MODELS OF INSURANCE CLAIM COUNTS WITH TIME DEPENDENCE BASED ON GENERALISATION OF POISSON AND NEGATIVE BINOMIAL DISTRIBUTIONS

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Introduction

General Topics

- Count Data;
- Risk Classification;
- Panel Data (Longitudinal Data).

Insurance

- *A priori* Ratemaking;
- *A posteriori* Ratemaking;
- Number of Claims.

Résumé

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Plan

Overview

- Old Ratemaking Technique;
- Introduced by Bailey and Simon(1960) and Bailey(1963);
- Has been shown the similarities with some statistical distributions. See:
  - Brown(1988);
  - Mildenhall(1999);
  - Holler and Sommer(1999) from the 9^{th} exam of the Casualty Actuarial Society, that expose the link between the GLM and the minimum bias technique.

Poisson Distribution

Probability Distribution

$$\Pr[N_{i,t} = n_{i,t}] = \frac{e^{-\lambda_{i,t}} \lambda_{i,t}^{n_{i,t}}}{n_{i,t}!}, \lambda_{i,t} = \exp(x'_{i,t}\beta)$$

Overview

- Regressors are introduced by a score function;
- *Law of small numbers*;
- Exponential Family of Distributions : direct application of the GLM theory;
- Equidispersion property;

Negative Binomial Distributions

Probability Distributions - NB2 and NB1

$$NB2 \quad : \quad \Pr[N_{i,t} = n_{i,t}] = \frac{\Gamma(n_{i,t} + \alpha^{-1})}{\Gamma(n_{i,t}+1)\Gamma(\alpha^{-1})} \left(\frac{\lambda_{i,t}}{\alpha^{-1} + \lambda_{i,t}} \right)^{n_{i,t}} \left(\frac{\alpha^{-1}}{\alpha^{-1} + \lambda_{i,t}} \right)^{\alpha^{-1}}$$

$$NB1 \quad : \quad \Pr[N_{i,t} = n_{i,t}] = \frac{\Gamma(n_{i,t} + \alpha^{-1}\lambda_{i,t})}{\Gamma(n_{i,t}+1)\Gamma(\alpha^{-1}\lambda_{i,t})} (1+\alpha)^{-\lambda_{i,t}/\alpha} (1+\alpha^{-1})^{-n_{i,t}}.$$

Overview

- Obtained by adding an heterogeneity term θ to the mean parameter of the Poisson distribution, when θ follows a gamma distribution;
- NB2 : $Var[N_{i,t}] = \lambda_{i,t} + \alpha\lambda_{i,t}^2 > E[N_{i,t}]$;
- NB1 : $Var[N_{i,t}] = \lambda_{i,t} + \alpha\lambda_{i,t} > E[N_{i,t}]$;
- Poisson distribution is the limiting case of NB distributions when $\alpha \to 0$.

Plan

Time Dependence

Ideas

- Classic Poisson and Negative Binomial distributions suppose independence between all the contracts of the same insured;
- There are advantages of using the information on each policyholder along time for modeling the number of claims;
- Allowing for time dependence between observations are closer to the data generating process that one can find in practice;
- Future premiums given the past observations can be calculated (credibility theory).

Random Effects

Construction

- Missing of some important classification variables (swiftness of reflexes, aggressiveness behind the wheel, consumption of drugs, etc.) in the classification;
- Hidden features captured by an individual random heterogeneity term θ_i;
- Given θ_i, the annual claim numbers $N_{i,1}, N_{i,2}, \ldots, N_{i,T}$ are independent.
- The joint probability function of $N_{i,1}, ..., N_{i,T}$ is given by

$$\begin{aligned}\Pr[N_{i,1} = n_{i,1}, ..., N_{i,T} = n_{i,T}] &= \int_0^\infty \Pr[N_{i,1} = n_{i,1}, ..., N_{i,T} = n_{i,T}|\theta_i]g(\theta_i)d\theta_i \\ &= \int_0^\infty \left(\prod_{t=1}^{T} \Pr[N_{i,t} = n_{i,t}|\theta_i]\right) g(\theta_i)d\theta_i.\end{aligned}$$

- Models depend on the choices of the conditional distribution of the $N_{i,t}$ and the distribution of θ_i.

Multivariate Negative Binomial Distribution (MVNB)

When $N_{i,t}$ is conditionaly distributed as a Poisson distribution with random effects following a gamma distribution:

$$\Pr[N_{i,1} = n_{i,1}, ..., N_{i,T} = n_{i,T}] =$$

$$\left[\prod_{t=1}^{T} \frac{(\lambda_{i,t})^{n_{i,t}}}{n_{i,t}!}\right] \frac{\Gamma(\sum_{t=1}^{T} n_{i,t} + 1/\alpha)}{\Gamma(1/\alpha)} \left(\frac{1/\alpha}{\sum_{t=1}^{T} \lambda_{i,t} + 1/\alpha}\right)^{1/\alpha} (\sum_{t=1}^{T} \lambda_{i,t} + 1/\alpha)^{-\sum_{t=1}^{T} n_{i,t}}.$$

Negative Binomial-Beta distribution (NB-Beta)

When $N_{i,t}$ is conditionaly distributed as a NB1 distribution with random effects following a beta distribution:

$$\Pr[N_{i,1} = n_{i,1}, ..., N_{i,T} = n_{i,T}] =$$

$$\frac{\Gamma(a+b)\Gamma(a+\sum_t \lambda_{i,t})\Gamma(b+\sum_t n_{i,t})}{\Gamma(a)\Gamma(b)\Gamma(a+b+\sum_t \lambda_{i,t}+\sum_t n_{i,t})} \prod_{t}^{T} \frac{\Gamma(\lambda_{i,t}+n_{i,t})}{\Gamma(\lambda_{i,t})\Gamma(n_{i,t}+1)}.$$

Predictive Distribution and Predictive Premiums

- Predictive distributions of panel data with random effects involve Bayesian analysis;
- At each insured period, the random effects can be updated for past claim experience, revealing some insured-specific informations.

$$\Pr[N_{i,T+1} = n_{i,T+1} | n_{i,1}, ..., n_{i,T}] = \int \Pr(n_{i,T+1}|\theta_i) g(\theta_i | n_{i,1}, ..., n_{i,T}) d\theta_i,$$

where $g(\theta_i | n_{i,1}, ..., n_{i,T})$ is the *a posteriori* distribution of the random effects θ_i.

$$(MVNB) : E[N_{i,T+1} | N_{i,1}, ..., N_{i,T}] = \lambda_{i,T+1} \frac{\sum_t^T n_{i,t} + 1/\alpha}{\sum_t^T \lambda_{i,t} + 1/\alpha}.$$

$$(NB - Beta) : E[N_{i,T+1} | N_{i,1}, ..., N_{i,T}] = \lambda_{i,T+1} \frac{\sum_t^T n_{i,t} + b}{\sum_t^T \lambda_{i,t} + a - 1},$$

Past Experience

Intuitive Approach

- An intuitive approach can be the introduction of the past experience of the insured as covariate as done by Gerber and Jones(1975) or Sundt(1988);
- Stationarity properties of this model cannot be established and premiums for new insureds cannot be computed.

Conditional Distributions with Artificial Marginal Distribution

- N_1 is distributed with parameter $\lambda_1 = \exp(x_1\beta)$, while $N_t|n_{t-1}$, $t = 2, ..., T$ are distributed with mean $\lambda_t = \exp(x_t'\beta + \phi n_{t-1})$;
- The joint distribution is expressed as $\prod_{t=1}^{T} \frac{\lambda_t^{n_t} e^{-\lambda_t}}{n_t!}$;
- Can be done with Poisson and NB1 distributions;
- The model collapses to the product of independent distributions when the parameter ϕ is set to 0;
- Models have a Markovian property and only depend on the number of reported claims of the last insured period;
- The predictive distributions of this kind of model, conditionaly on a past insured period, are standard Poisson and Negative Binomial distributions.

Integer Valued Autoregression Models (INAR)

Model

- The Integer Valued Autoregression Models of order 1 is defined by the recursive equation :

$$n_{i,t} = \rho \circ n_{i,t-1} + I_{i,t},$$

  where $\circ$ is defined as $\rho \circ n = \sum_{i=1}^{n} Y_i = B(\rho, n)$.
- $\rho \circ n_{i,t-1}$: the number of claims due to the impact of the insured's past claims on his driving behavior;
- $I_{i,t}$: the number of claims caused by random variation.
- The INAR(1) model has Markovian property which facilitates the expression of the joint density :

$$\Pr[n_{i,1}, ..., n_{i,T}] = \Pr[n_{i,T}|n_{i,T-1}] ... \Pr[n_{i,2}|n_{i,1}] \Pr[n_{i,1}];$$

- Can be done with Poisson and NB1 distributions;
- The INAR models are directly constructed to be used as predictive distributions.

Common Shock Models

Model

- Based on a convolution structure;
- The dependence between contracts of the same insured comes from a common individual random variable that is added to each time period;
- $T+1$ random variables $M_1, M_2, ..., M_T$ and U with respective parameters $\lambda_{i,1}, \lambda_{i,2}, ..., \lambda_{i,T}$ and μ_i. The common schock model is the T-joint distribution of $M_{i,1} + U_i, M_{i,2} + U_i, ..., M_{i,T} + U_i$;
- Can be done with Poisson and NB1 distributions;
- The common shock in models cannot exceed any of the $n_{i,t}$;
- Not useful for modeling the number of reported claims in insurance.

Copula Approach

Model

- A copula is defined as a multivariate distribution whose one-dimensional margins are uniform on [0,1];
- Many copulas can be used. For example (Archimedean copulas):

$$
\begin{aligned}
\textit{Frank}: C_\theta(u_1,...,u_T) &= -\frac{1}{\theta}\log\left[1+\frac{\prod_{i=1}^{T}(e^{-\theta u_i}-1)}{(e^{-\theta}-1)^{T-1}}\right] \\
\textit{Clayton}: C_\theta(u_1,...,u_T) &= \left[u_1^{-\theta}+...+u_T^{-\theta}-T+1\right]^{-1/\theta} \\
\textit{Joe}: C_\theta(u_1,...,u_T) &= 1-\left[1-\prod_{i=1}^{T}\left(1-(1-u_i)^{\theta}\right)\right]^{1/\theta}.
\end{aligned}
$$

Exchangeable Dependence

- Joint distribution:

$$\Pr[n_{i,1}, ..., n_{i,T}] = \sum_{l_1=n_{i,1}-1}^{n_{i,1}} ... \sum_{l_T=n_{i,T}-1}^{n_{i,T}} (-1)^{\sum_j (n_{i,j}-l_j)} C(F_{i,1}(l_1), ..., F_{i,T}(l_T)).$$

- The use of this kind of model does not allow predictions because we cannot suppose the kind of dependence implied for periods $T+j$, $j=1,2,....$

Autoregressive Dependence

- The probability needed for the joint distribution can be expressed as a function of a bivariate copula:

$$\Pr(N_{i,t} \leq n_{i,t} | n_{i,t-1}) = \frac{C[F_{t-1}(n_{i,t-1}), F_t(n_{i,t})] - C[F_{t-1}(n_{i,t-1}-1), F_t(n_{i,t})]}{\Pr_{t-1}(N_{i,t-1} = n_{i,t-1})},$$

- It defines the predictive distribution directly and only depends on the number of reported claims of the last insured period.

Plan

A priori Premiums

Models		Low Risk Mean	Low Risk Variance	Medium Risk Mean	Medium Risk Variance	High Risk Mean	High Risk Variance
Time	Poisson	0.0564	0.0564	0.0666	0.0666	0.1100	0.1100
Ind.	NB2	0.0563	0.0613	0.0666	0.0736	0.1102	0.1293
	NB1	0.0558	0.0619	0.0670	0.0744	0.1122	0.1245
Random	MVNB	0.0565	0.0593	0.0661	0.0699	0.1077	0.1179
Effects	NB−Beta	0.0560	0.0630	0.0664	0.0753	0.1100	0.1283
AM	Poisson	0.0531	0.0531	0.0626	0.0626	0.1041	0.1041
	NB1	0.0525	0.0581	0.0629	0.0696	0.1061	0.1173
INAR	Poisson	0.0565	0.0565	0.0663	0.0663	0.1088	0.1088
	NB1	0.0559	0.0623	0.0665	0.0741	0.1108	0.1235
Copula	Poisson	0.0540	0.0540	0.0669	0.0669	0.1119	0.1119
Frank	NB1	0.0530	0.0586	0.0669	0.0743	0.1114	0.1237
Copula	Poisson	0.0569	0.0569	0.0669	0.0669	0.1118	0.1118
Clayton	NB1	0.0559	0.0620	0.0669	0.0743	0.1113	0.1236
Copula	Poisson	0.0562	0.0562	0.0663	0.0663	0.1108	0.1108
Joe	NB1	0.0562	0.0627	0.0672	0.0751	0.1110	0.1239

A posteriori Premiums

Models		Last Year's Number of Reported Claims 0 Mean	0 Variance	1 Mean	1 Variance	2 Mean	2 Variance
Random	MVNB	0.0624	0.0655	0.1174	0.1377	0.1723	0.2366
Effects	NB−Beta	0.0606	0.0713	0.1054	0.1267	0.1501	0.1843
AM	Poisson	0.0626	0.0626	0.1311	0.1311	0.2746	0.2746
	NB1	0.0629	0.0696	0.1330	0.1472	0.2814	0.3114
INAR	Poisson	0.0614	0.0614	0.1354	0.1299	0.2094	0.1984
	NB1	0.0618	0.0689	0.1321	0.1342	0.2023	0.1995
Copula	Poisson	0.0612	0.0615	0.1489	0.1373	0.1586	0.1448
Frank	NB1	0.0614	0.0684	0.1505	0.1559	0.1599	0.1644
Copula	Poisson	0.0612	0.0615	0.1493	0.1374	0.1557	0.1423
Clayton	NB1	0.0614	0.0685	0.1506	0.1555	0.1568	0.1611
Copula	Poisson	0.0625	0.0622	0.1148	0.1135	0.2905	0.3370
Joe	NB1	0.0618	0.0678	0.1319	0.1479	0.3117	0.4113

Models		Last Year's Number of Reported Claims 3 Mean	3 Variance	4 Mean	4 Variance
Random	MVNB	0.2272	0.3748	0.2821	0.5647
Effects	NB−Beta	0.1949	0.2442	0.2397	0.3064
AM	Poisson	0.5751	0.5751	1.2042	1.2042
	NB1	0.5954	0.6587	1.2594	1.3933
INAR	Poisson	0.2833	0.2669	0.3573	0.3354
	NB1	0.2726	0.2648	0.3428	0.3302
Copula	Poisson	0.1590	0.1451	0.1590	0.1451
Frank	NB1	0.1607	0.1651	0.1608	0.1652
Copula	Poisson	0.1559	0.1425	0.1559	0.1425
Clayton	NB1	0.1573	0.1616	0.1574	0.1616
Copula	Poisson	0.6136	0.8830	1.0737	1.7997
Joe	NB1	0.5983	0.9606	0.9788	1.8389

Model	A priori	Number of Reported Claims over the past 10 years: 0	1	2	3	5	10
MVNB	0.0661	0.0418	0.0785	0.1153	0.1520	0.2256	0.4093
NB−Beta	0.0664	0.0432	0.0751	0.1070	0.1389	0.2028	0.3623

Plan

Nested Models

Specification Tests

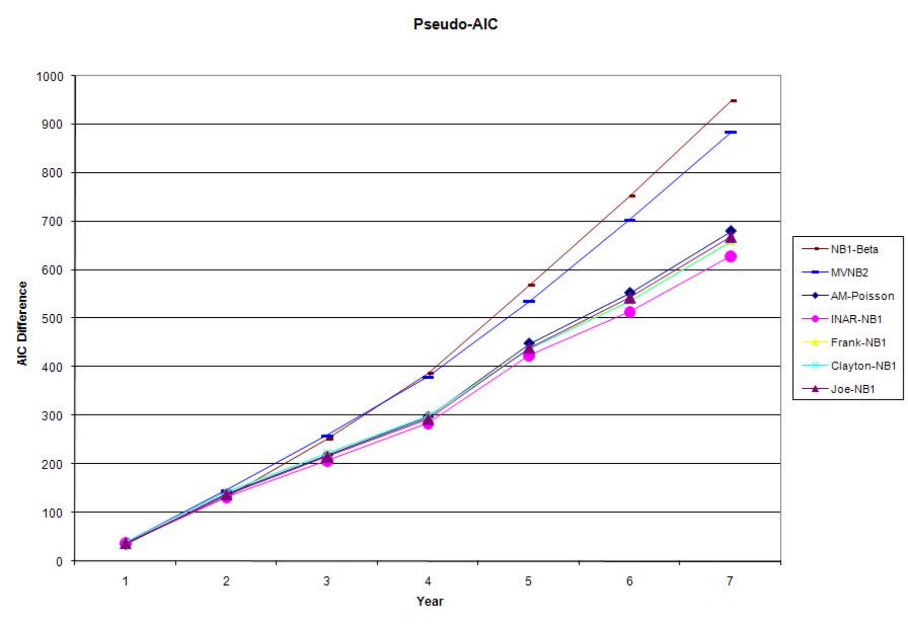
Three kinds of links can be tested by the tests of nested models :

- Overdispersion test as Poisson marginal against Negative Binomial marginal;
- Poisson against their generalized models allowing for time dependence;
- Negative Binomial against their generalized models allowing for time dependence.

Non-Nested Models

Information Criteria

Models	LogLikelihood	AIC	BIC
MVNB	−26,690.9	53,395.78	53,462.79
Pseudo MVNB1	−26,689.8	53,393.57	53,460.58
NB1−Beta	−26,657.4	53,330.73	53,407.32
MNB1AM	−26,790.7	53,599.32	53,685.48
INAR−NB1	−26,816.0	53,649.93	53,736.09
Frank−NB1	−26,800.2	53,618.48	53,704.64
Clayton−NB1	−26,801.2	53,620.31	53,706.48
Joe−NB1	−26,796.3	53,610.61	53,696.77


Pseudo-AIC
AIC Difference
Year
1000
900
800
700
600
500
400
300
200
100
0
1
2
3
4
5
6
7
NB1-Beta
MVNB2
AM-Poisson
INAR-NB1
Frank-NB1
Clayton-NB1
Joe-NB1

Plan

Conclusion

Summary

- A wide selection of models can be used to model the dependence between contracts of the same insured;
- Each model can be interpreted differently;
- The choice of a model has a great impact on the predictive premiums.

Best Models

- Random effects models are the better ones to fit the data;
- They are more flexible to compute the next year premium, that depends not only on the past insured period but on the sum of all reported claims.

Publications

J.-P. Boucher and M. Denuit (2006).
Fixed versus Random Effects in Poisson Regression Models for Claim Counts: Case Study with Motor Insurance.
ASTIN Bulletin, 36, 285-301

J.-P. Boucher and M. Denuit and M. Guillén (2007).
Risk Classification for Claim Counts: Mixed Poisson, Zero-Inflated Mixed Poisson and Hurdle Models.
North American Actuarial Journal, 11-4, 110-131

J.-P. Boucher and M. Denuit and M. Guillén (2008).
Models of Insurance Claim Counts with Time Dependence Based on Generalisation of Poisson and Negative Binomial Distributions.
Will be published in Variance