(Expected)
- Variance of expected depends on distribution family

Use Plot of Residual vs Fitted to Identify NonLinearity



Residuals After Log Transform



Output with No Transformation

```
all: glm(formula = SimPaid.ln ~ LogInitRes, family
 = gaussian, link = log, na.omit.p = T)

Deviance Residuals:
       Min       1Q    Median        3Q      Max
 -301858.6 -242113 -161786.5 -42375.19 8449785

Coefficients:
                 Value Std. Error   t value
(Intercept) -1180218.1  61191.817 -19.28719
 LogInitRes   149096.6   6220.791  23.96746


    Null Deviance: 8.99586e+014 on 1930 degrees
of freedom

Residual Deviance: 6.93167e+014 on 1929 degrees
of freedom
```

Output After Log Transform

```
all: glm(formula = logPaid ~ LogInitRes, family
gaussian, na.omit.p = T)

Deviance Residuals:
      Min        1Q     Median       3Q
 -4.272846 -0.4268939 -0.1831501 0.2630763

     Max
 3.723691

Coefficients:
               Value       Std. Error       t value
(Intercept) 7.160387  0.07455613  96.04023
 LogInitRes 0.453788  0.0075794  59.87116

(Dispersion Parameter for Gaussian family taken
to be 0.5334401 )

    Null Deviance: 2941.152 on 1930 degrees of f
reedom

Residual Deviance: 1029.006 on 1929 degrees of f
reedom
```

Real Example of Transformation

- Previous example used simulated data
- When using real data need right transforms for both dependent and independent variables
- For heavy tailed data, log transform for dependent is common
- For volatile predictor variables: often bin the data