The Marginal Cost of Risk, Risk Measures, and Capital Allocation

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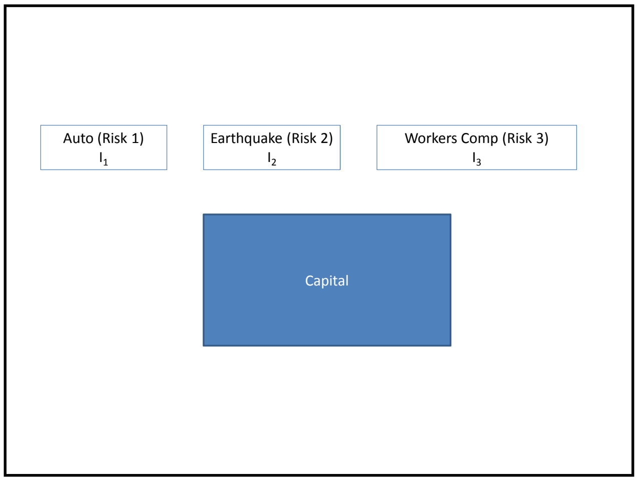
Cost of risk: Intellectual Evolution

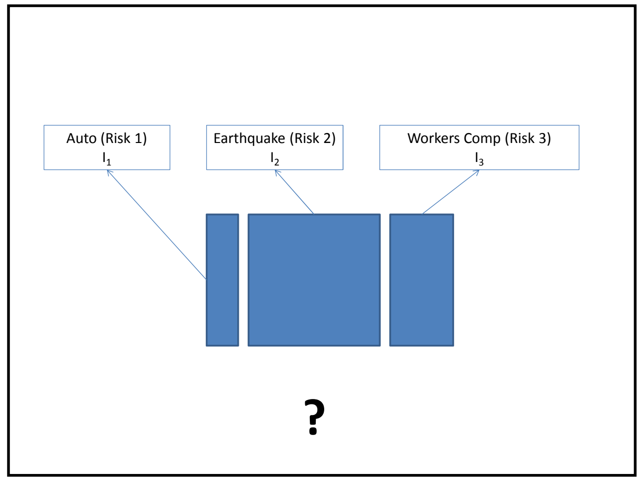
- Traditional actuarial methods
  - Profit load (5%)
  - Catastrophe load
- Equilibrium asset pricing methods: CAPM
  - Empirical problems
  - Practical problems
- Capital allocation
  - Target ROE must be recovered from business

Auto (Risk 1) | Earthquake (Risk 2) | Workers Comp (Risk 3)

Auto (Risk 1) I_1 | Earthquake (Risk 2) I_2 | Workers Comp (Risk 3) I_3



Auto (Risk 1)
I_1
Earthquake (Risk 2)
I_2
Workers Comp (Risk 3)
I_3
Capital



Auto (Risk 1)
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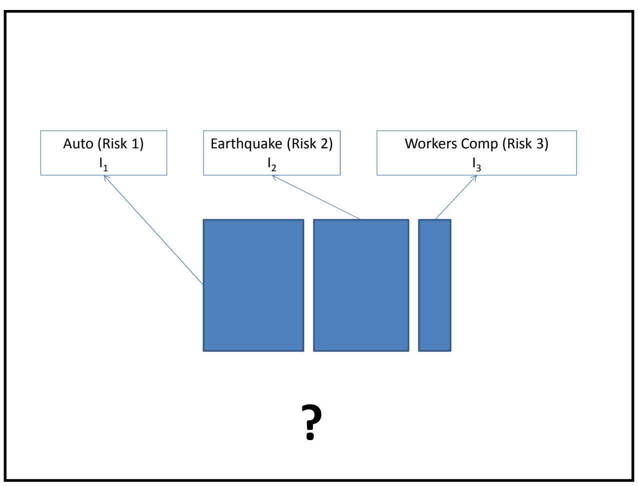
Earthquake (Risk 2)
I_2
Workers Comp (Risk 3)
I_3
?



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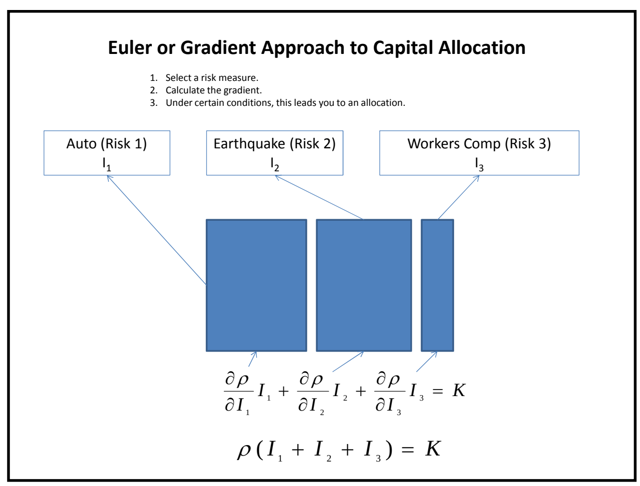


Euler or Gradient Approach to Capital Allocation
1. Select a risk measure.
2. Calculate the gradient.
3. Under certain conditions, this leads you to an allocation.
Auto (Risk 1)
I_1
Earthquake (Risk 2)
I_2
Workers Comp (Risk 3)
I_3
$\frac{\partial \rho}{\partial I_1} I_1 + \frac{\partial \rho}{\partial I_2} I_2 + \frac{\partial \rho}{\partial I_3} I_3 = K$
$\rho(I_1 + I_2 + I_3) = K$

Euler or Gradient Approach to Capital Allocation

- Advantage 1: Ease of implementation

- "Advantage" 2: Billed as "economic"; supposedly connected to "marginal cost"

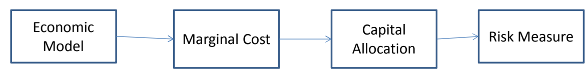
  In brief: max profits subject to risk-measure based constraint …. implies the Euler allocation of capital

- Problem: Selection of risk measure is *ad hoc*. No theoretical economic foundation.

What we do: The opposite

- Standard approach is to pick a risk measure and use it to get to marginal cost and an allocation of capital.

- We start with a primitive economic model of profit maximizing insurer, calculate marginal cost and the implied capital allocation, and then figure out what risk measure would yield the correct allocation.



Preview of Results

- Study social planning problem with risk averse policyholders. The source of solvency discipline is thus the policyholder

- Marginal cost of risk/capital allocation ends up being driven by policyholder concerns about solvency. Each policyholder's risk is penalized according to the externalities it generates on *other* policyholders.

- Capital allocation is determined by the marginal impact of each risk on the total value of policyholder recoveries from the insurer in states of default.
  - Polar case of risk-neutral policyholders: Each policyholder is allocated capital according to her share of expected recoveries from the company in states of default.
  - General case with risk aversion: Recovery values are adjusted according to the average marginal utility of policyholder consumption in the various states of default (e.g., more severe states are penalized more heavily than less severe ones). So each policyholder gets capital allocated according to her share of value-adjusted expected recoveries from the company in states of default.

- Extension – Can you do this with a risk measure? Yes.

$$\tilde{\rho}(X) = \exp\left\{\mathbb{E}^{\tilde{\mathbb{P}}}\left[\log\{X\}\right]\right\}$$

Literature Review

- Merton and Perold (1993) – allocation is infeasible
- Phillips, Cummins, and Allen (1998) – allocation is unnecessary (given complete markets and no frictional costs)

- Myers/Read (2001); Tasche (2000); Schmock/Straumann (1999); Zanjani (2002); Denault (2001); Kalkbrener (2005); Mildenhall (2006); Powers (2007)
  - allocation is feasible when considering marginal changes
  - gradient method applied to risk measure; Aumann-Shapley values in game theoretic approaches

- But how to pick the risk measure????
  - Why would a self-interested firm use one?
  - If measure is chosen by regulators/rating agencies, which should they choose?

Social Planning Problem
Setup and Notation

- *N* consumers, each exposed to a loss of size L_i with probability p_i
- State of the world is a vector **x** with elements taking a value of <u>one</u> or <u>zero</u>:
  - **X**(i)=1 means consumer i experienced a loss in that state
  - **X**(i)=0 means consumer i <u>did not</u> experience a loss in that state

$\Omega:$ Set of all possible states

$\Omega^i = \{\mathbf{x} : \mathbf{x}(i) = 1\}$ Set of all states where consumer *i* loses

$\Gamma(\mathbf{x}) = \{i : \mathbf{x}(i) = 1\}$ Set of all consumers who lose in state **x**

$$p_i = \sum_{\mathbf{x} \in \Omega^i} \Pr(\mathbf{x})$$

Social Planning Problem
Setup and Notation

- Social planner chooses
  - Policy limits for each consumer: $\mathbf{I_i}$
  - Assets for the insurance company: **A**
  - Cost allocation (premiums) for each consumer: $\mathbf{P_i}$

Consumer *i* recovery in state **x**: $R_i^{\mathbf{x}} \equiv \min\left\{ I_i, \frac{A}{\sum_{j \in \Gamma(\mathbf{x})} I_j} I_i \right\}$ for $\mathbf{x} \in \Omega^i$, 0 otherwise

Expected value of recoveries: $E_i = \sum_{\mathbf{x} \in \Omega} \Pr(\mathbf{x}) R_i^{\mathbf{x}}$

Set of states where firm defaults $\Omega^D = \{\mathbf{x} : \sum_{j \in \Gamma(\mathbf{x})} I_j \geq A\}$, $E_i^D = \sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x}) R_i^{\mathbf{x}}$

Set of states where firm is solvent $\Omega^Z = \{\mathbf{x} : \sum_{j \in \Gamma(\mathbf{x})} I_j < A\}$, $E_i^Z = \sum_{\mathbf{x} \in \Omega^Z} \Pr(\mathbf{x}) R_i^{\mathbf{x}}$

$$E_i = E_i^Z + E_i^D$$

Social Planning Problem
Setup and Notation

VNM Utility

$$V_i(A, W_i - P_i, I_1, \ldots, I_N) \equiv \sum_{\mathbf{x} \notin \Omega^i} \Pr(\mathbf{x}) U_i(W_i - P_i) + \sum_{\mathbf{x} \in \Omega^i} \Pr(\mathbf{x}) U_i(W_i - P_i - L_i + R_i^{\mathbf{x}})$$

$$\max_{A, I_1, \ldots, I_N, P_1, \ldots, P_N} \left\{ \sum_i V_i \right\}$$

subject to a budget constraint

$$\sum_i P_i = \sum_i E_i + \tau A$$

Tax on assets---so we'll allocate assets rather than capital. See the Appendix for capital allocation.

Social Planning Problem
Optimality Conditions*

$$[I_i]: \ \frac{\partial V_i}{\partial I_i} + \frac{\partial V_i}{\partial W}\left(\frac{\partial E_i}{\partial I_i} + \sum_{j \neq i} \frac{\partial E_j}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0$$

$$[A]: \ \sum_k \frac{\partial V_k}{\partial A} + \frac{\partial V_i}{\partial W}\left(\sum_k \frac{\partial E_k}{\partial A} + \tau \right) = 0$$

$$[P_i]: \ \frac{\partial V_i}{\partial W} - \frac{\partial V_j}{\partial W} = 0$$

* Conditions assume optimum is in smooth part of function. See paper for general conditions encompassing optima at non-differentiable points.

Social Planning Problem

Pricing Function

$$\frac{\partial P_i^*}{\partial I_i} = \frac{\partial E_i^Z}{\partial I_i} - \frac{\displaystyle\sum_{j \neq i} \frac{\partial V_j}{\partial I_i} - \sum_{\mathbf{x} \in \Omega^i \cap \Omega^D} \Pr(\mathbf{x}) \mu_i^{\mathbf{x}} \frac{A}{\left(\displaystyle\sum_{j \in \Gamma(\mathbf{x})} I_j\right)^2} I_i}{\dfrac{\partial V_i}{\partial W}}$$

where $\mu_i^{\mathbf{x}} = \dfrac{\partial U_i(W_i - P_i - L_i + R_i^{\mathbf{x}})}{\partial W}$

This simplifies to…

Social Planning Problem

Pricing Function

$$\frac{\partial P_i^*}{\partial I_i} = \frac{\partial E_i^Z}{\partial I_i} + \phi_i A \left(\sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x}) + \tau \right)$$

$$\text{where } \phi_i = \frac{\displaystyle\sum_{\mathbf{x} \in \Omega^D \cap \Omega^i} \Pr(\mathbf{x}) \left(\sum_{k \in \Gamma(\mathbf{x})} \mu_k^{\mathbf{x}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right) \right) \left(\frac{1}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right)}{\displaystyle\sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x}) \left(\sum_{k \in \Gamma(\mathbf{x})} \mu_k^{\mathbf{x}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right) \right)}$$

Social Planning Problem

Pricing Function

It adds up: $\displaystyle\sum_i \phi_i I_i = 1$

$$\frac{\partial P_i^*}{\partial I_i} I_i = \frac{\partial E_i^Z}{\partial I_i} I_i + \phi_i I_i \left(\sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x}) A + \tau A \right)$$

- Consumer i's expected claims recoveries in states where insurer is solvent
- *Total* consumer expected claims recoveries in states where insurer is *insolvent*
- Total frictional costs

Thus, with binary loss distributions, by pricing at marginal cost you cover all costs and get to a social optimum for some initial distribution of wealth.

Social Planning Problem

So what is the allocation rule exactly?

$$\phi_i = \frac{\displaystyle\sum_{\mathbf{x} \in \Omega^D \cap \Omega^i} \Pr(x) \left(\sum_{k \in \Gamma(\mathbf{x})} \mu_k^{\mathbf{x}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right) \right) \left(\frac{1}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right)}{\displaystyle\sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x}) \left(\sum_{k \in \Gamma(\mathbf{x})} \mu_k^{\mathbf{x}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right) \right)}$$

If there is only one state of default, or if we ignore the marginal utility effects:

$$\phi_i I_i = \frac{\displaystyle\sum_{\mathbf{x} \in \Omega^D \cap \Omega^i} \Pr(x) \left(\frac{I_i}{\sum_{j \in \Gamma(\mathbf{x})} I_j} \right)}{\displaystyle\sum_{\mathbf{x} \in \Omega^D} \Pr(\mathbf{x})} = \frac{E_i^D}{\sum_j E_j^D}$$

So the starting point for allocation is the consumer's share of recoveries in states where the insurer defaults. But that's only the starting point…

Social Planning Problem

So what is the allocation rule exactly?

$$\phi_i = \frac{\sum_{\mathbf{x}\in\Omega^D\cap\Omega^i} \Pr(x)\left(\sum_{k\in\Gamma(\mathbf{x})} \mu_k^{\mathbf{x}}\left(\frac{I_k}{\sum_{j\in\Gamma(\mathbf{x})} I_j}\right)\right)\left(\frac{1}{\sum_{j\in\Gamma(\mathbf{x})} I_j}\right)}{\sum_{\mathbf{x}\in\Omega^D} \Pr(\mathbf{x})\left(\sum_{k\in\Gamma(\mathbf{x})} \mu_k^{\mathbf{x}}\left(\frac{I_k}{\sum_{j\in\Gamma(\mathbf{x})} I_j}\right)\right)}$$

Marginal social value of an extra dollar in state **x**

The marginal social value of an extra dollar in state **x** varies. This has the effect of weighting default states according to severity.

- Expected claims recovery component of price thus may deviate from expected value.
- Weights represent prices from an internal company market for contingent claims in states of default

Allocation Basis: Value-weighted expected recoveries when insurer is insolvent

Extensions

- ✓ Security markets
- ✓ Allocating capital or assets
- ✓ Generalized (smooth) loss distributions*
- ✓ Profit maximization instead of social planning *
- ✓ Implementation via gradient method applied to risk measure *

* Bauer, D. and Zanjani, G. (2012), "The Marginal Cost of Risk, Risk Measures, and Capital Allocation," working paper.

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A "reverse-engineered" risk measure

$$\exp\left\{E\left[\varphi(I)\log(I)\mid I > A\right]\right\}$$

$$\varphi(\mathrm{I}) = \frac{\sum_{\mathbf{x}\in\{\mathbf{x}\ \mid \sum_{j\in\Gamma(\mathbf{x})} I_j = I\}} \Pr(x)\left(\sum_{k\in\Gamma(\mathbf{x})} \mu_k^{\mathbf{x}}\left(\frac{I_k}{\sum_{j\in\Gamma(\mathbf{x})} I_j}\right)\right)}{E\left[\sum_{\mathbf{x}\in\Omega^D} \Pr(\mathbf{x}) \sum_{k\in\Gamma(\mathbf{x})} \mu_k^{\mathbf{x}}\left(\frac{I_k}{\sum_{j\in\Gamma(\mathbf{x})} I_j}\right)\right]}$$

Not coherent! Not convex!

Concluding Remarks

- Capital allocation can be and should be grounded in an economic context.

- We do not need to, nor should we, rely on *ad hoc* risk measures. Even "axiomatic" ones.

- Lesson for practitioners and regulators: We can still use risk measures, but we need to design them appropriately.

<u>We can and should formulate capital allocation from first principles based on the objectives of the organization.</u>