On the Subadditivity of Tail Value at Risk

An Investigation with Copulas

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Outline

- Introduction
- Residual risk of conglomerates and stand-alones
- Copulas
- Measures of dependence
- Examples
- Analysis of residual risk when using TVaR
- Conclusion

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Introduction

- Assume the loss incurred by an insurer is denoted by a random variable X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- To protect the insured, the regulators demand that the insurer holds “enough” money to be able to pay the policyholders with a “high” probability

Risk measures

- Value-at-Risk (Quantile):

$$\text{VaR}_p[X] = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, 0 < p < 1,$$

  where $F_X(x) = \mathbb{P}[X \leq x]$ is the cumulative density function of X.
  - Most widely used risk measure, very popular in banking
  - There is only a chance of $1 - p$ to have larger losses
- Risk Measures: $\rho : \Gamma \rightarrow \mathbb{R} \cup \{\infty\}$. where Γ is a non-empty set of $\mathcal{F}$-measurable random variables

Properties of risk measures

- Translation Invariance: $\forall X \in \Gamma, \forall b \in \mathbb{R} : \rho[X + b] = \rho[X] + b$
- Homogeneity: $\forall X \in \Gamma, \forall a \in \mathbb{R}_0^+ : \rho[aX] = a\rho[X]$
- Monotonicity: $\forall X_1, X_2 \in \Gamma$ with $\mathbb{P}[X_1 \leq X_2] = 1 : \rho[X_1] \leq \rho[X_2]$
- Sub-additivity: $\forall X_1, X_2 \in \Gamma : \rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$
- A risk measure which satisfies each of these four properties is called coherent in the sense of Artzner et al. (1999)
- It is well-known that the VaR is not sub-additive

Some popular risk measures

$$\text{TVaR}_p[X] = \frac{1}{1-p}\int_p^1 \text{VaR}_q[X]dq, 0 < p < 1$$
$$\text{CTE}_p[X] = \mathbb{E}[X|X > \text{VaR}_p[X]], 0 < p < 1$$

- TVaR at level p = average of all quantiles above p
- TVaR is the most popular coherent risk measure in practice
- CTE is not a coherent risk measure
- For continuous random variables, $\text{TVaR}_p[X] = \text{CTE}_p[X]$ for all $p \in]0, 1[$

Residual risk

- The regulator wants to minimize the residual risk: $RR_X = \max(0, X - \rho[X]) = (X - \rho[X])_+$
- For a merger, the following inequality holds with probability one:

$$(X_1 + X_2 - \rho[X_1] - \rho[X_2])_+ \leq (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ \qquad (1)$$

$\Rightarrow$ To avoid shortfall: aggregation of risk is to be preferred

- However, investors will be attracted by a stand-alone situation because the following inequality holds with probability one:

$$(\rho[X_1] + \rho[X_2] - X_1 - X_2)_+ \leq (\rho[X_1] - X_1)_+ + (\rho[X_2] - X_2)_+ \qquad (2)$$

due to fire-walls between risks X_1 and X_2.

Towards a compromise between regulators and shareholders

- Investors may have incentives to invest in a merger once

$$\rho[X_1 + X_2] \leq \rho[X_1] + \rho[X_2]$$

- However, for such a risk measure, we do not necessarily have:

$$(X_1 + X_2 - \rho[X_1 + X_2])_+ \quad \leq \quad (X_1 - \rho[X_1])_+ + (X_2 - \rho[X_2])_+ \qquad (3)$$

  for all outcomes of X_1 and X_2

- Condition (3) limits the range of risk measures considerably: If for (X_1, X_2), we have that $\mathbb{P}[X_1 > \rho[X_1], X_2 > \rho[X_2]] > 0$ and that equation (3) is satisfied for all outcomes of X_1 and X_2, then we need to have that $\rho[X_1 + X_2] \geq \rho[X_1] + \rho[X_2]$

Towards a compromise between regulators and shareholders

- Dhaene et al. (2006) analyzed the possibility of weakening condition (3) to:

$$\mathbb{E}(X_1 + X_2 - \rho[X_1 + X_2])_+ \quad \leq \quad \mathbb{E}(X_1 - \rho[X_1])_+ + \mathbb{E}(X_2 - \rho[X_2])_+ \qquad (4)$$

- They showed that:
  - All translation invariant and positively homogeneous risk measures satisfy condition (4) for every bivariate elliptical distribution
  - Condition (4) does not always hold in general for the TVaR

Purpose

- Useful measures to analyze the residual risk
- Characterize different aspects of diversification benefit
- Show that the TVaR can provide a framework for compromise between the expectations of the investors and the regulator under a wide range of dependence structures and margins
- Analyze diversification benefit for different copulas

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Risk measures of residual risk

- Let $X = \sum_{i=1}^{K} X_i$ denote a merger of K subsidiaries
- Let $X_{1;K}$ denote a set of K stand-alones
- We compare several risk measures ψ of the residual risk
  - Merger: $\psi[RR_X] = \psi[(X - \rho[X])_+]$
  - Set of stand-alones: $\psi[RR_{X_{1;K}}] = \psi[\sum_{i=1}^{K}(X_i - \rho[X_i])_+]$
- Possible risk measures for ψ
  - Moments (mean, variance, skewness, kurtosis)
  - Probability that the residual risk is larger than zero

2 exponential risks (i.i.d.)

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- Let $X_i \overset{\text{i.i.d.}}{\sim} Expo(\lambda)$, where $\lambda = 1/50$ for $i \in \{1,2\}$.
  $\Rightarrow$ $\text{TVaR}_{0.95}[X_i] = 200$ and $\text{TVaR}_{0.95}[X] = 296$
  $\Rightarrow$ $\text{TVaR}_{0.99}[X_i] = 280$ and $\text{TVaR}_{0.99}[X] = 388$
- Then we have:

Risk Measure	$TVaR_{0.95}$		$TVaR_{0.99}$	
	$X_{1;2}$	$X_1 + X_2$	$X_{1;2}$	$X_1 + X_2$
$\mathbb{E}[RR]$	1.839	1.065	0.368	0.206
$\sigma[RR]$	13.450	10.902	6.060	4.765
$\gamma[RR]$	11.011	15.156	24.708	34.335
$\kappa[RR]$	260.252	306.018	815.487	1563.420
$\mathbb{P}[RR > 0]$	3.6%	1.9%	0.7%	0.4%

5 or 10 exponential risks (i.i.d.)

- TVaR$_{0.99}[X_i] = 280$
- TVaR$_{0.99}[\sum_{i=1}^{5} X_i] = 650$
- TVaR$_{0.99}[\sum_{i=1}^{10} X_i] = 1024$

Risk Measure	$TVaR_{0.99}$		$TVaR_{0.99}$	
	$X_{1;5}$	$X = \sum_{i=1}^{5} X_i$	$X_{1;10}$	$X = \sum_{i=1}^{10} X_i$
$\mathbb{E}[RR]$	0.920	0.252	1.839	0.305
$\sigma[RR]$	9.581	5.758	13.550	6.913
$\gamma[RR]$	15.627	33.550	11.050	33.013
$\kappa[RR]$	326.195	1478.030	163.097	1420.910
$\mathbb{P}[RR > 0]$	1.8%	0.4%	3.6%	0.4%

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Copulas

- A d-dimensional copula $C(u_1, \ldots, u_d)$ is a joint distribution function of a random vector on the unit cube $[0, 1]^d$
- **Theorem 1 (Sklar's Theorem in d-dimensions)** *Let F be a d-dimensional distribution function with marginal distribution functions F_1,...,F_d. Then there is a d-dimensional copula C such that for all $x \in \mathbb{R}^d$:*

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)). \tag{5}$$

If F_1,...,F_d are all continuous, then C is unique. Conversely, if C is a d-dimensional copula, and F_1,...,F_d are distribution functions, then F defined by (5) is a d-dimensional distribution with margins F_1,...,F_d.

Examples

- Every d-dimensional copula C satisfies for all $(u_1, \ldots, u_d) \in [0,1]^d$:

$$\max\left\{0, \sum_{i=1}^{d} u_i - (n-1)\right\} \leq C(u_1, \ldots, u_d) \leq \min\{u_1, \ldots, u_d\}, . \quad (6)$$

- The right-hand side of (6) is called the **comonotonic copula** C_U
- For $d \geq 3$, the left-hand side of (6) is not a copula.
  For $d = 2$, this is called the **countermonotonic copula** C_L.
- **Independence copula**: $C_I(u_1, \ldots, u_d) = \Pi_{i=1}^{d} u_i, (u_1, \ldots, u_d) \in [0,1]^d$

Copulas of multivariate distributions

- **Normal Copula:**
  The d-dimensional normal copula with correlation matrix Σ is defined as:

$$C_\Sigma(u_1, \ldots, u_d) = \nu_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)), \text{ for all } (u_1, \ldots, u_d) \in [0,1]^d$$

- **Student Copula:**
  The d-dimensional Student copula with correlation matrix Σ and m degrees of freedom ($m > 0$) is defined as:

$$C_{m,\Sigma}(u_1, \ldots, u_d) = t_{m,\Sigma}(t_m^{-1}(u_1), \ldots, t_m^{-1}(u_d)), \text{ for all } (u_1, \ldots, u_d) \in [0,1]^d.$$

Archimedean copulas

- **Clayton's copula** ($\alpha > 0$):

$$C_{C,\alpha}(u_1, \ldots, u_d) = (u_1^{-\alpha} + \ldots + u_d^{-\alpha} - d + 1)^{-1/\alpha}$$

- **Frank copula** ($\alpha > 0$ if $d > 2$ and $\alpha \in \mathbb{R}_0$ if $d = 2$):

$$C_{F,\alpha}(u_1, \ldots, u_d) = -\frac{1}{\alpha} \ln \left(1 + \frac{\Pi_{i=1}^{d}(\exp(-\alpha u_i) - 1)}{\exp(-\alpha) - 1} \right)$$

- **Gumbel-Hougaard copula** ($\alpha > 1$):

$$C_{G,\alpha}(u_1, \ldots, u_d) = \exp(-((-\ln(u_1)^{\alpha} + \ldots + (-\ln(u_d)^{\alpha}))^{1/\alpha})$$

Survival copulas

- **Survival copula** of a copula C:
  - Define
    $C_S(u_1, \ldots, u_d) = \mathbb{P}[U_1 > u_1, \ldots, U_d > u_d], (u_1, \ldots, u_d) \in [0,1]^d$
  - Then the survival copula is defined and denoted as
    $\overline{C}(u_1, \ldots, u_d) = C_S(1 - u_1, \ldots, 1 - u_d), (u_1, \ldots, u_d) \in [0,1]^d$

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Measures of dependence

- Pearson's correlation:

$$\rho_P(X_1, X_2) = \frac{\mathbb{C}ov[X_1, X_2]}{\sqrt{\mathbb{V}ar[X_1]\mathbb{V}ar[X_2]}}$$

- Kendall's tau:

$$\begin{aligned}\rho_\tau(X_1, X_2) &= \mathbb{E}[\text{sign}[(X_1 - Y_1)(X_2 - Y_2)]] \\ &= \mathbb{P}[(X_1 - Y_1)(X_2 - Y_2) > 0] - \mathbb{P}[(X_1 - Y_1)(X_2 - Y_2) < 0]\end{aligned}$$

where (Y_1, Y_2) and (X_1, X_2) are i.i.d.

- Tail dependence:

$$\begin{aligned}\lambda_U &= \lim_{v \to 0} \mathbb{P}[X_1 > \overline{F}_1^{-1}(v) | X_2 > \overline{F}_2^{-1}(v)] \\ \lambda_L &= \lim_{v \to 0} \mathbb{P}[X_1 \leq F_1^{-1}(v) | X_2 \leq F_2^{-1}(v)]\end{aligned}$$

Measures of dependence

Copula	ρ_τ	λ_L	λ_U
C_I	0	0	0
C_U	1	1	1
C_L	-1	0	0
C_α	$2\arcsin(\alpha)/\pi$	0 if $\alpha < 1$ and 1 if $\alpha = 1$	
$C_{m,\alpha}$	$2\arcsin(\alpha)/\pi$	$2t_{m+1}\left(-\sqrt{m+1}\sqrt{\frac{1-\alpha}{1+\alpha}}\right)$	
$C_{C,\alpha}$	$\frac{\alpha}{\alpha+2}$	$2^{-1/\alpha}$	0
$C_{F,\alpha}$	$1-\frac{4}{\alpha}+\frac{4}{\alpha^2}\int_0^\alpha \frac{t}{e^t-1}dt$	0	0
$C_{G,\alpha}$	$1-\frac{1}{\alpha}$	0	$2-2^{1/\alpha}$

Tail dependence (Kendall tau of 0.5)

Copula	ρ_τ	α	λ_L	λ_U
C_α	0.5	0.707	0	0
$C_{4,\alpha}$	0.5	0.707	0.397	0.397
$C_{C,\alpha}$	0.5	2	0.707	0
$C_{F,\alpha}$	0.5	5.736	0	0
$C_{G,\alpha}$	0.5	2	0	0.586

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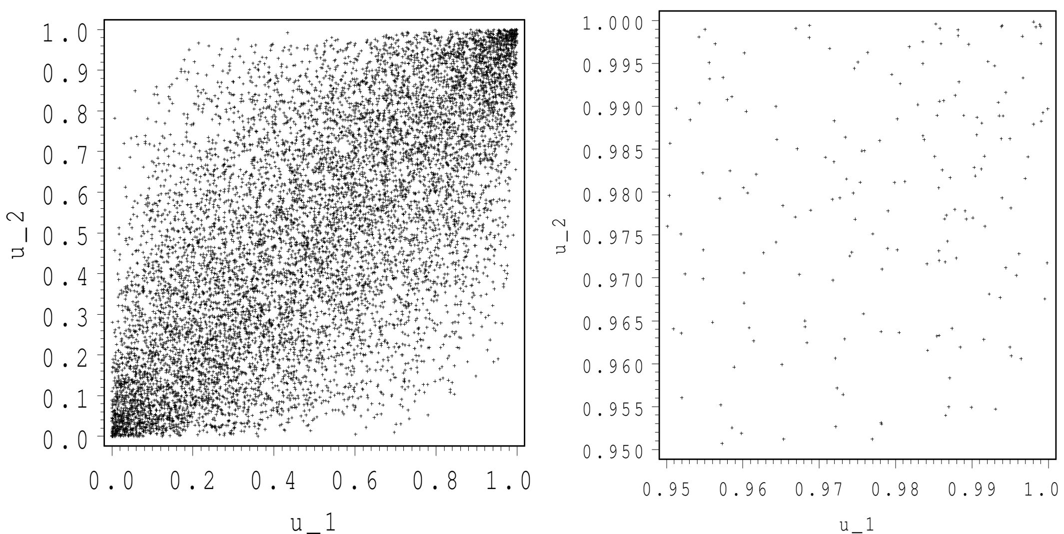
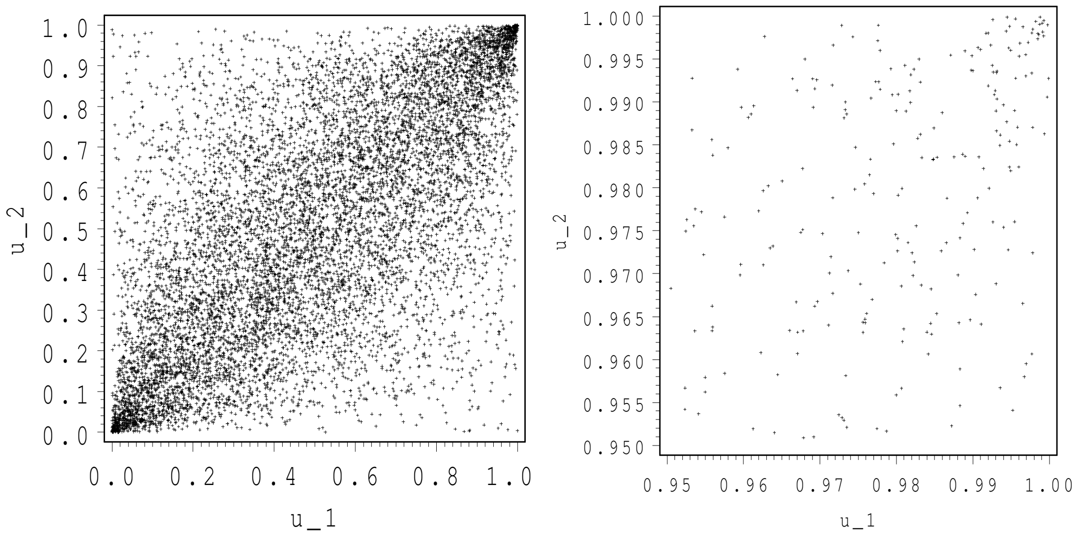
Normal copula

- Copula: $C_{0.707}$
- $\rho_\tau = 0.5$ and $\lambda_L = \lambda_U = 0$



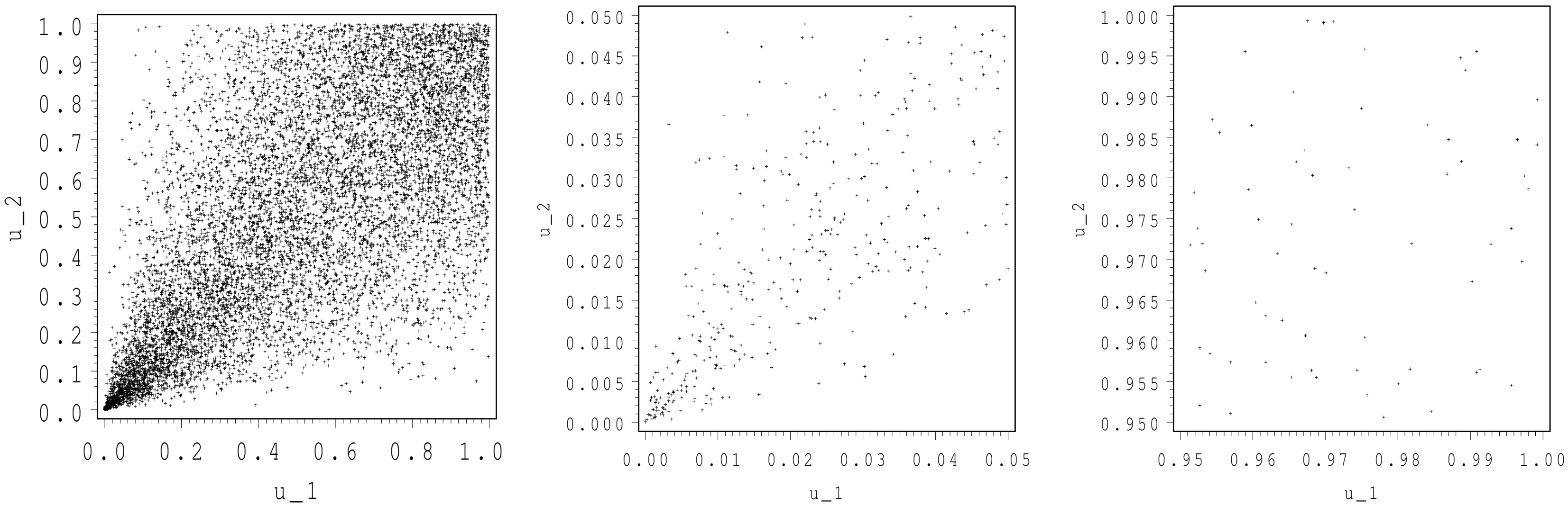
Student copula

- Copula: $C_{4,\ 0.707}$
- $\rho_\tau = 0.5 \Rightarrow \lambda_L = \lambda_U = 0.397$



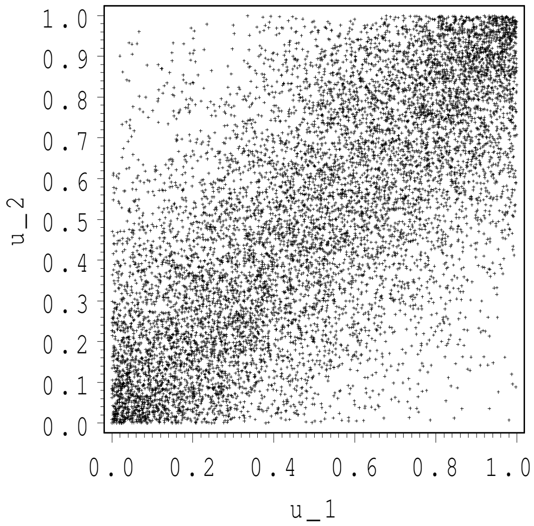
Clayton copula

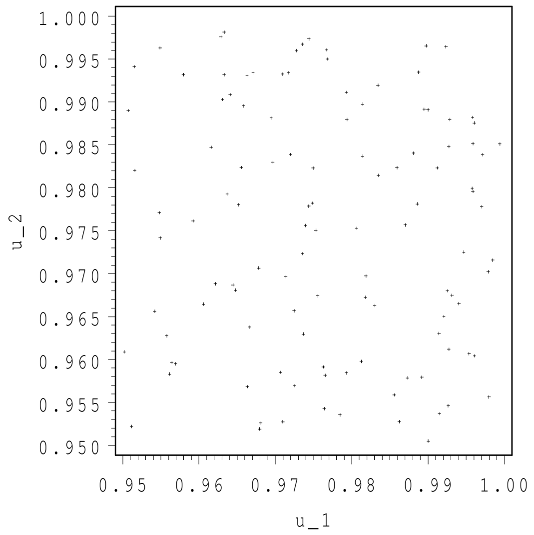
- Copula: $C_{C,\,2}$
- $\rho_\tau = 0.5 \Rightarrow \lambda_L = 0.707$ and $\lambda_U = 0$



Frank copula

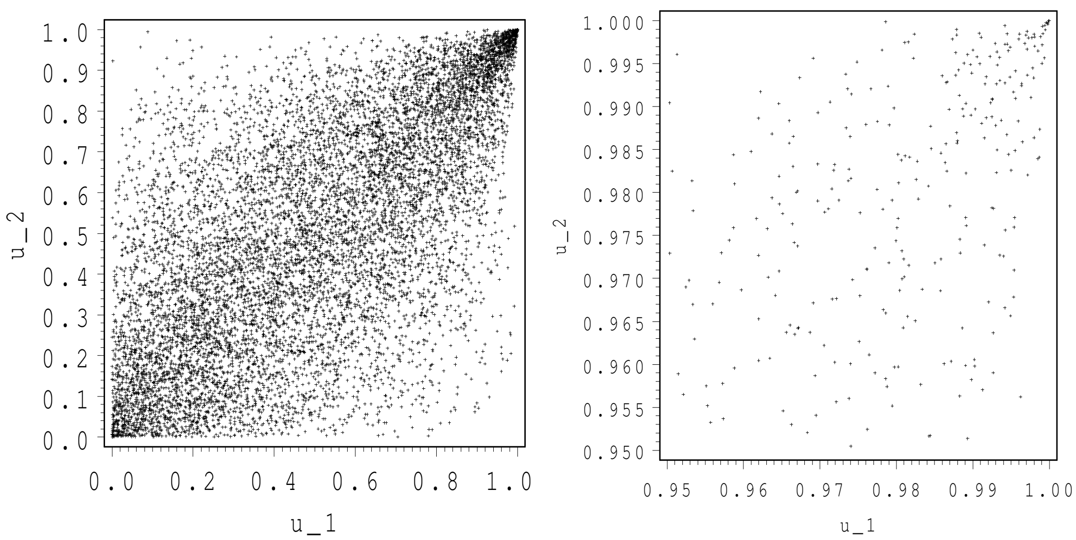
- Copula: $C_{F,\ 5.736}$
- $\rho_\tau = 0.5$ and $\lambda_L = \lambda_U = 0$





Gumbel-Hougaard copula

- Copula: $C_{G,\ 2}$
- $\rho_\tau = 0.5 \Rightarrow \lambda_L = 0$ and $\lambda_U = 0.586$



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Analysis of residual risk when using TVaR

- Marginal distributions (identical to have symmetry)
  - Exponential (mean = standard deviation = 50)
  - Lognormal:
    - ◇ Mean = standard deviation = 50
    - ◇ Mean = 50, coefficient of variation of 0.25
- Copulas:
  - Kendall's tau of 0.5
  - Kendall's tau of 0.25
- Dimensions: 2D or 5D
- Risk measure: TVaR at level 0.95 or 0.99

Analysis of residual risk when using TVaR

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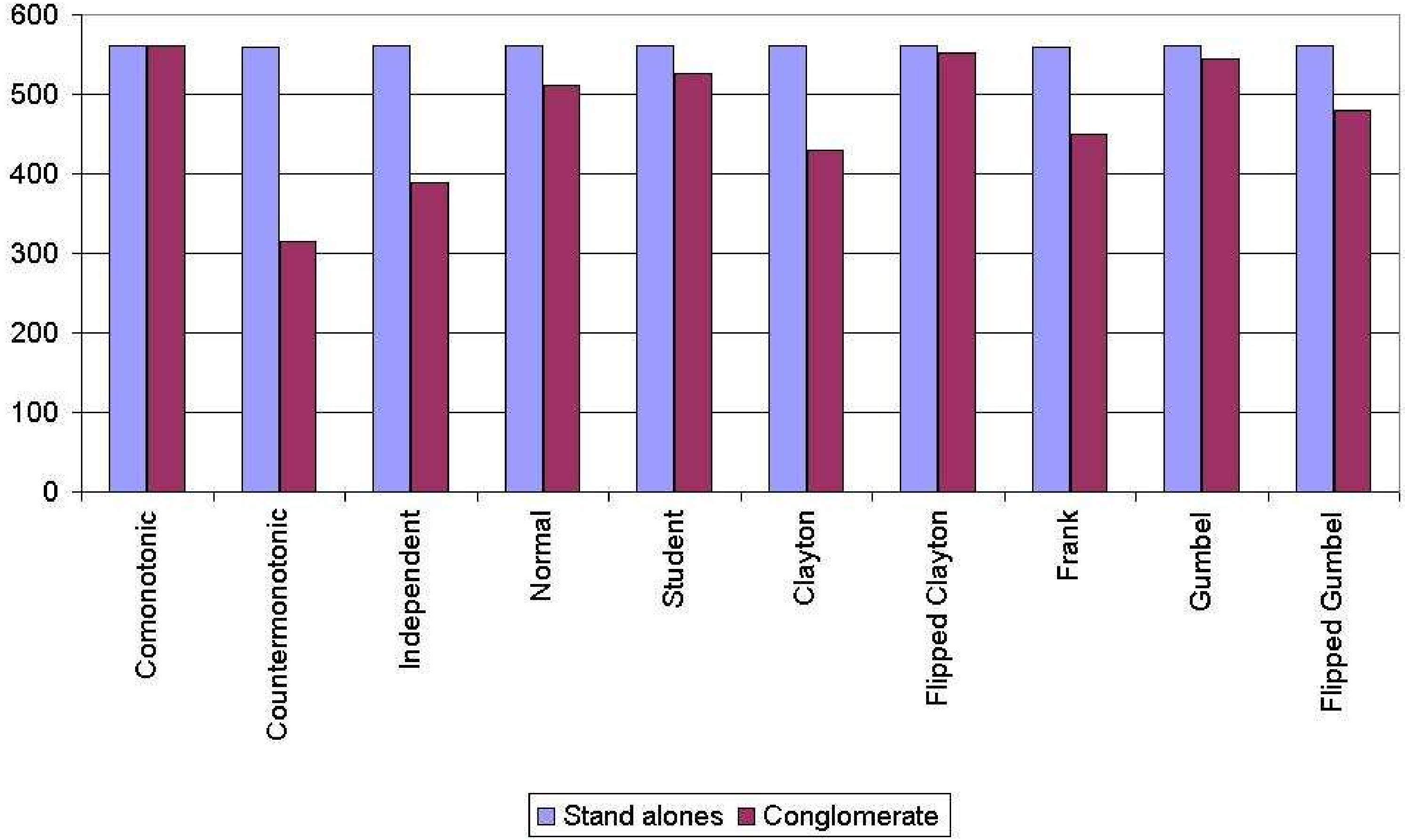
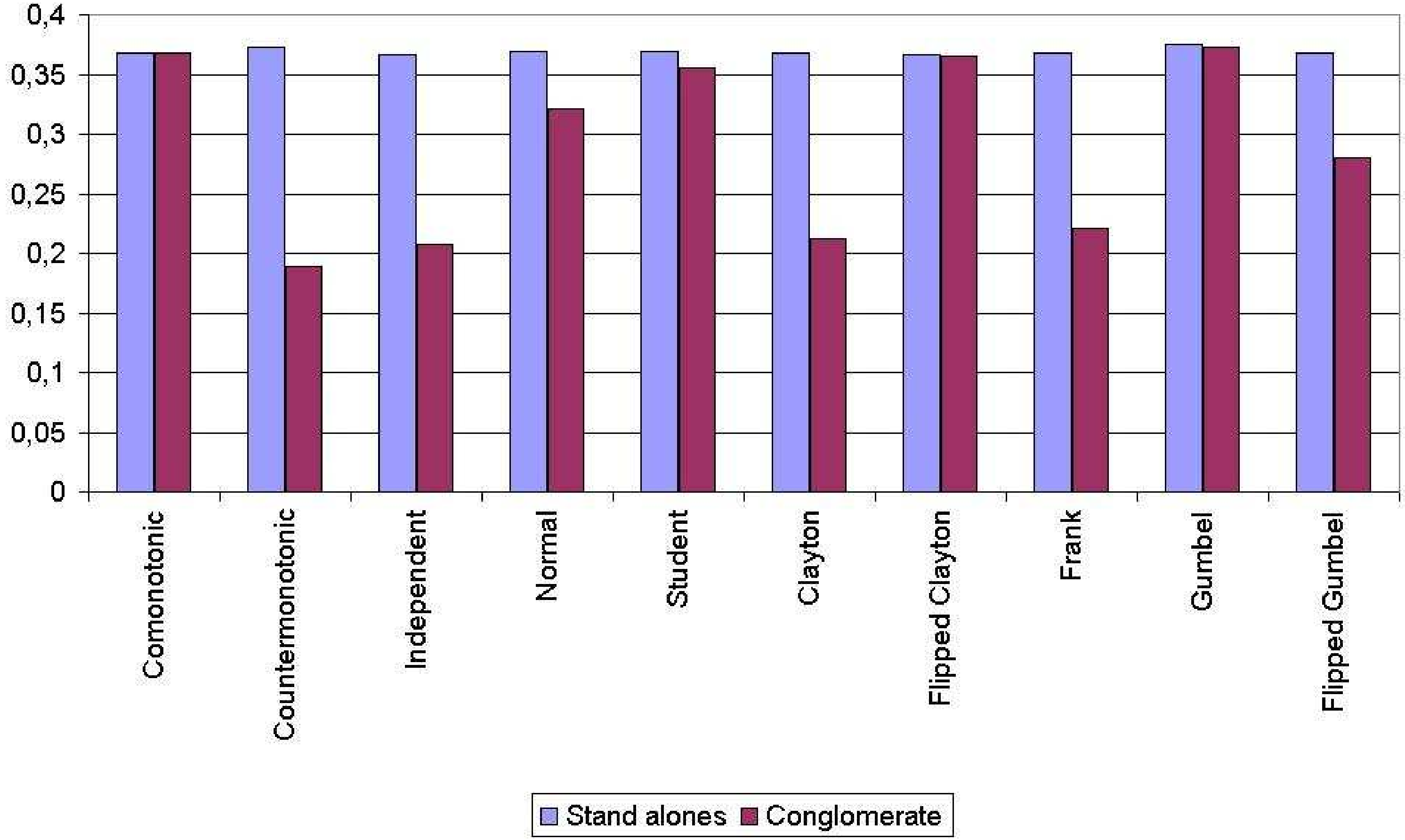
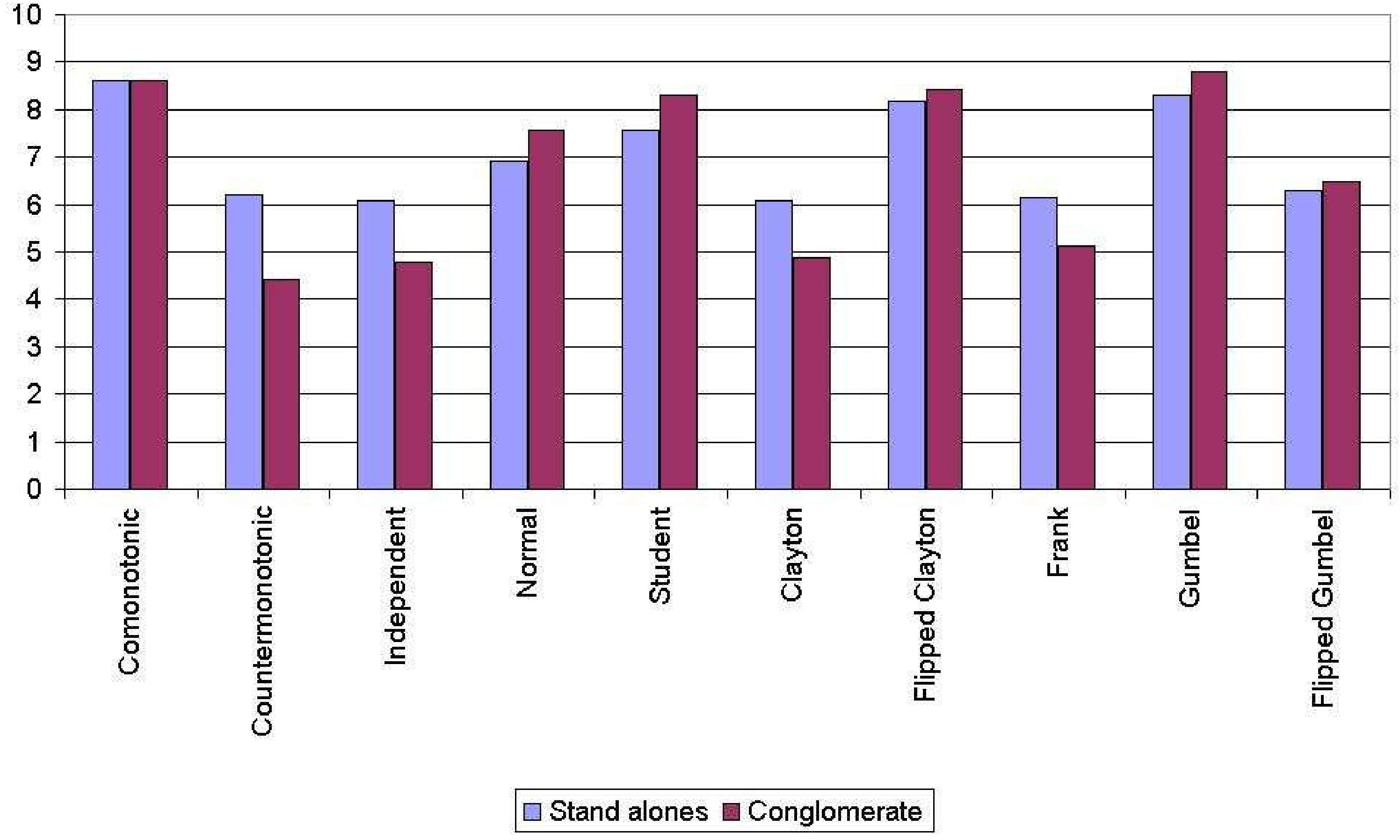
TVaR at level 0.99



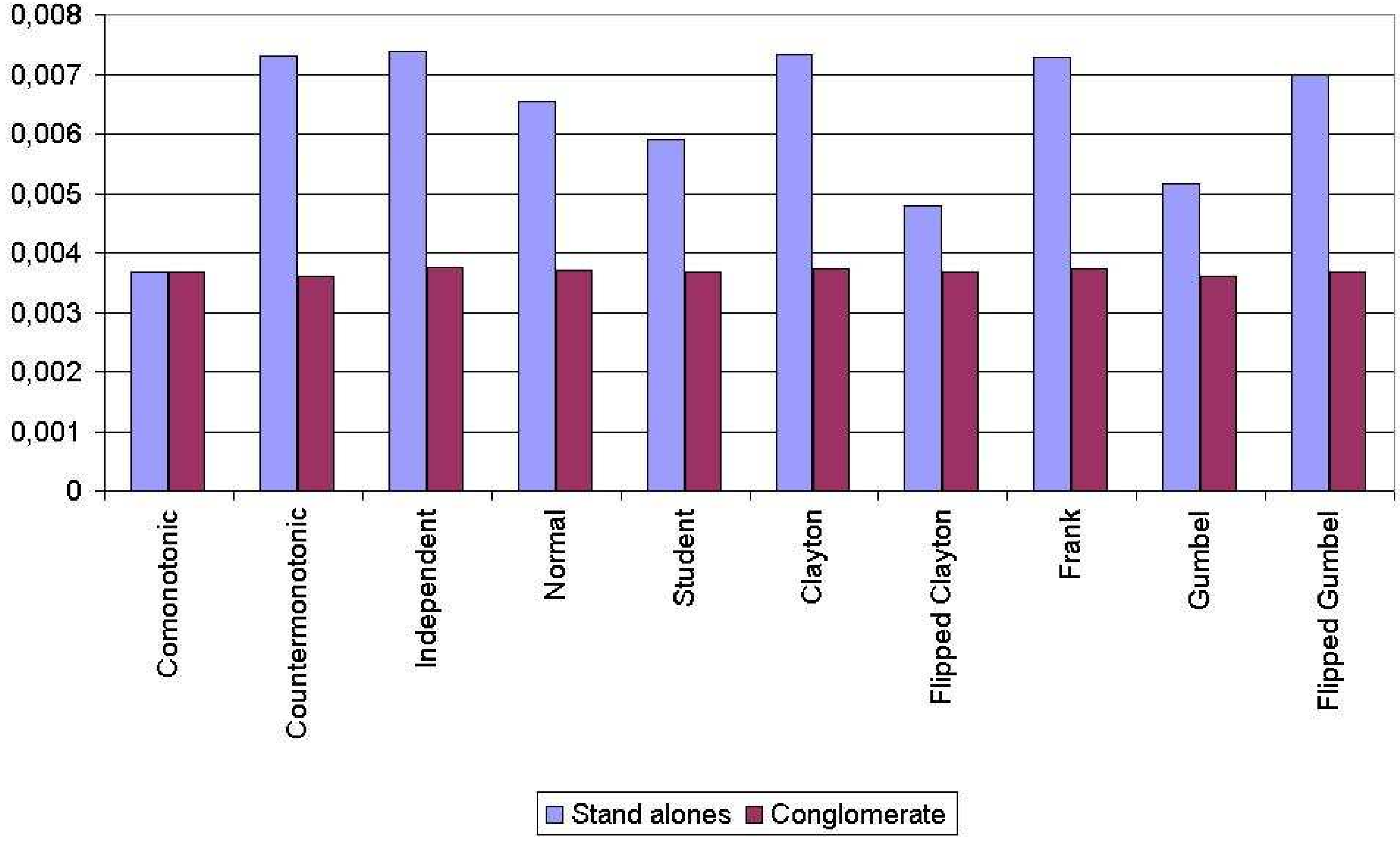
Mean residual risk



Standard deviation residual risk



Default probability



Main observations

- Merging risks (and using TVaR for solvency buffer) allows for important diversification benefit on:
  - TVaR
  - Mean residual risk
  - Default probability
- Tail dependence has important impact on diversification possibilities
- If average residual risk or default probability are used as a benchmark, TVaR is not too subadditive under a wide range of dependence structures

Analysis of residual risk when using TVaR

- Marginal distributions (identical to have symmetry)
  - Exponential (mean = standard deviation = 50)
  - **Lognormal:**
    - ◇ **Mean = standard deviation = 50**
    - ◇ Mean = 50, coefficient of variation of 0.25
- Copulas:
  - Kendall's tau of 0.5
  - **Kendall's tau of 0.25**
- Dimensions: **2D** or 5D
- Risk measure: TVaR at level 0.95 or 0.99

2 Lognormal risks
Kendall tau of 0.25

- Lognormal distribution has larger tails than exponential:
  $\Rightarrow$ TVaR increases
  $\Rightarrow$ Default probabilities are slightly lower
- Minimum DB for survival Clayton copula:
  +/- 9% on TVaR and average residual risk
- Student copula: DB now increases with probability level
- Survival Clayton, Student and Gumbel copula:
  DB on TVaR and residual risk nearly constant with probability level

5-Dimensional results

- **5 Exponential risks with Kendall tau of 0.5:**
  - DB on the TVaR always larger than in 2D
  - Default probability for the stand-alones is substantially larger than in 2D (mainly in cases with weak (tail)-dependence)
  - Larger DB for average residual risk
  - Comparable conclusions as in 2D
- **5 Lognormal risks with Kendall tau of 0.25:**
  Comparable conclusions as for the comparison of the 2D and 5D situation for exponential risks

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Conclusion

- When merging risks, there can be a diversification benefit on:
  - Required capital
  - Residual risk
  - Default probability
- TVaR is a basis for compromise between the interests of the regulator and the investors
- When using the average residual risk or the default probability, the TVaR is not too subadditive under a wide range of dependence structures

Conclusion

- The diversification benefit for different copulas with the same Kendall tau can be very different
  - $\Rightarrow$ Tails in general (and tail dependence in particular) are important when looking at capital requirements
  - $\Rightarrow$ If tail dependence is being ignored by using a simplified dependence assumption, the DB may be substantially over-estimated
- Positive upper tail dependence and high Kendall's tau (0.5):
  $\Rightarrow$ DB decreases with increasing solvency level
- Kendall's tau of 0.25 and lower upper tail dependence:
  $\Rightarrow$ this is not necessarily true

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