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Insurance Companies

Mixture Distribution and Its Applications on P&C Insurance Data

Luyang Fu, Ph.D., FCAS, MAAA
Doug Pirtle, FCAS

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Agenda

- Introduction
- Mixture Distribution
- Finite Mixture Model
- Case Study
- Conclusions
- Q&A

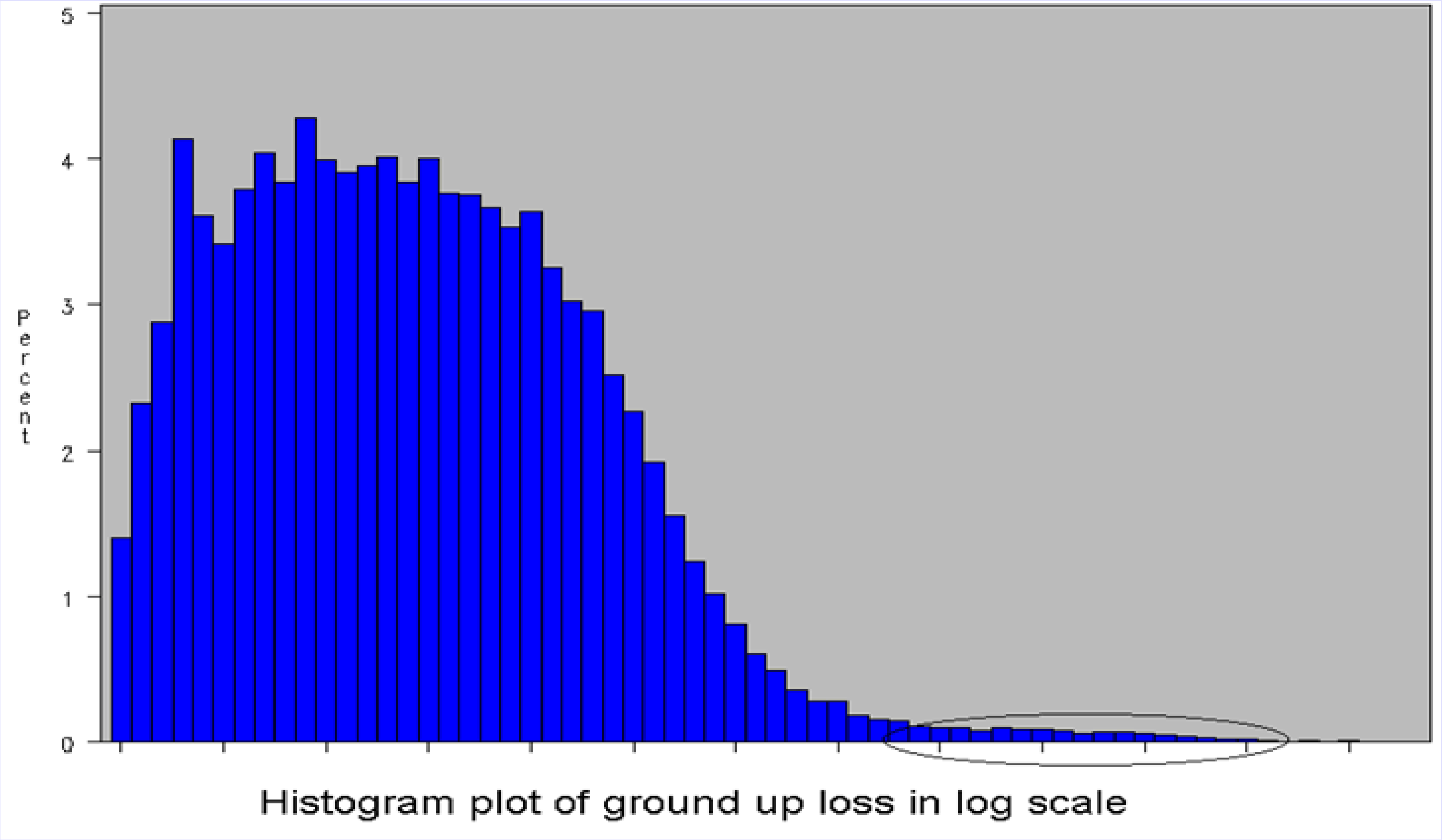
Introduction

Skewed Insurance Data

- Skewed and asymmetric
- Heavy tails
- Mixed: typical and extreme
  - Investment return: normal and crisis
  - Claim amount: typical and large losses

Introduction

HO by-peril example: heavier tail than lognormal

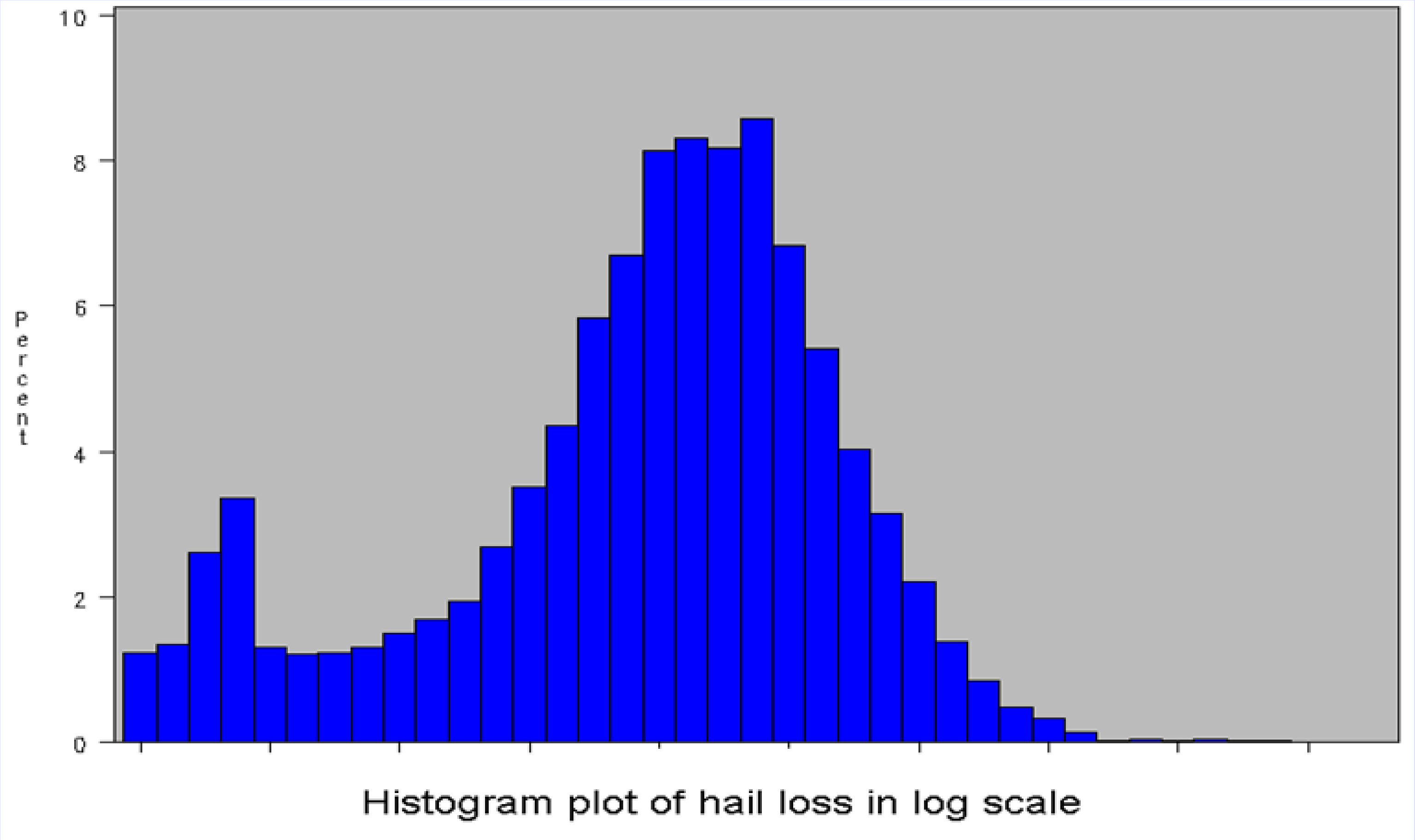


Introduction

HO by-peril example: multiple peaks

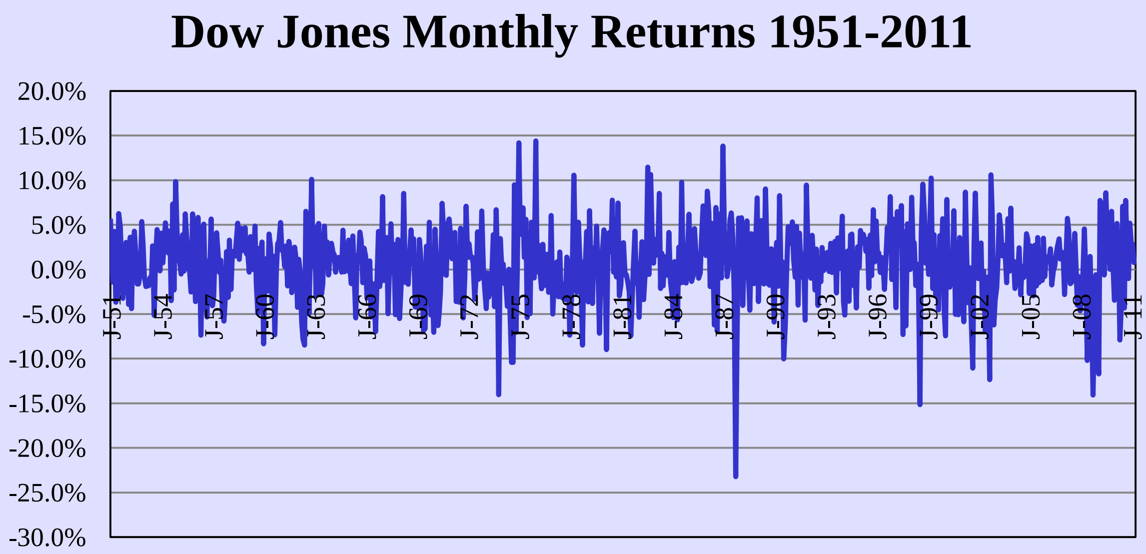
Introduction

HO by-peril example: multiple peaks



Introduction

Investment example in DFA



- Assuming normal distribution, the likelihood of monthly loss over 14.1% (largest monthly drop in Deep Recession) is 0.02%; actual observation is 0.55%.

Mixture Distribution

- Single distribution does not fit insurance data well
- A combination of multiple distributions can represent data better
- Mixture distributions:

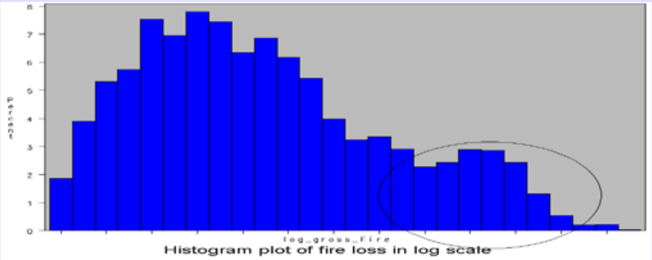
$$f(x, \pi_1, \pi_2, \ldots \pi_n, \beta_1, \beta_2, \ldots \beta_n) = \sum_i^n \pi_i \cdot f_i(x, \beta_i)$$

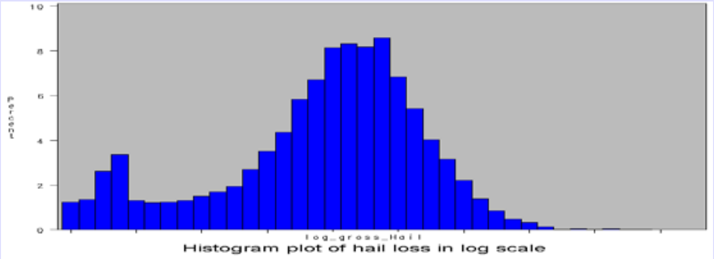
$$where \quad \sum_i^n \pi_i = 1$$

Mixture Distribution

Typical mixture distributions in insurance

- Claims count: Zero + Poisson
- Claim amount: gamma + lognormal or gamma + Pareto





Peril	π	α	β	μ	σ
Fire	0.785	0.51	10500	11.5	0.83
Hail	0.148	1.19	520	8.8	0.61

Mixture Distribution

- Regime-Switching Models of Equity Returns;
- Two lognormal distributions with low and high volatilities;
- Two regimes may switch by a matrix of transition probabilities;
- Hamilton (1990), Hardy (2001), Ahlgrim, D'Arcy, and Gorvett (2004).

	Low Volatility	High Volatility
Mean	0.96%	-2.20%
Standard Deviation	3.59%	7.17%
Probability of Switching	3.37%	30.87%

The likelihood of penetrating -14.1% by regime-switching model is 0.41%.

Finite Mixture Model

$$f(y \mid X; \pi_1, \pi_2, \ldots \pi_n, \theta_1, \theta_2, \ldots \theta_n) = \sum_i^n \pi_i \cdot f_i(X, \theta_i)$$

$$where \quad \sum_i^n \pi_i = 1$$

- y: response variable; X: explanatory variables
- A finite mixture model can be thought as a mixture of multiple GLMs
  - $f_1(y \mid X; \theta_1)$ is a GLM for smaller fire loss assuming gamma
  - $f_2(y \mid X; \theta_2)$ is a GLM for large fire loss assuming lognormal
- Often named as latent class model in economics

Finite Mixture Model

- Improvements on GLM
  - Expand distribution assumptions:
    Single exponential-family distribution vs. mixture
  - Expand model structure:
    Single regression formula vs. multiple models
  - Better fits on insurance data with heavy-tails, multimodal , excessive zeros, and other complex error distributions

AOI Group	**5%** Deductible Factors for Hail	
	GLM gamma	FMM
2	0.359	0.419
18	0.187	0.348

Finite Mixture Model

Numerical Solution

- Solving maximum likelihood function

$$\underset{\pi,\theta}{Max}\sum_{j=1}^{N}\log(\sum_{i=1}^{n}\pi_i f_i(y_j \mid X_j;\theta_i))$$

with constraint $\pi_i > 0 \quad and \quad \sum_{i}^{n}\pi_i = 1$

- EM (Expectation-Maximization) Algorithm
- Quasi-Newton Method
- Bayesian MCMC

Case Study: Data Description

- Simulated Hurricane Model Output
- 8,500 of 10,000 years with hurricane losses.
- Mean Aggregate Severity = $57,000,000
- Standard Deviation = $136,000,000
- Skewness = 6.5

- Positive skewness suggests an asymmetric distribution
  - Lognormal
  - Gamma

Case Study: Simple Distributions Fit Poorly

- Lognormal: Determine Parameters
  - Maximum Likelihood Estimation (MLE)
  - Method of Moments (MOM)
- Intuitive Test: MLE and MOM parameter estimates differ implying Lognormal is not a good fit.
- Chi-Square Test:
  - Critical Value at 95% = 11.1
  - Test Statistic Value = 419.0
  - Since 419.0>11.1 we reject the null hypothesis that the data were drawn from a Lognormal distribution with the fitted parameters.

Case Study: Simple Distributions Fit Poorly

Lognormal MLE

- Mean of log(loss) is 16.03 and Standard deviation is 2.50
  - Implied Mean = $ 207,000,000
  - Implied Stdev = $4,681,000,000
  - Max observed value = $3,053,000,000
- Excess small losses (81 losses <=$3000) make the error from model misspecification extreme.
  - Lognormal assumes log(loss) are symmetric
  - Log($3000)=8.01. The symmetric point on the other side of mean is 24.05, or $27,800,000,000
  - The losses are positively skewed with a heavy right tail; log(loss) is negatively skewed with heavy left tail. Lognormal cannot address this specific shape of distribution.

Case Study: Simple Distributions Fit Poorly

- Gamma: Determine Parameters
  - MLE fit
  - MOM fit
- Intuitive Test: MLE and MOM parameter estimates differ implying Gamma is not a good fit.
- Chi-Square Test:
  - Critical Value at 95% = 11.1
  - Test Statistic Value = 683.3
  - Since 683.3>11.1 we reject the null hypothesis that the data were drawn from a Gamma distribution with the fitted parameters.

Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: Determine Parameters
  - Density:

$$f(x,\alpha_1,\beta_1,\pi_1,\mu_2,\sigma_2) = \pi_1 * f_1(x,\alpha_1,\beta_1) + (1-\pi_1) * f_2(x,\mu_2,\sigma_2)$$

  - Likelihood:

$$L(\alpha_1,\beta_1,\pi_1,\mu_2,\sigma_2) = \prod_{i=1}^{8500} f(x_i,\alpha_1,\beta_1,\pi_1,\mu_2,\sigma_2)$$

  - Log-Likelihood:

$$l(\alpha_1,\beta_1,\pi_1,\mu_2,\sigma_2) = \sum_{i=1}^{8500} \ln(f(x_i,\alpha_1,\beta_1,\pi_1,\mu_2,\sigma_2))$$

Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: MLE Parameters

$$\alpha_1 = .446, \beta_1 = 57.9M$$
$$\pi_1 = 0.884$$
$$\mu_2 = 19.221, \sigma_2 = 0.789$$

- Intuition: Aggregate Severity is drawn from:
  - 88.4% of time Gamma (Mean=26M, Stdev=39M)
  - 11.6% of time Lognormal (Mean=304M, Stdev=282M)

- Match to 1st two moments:
  - Mean of mixture matches data within 0.2%.
  - Standard deviation of mixture matches data within -0.7%.

Case Study: Mixed Distributions Fit Better

- Mixed Gamma-Lognormal: Significance?

- Likelihood Ratio Test 95% Critical Value=7.8
  - Mixed vs. Gamma Test Statistic = 668
  - Mixed vs. Lognormal Test Statistic = 1331

- Since test statistics > critical value the mixed distribution provides a significantly better fit to the data than either of the simple distributions.

Case Study: Fitting Mixtures

- Tools Available to Fit Mixed Distributions
  - Microsoft Excel SOLVER
  - R
  - SAS
  - Other
- Steps to Fit Mixed Distributions
  - Write the Mixed Density Function
  - Specify Initial Parameter Values
  - Write the Log-Likelihood Function
  - Maximize the Log-Likelihood by Changing Parameters

Case Study: Fitting Mixtures

- Mixed Gamma-Gamma:
  - Density:

$$f(x,\alpha_1,\beta_1,\pi_1,\alpha_2,\beta_2) = \pi_1 * f_1(x,\alpha_1,\beta_1) + (1-\pi_1) * f_2(x,\alpha_2,\beta_2)$$

  - Specify Initial Parameter Values
  - Likelihood:

$$L(\alpha_1,\beta_1,\pi_1,\alpha_2,\beta_2) = \prod_{i=1}^{8500} f(x_i,\alpha_1,\beta_1,\pi_1,\alpha_2,\beta_2)$$

  - Log-Likelihood:

$$l(\alpha_1,\beta_1,\pi_1,\alpha_2,\beta_2) = \sum_{i=1}^{8500} \ln(f(x_i,\alpha_1,\beta_1,\pi_1,\alpha_2,\beta_2))$$

Case Study: Fitting Mixtures

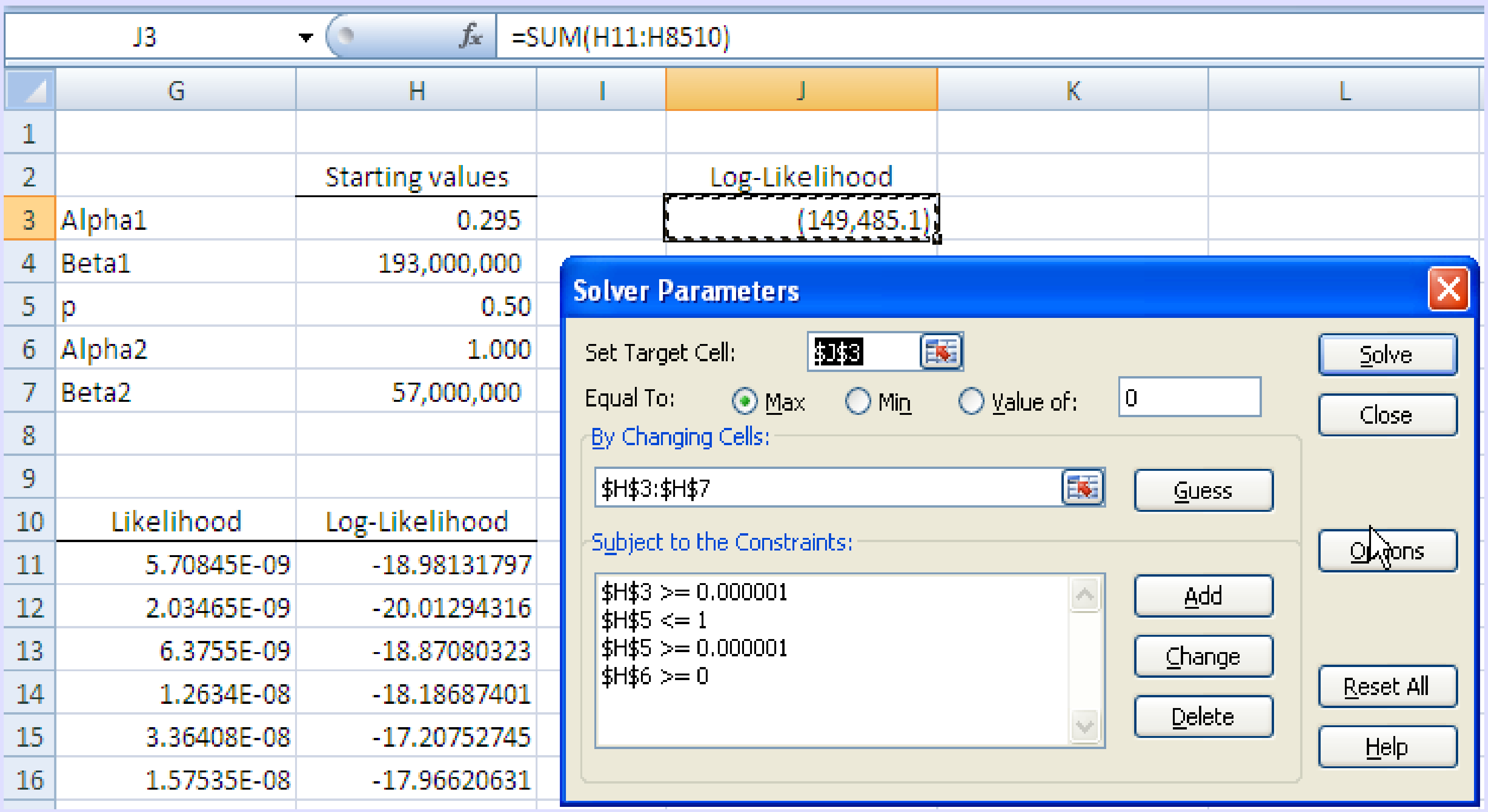
- Maximize Log-Likelihood: Excel SOLVER

=$H$5*GAMMADIST(E11,$H$3,$H$4,FALSE)+(1-$H$5)*GAMMADIST(E11,$H$6,$H$7,FALSE)

D	E	F	G	H	I	J	K
				Starting values		Log-Likelihood	
			Alpha1	0.295		(150,165.8)	
			Beta1	193,000,000			
			p	0.50			
			Alpha2	1.000			
			Beta2	57,000,000			
Year	AggLoss		Likelihood	Log-Likelihood			
1	46,452,953		=$H$5*GAMMADIST(E11,$H$3,$H$4,FALSE)+(1-$H$5)*GAMMADIST(E11,$H$6,$H$7,FALSE)				
2	108,518,889		2.03465E-09	-20.01294316			

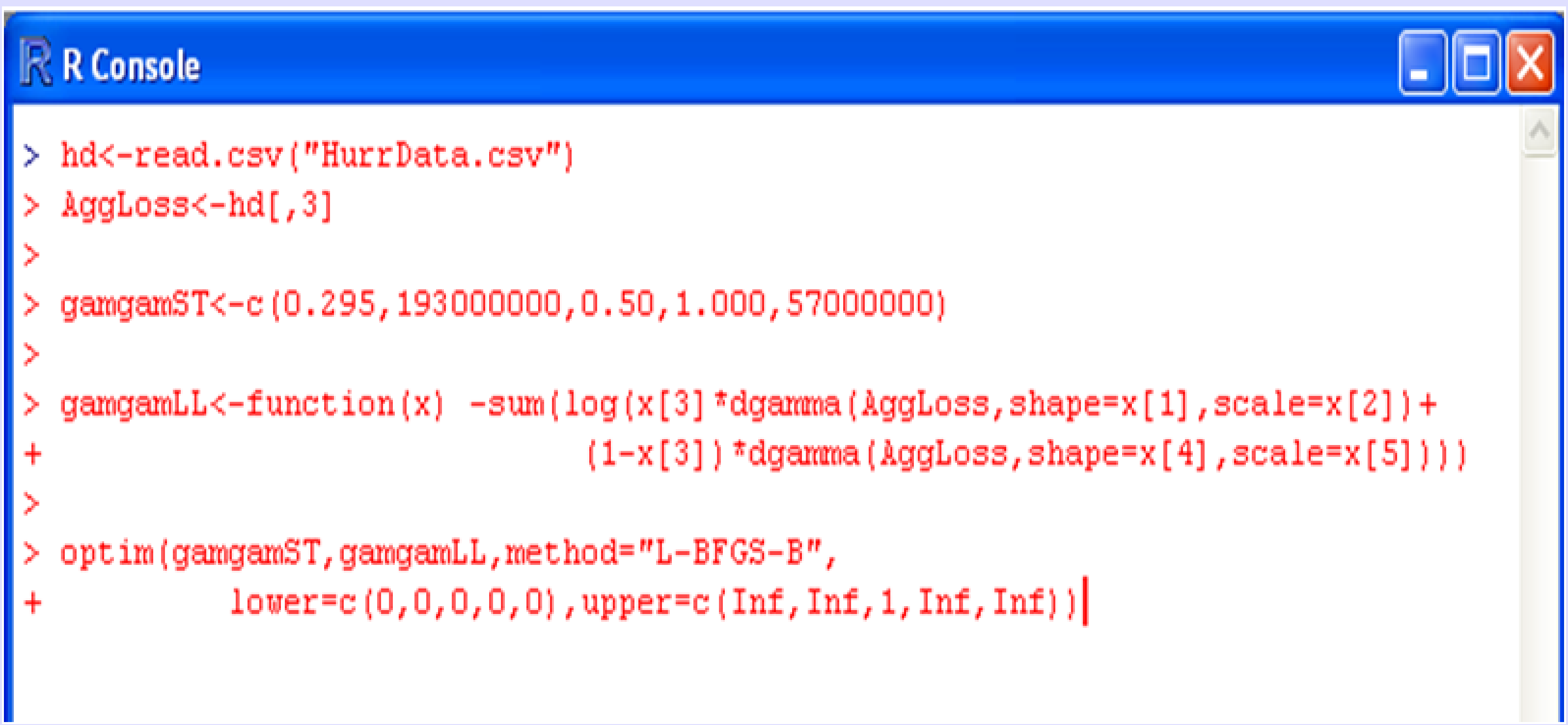
Case Study: Fitting Mixtures

- Maximize Log-Likelihood: Excel SOLVER



Case Study: Fitting Mixtures

- Maximize Log-Likelihood: R
  - http://www.r-project.org/



R Console

```
> hd<-read.csv("HurrData.csv")
> AggLoss<-hd[,3]
>
> gamgamST<-c(0.295,193000000,0.50,1.000,57000000)
>
> gamgamLL<-function(x) -sum(log(x[3]*dgamma(AggLoss,shape=x[1],scale=x[2])+
+                             (1-x[3])*dgamma(AggLoss,shape=x[4],scale=x[5])))
>
> optim(gamgamST,gamgamLL,method="L-BFGS-B",
+          lower=c(0,0,0,0,0),upper=c(Inf,Inf,1,Inf,Inf))
```

Case Study: Fitting Mixtures

- Parameter Risk: Sample Data
  - The second distribution could have low credibility.
  - Sensitivity test with slight data changes
  - Parameter uncertainties in cat modeling firms (AIR, RMS, EQECAT)

- Parameter Risk: Initial Values
  - Could lead to local maxima
  - Try different starting values
    - Start with 90%/10% weights
    - Use same distribution to infer starting means such as a mixture of 2-Gamma distributions.

Case Study: Fitting Mixtures

- Parameter Risk: Robustness
  - Remove 81 losses less than $3000, and refit MLE lognormal and gamma-lognormal distributions.
  - For lognormal, the fitted mean <u>decreased</u> by 29%; the fitted standard deviation <u>decreased</u> by 54%.

$$\mu = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i) \qquad E[X] = e^{\mu+\sigma^2/2}$$

  - For gamma-lognormal, the fitted mean <u>increased</u> 2%, the fitted standard deviation <u>decreased</u> by 0.1%.
  - Mixture distribution is more robust!

Case Study: Implications

- Expected Reinsurance Recovery
  - Low credibility for high layers
    - Hurricane output only contained 56 losses over $800M.
    - Only 5 losses over $1.6B.
  - Fitted distribution can help evaluate cost for higher layers

- Alternative Tail Estimates
  - Percentiles/VaR
  - TVaR

Conclusions

- Insurance data are skewed and heavy tailed.
- Single distribution in general cannot fit data well.
- Mixture distribution can represent insurance data with excess zeroes, multiple modes, and heavy tails.
- Finite mixture model improves GLM by assuming mixture distribution.
- Many insurance applications: ERM (PML, TVaR), asset management, reinsurance (cat, per risk), high deductible (worker comp, property), predictive modeling (frequency, severity).

Q & A

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