CAS Spring Meeting — Vancouver

A Family of Chain-Ladder Factor Models for Selected Link Ratios

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Introduction of the Chain Ladder Factor Model (CLFM) Framework

Motivation: What if an actuary selects a link ratio other than the straight average or volume weighted one?

- Practitioners today are “scaling” the Mack/Murphy results when employing a model with different selected link ratios
  - The CV implied by the straight average and volume weighted Mack/Murphy models apply to reserves that have been calculated in different ways
    - The presenters are concerned that the “scaling” technique might significantly understate the mse estimate
- Two relevant questions easily come to mind:
  - Should actuaries employ link ratios other than the ones based on some type of averages of the empirical data?
  - Can we expand the Mack/Murphy model framework to consider other types of selected link ratios?

Mack/Murphy model underlying structure

- The easiest way to think of this structure is to picture a linear regression across two consecutive columns in a loss triangle

$$C_{i,k+1} = f_k C_{i,k} + \sigma_k \varepsilon_{i,k} C_{i,k}^{a/2}, \quad where\, a = 0, 1 or\, 2$$

- i corresponds to accident year (i.e. row) and k corresponds to development year (i.e. column) of a triangle
- Alpha is assumed to be independent of k
- The random component of the error term is assumed to be independent and identically distributed (i.i.d.) around zero, with a variance of one
- The model is heteroscedastic since the variance of the error term is proportional to $\sigma_k^2 C_{i,k}^a$

Maximum likelihood solution for the Mack/Murphy model

$$
\begin{cases}
\hat{f}_k(\alpha) = \sum_{i=1}^{n-k} \dfrac{C_{i,k}^{1-\alpha}}{\sum_{j=1}^{n-k} C_{j,k}^{2-\alpha_k}} C_{i+1,k} = \sum_{i=1}^{n-k} w_{i,k}^{\alpha} \cdot F_{i,k}, \\
w_{i,k}^{\alpha} := \dfrac{C_{i,k}^{2-\alpha}}{\sum_{j=1}^{n-k} C_{j,k}^{2-\alpha}}, \quad F_{i,k} := \dfrac{C_{i,k+1}}{C_{i,k}}
\end{cases}
$$

Where $\hat{f}_k(a)$ represents the best linear unbiased estimator (BLUE) of the link ratio f_k from age k to age k+1

- What is the implied BLUE link ratio?
  - For alpha = 0 is the slope of a linear regression through the origin
  - For alpha = 1 is the all year volume weighted:

$$\hat{f}_k(a) = \sum_{i=1}^{n-k} C_{i,k+1} \Big/ \sum_{j=1}^{n-k} C_{j,k}$$

  - For alpha = 2 is the all year simple average:

$$\hat{f}_k(a) = \sum_{i=1}^{n-k} \frac{C_{i,k+1}}{C_{i,k}} \Big/ (n-k)$$

Why not use one of the "standard averages"?

- Actuaries can and should exercise judgment in the selection of the Chain Ladder link ratios
- According to ASOP No. 43 Section 3.6.2
  - *"The actuary should consider the reasonableness of the assumptions underlying each method or model used. Assumptions generally involve significant professional judgment as to the appropriateness of the methods and models used and the parameters underlying the application of such methods and models."*
- Moreover, according to Jacqueline Friedland's "Estimating Unpaid Claims using Basic Techniques" in Chapter 7
  - *"When the credibility of the insurer's own historical experience is limited, there may be a need to supplement the insurer's own historical experience with certain benchmarks. One possible benchmark includes experience from similar lines with similar claims handling practices within the insurer."*

CLFM Framework is intended to respond to that directive

- Simply an extension of the Mack/Murphy model

$$C_{i,k+1} = f_k C_{i,k} + \sigma_k \varepsilon_{i,k} C_{i,k}^{a_k/2}, \quad where\, a_k \in R$$

- Alpha is now dependent on k and is defined on the real line R
- We define the Link Ratio Function as the Maximum Likelihood solution

$$LR_k(\alpha) = \sum_{i=1}^{n-k} w_{i,k}^{\alpha} \cdot F_{i,k}$$

- Where all factors are defined the same way as in slide 3
- In essence we can think of the CLFM as a family of models identified by an index alpha defined in the real line
  - The Mack/Murphy model is a special case with α in {0,1,2}

Illustrative Example #1

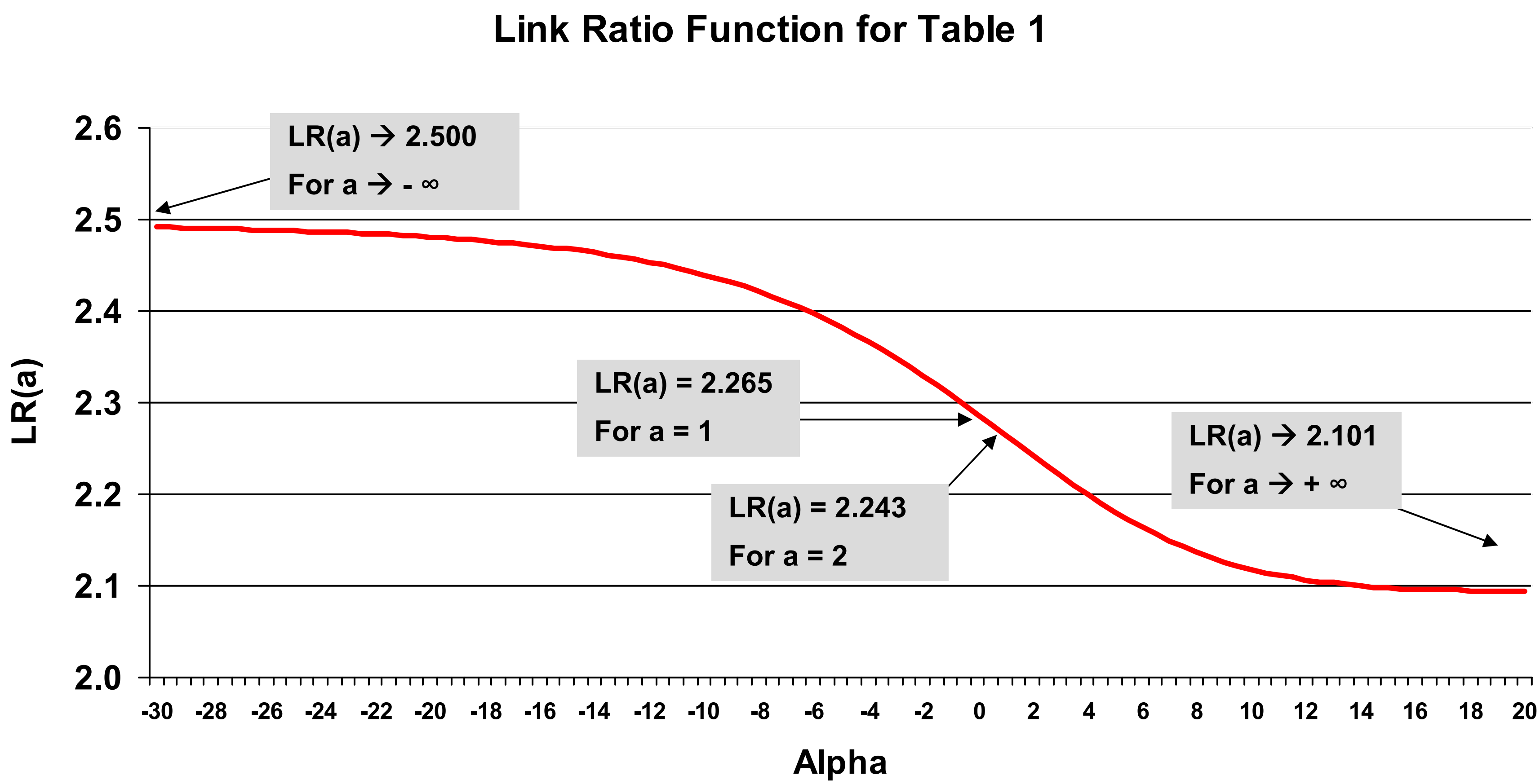
Table 1: Development Period Losses

$C_{i,j}$	j=1	j=2	$F_{i,1}$
j=1	280	680	2.429
j=2	250	550	2.200
j=3	300	750	2.500
j=4	235	466	1.983
j=5	207	435	2.101
	Volume Weighted Avg.		2.265
	Simple Avg.		2.243

- $LR_k(1.000) = 2.265$
- $LR_k(2.000) = 2.243$

- Given a link ratio, the alpha can be calculated through a numerical approximation technique (i.e. Newton-Rapshon)

Illustrative Example #1 (continued)



- The Link Ratio Function is asymptotic to the y = 2.500 and y = 2.101 lines

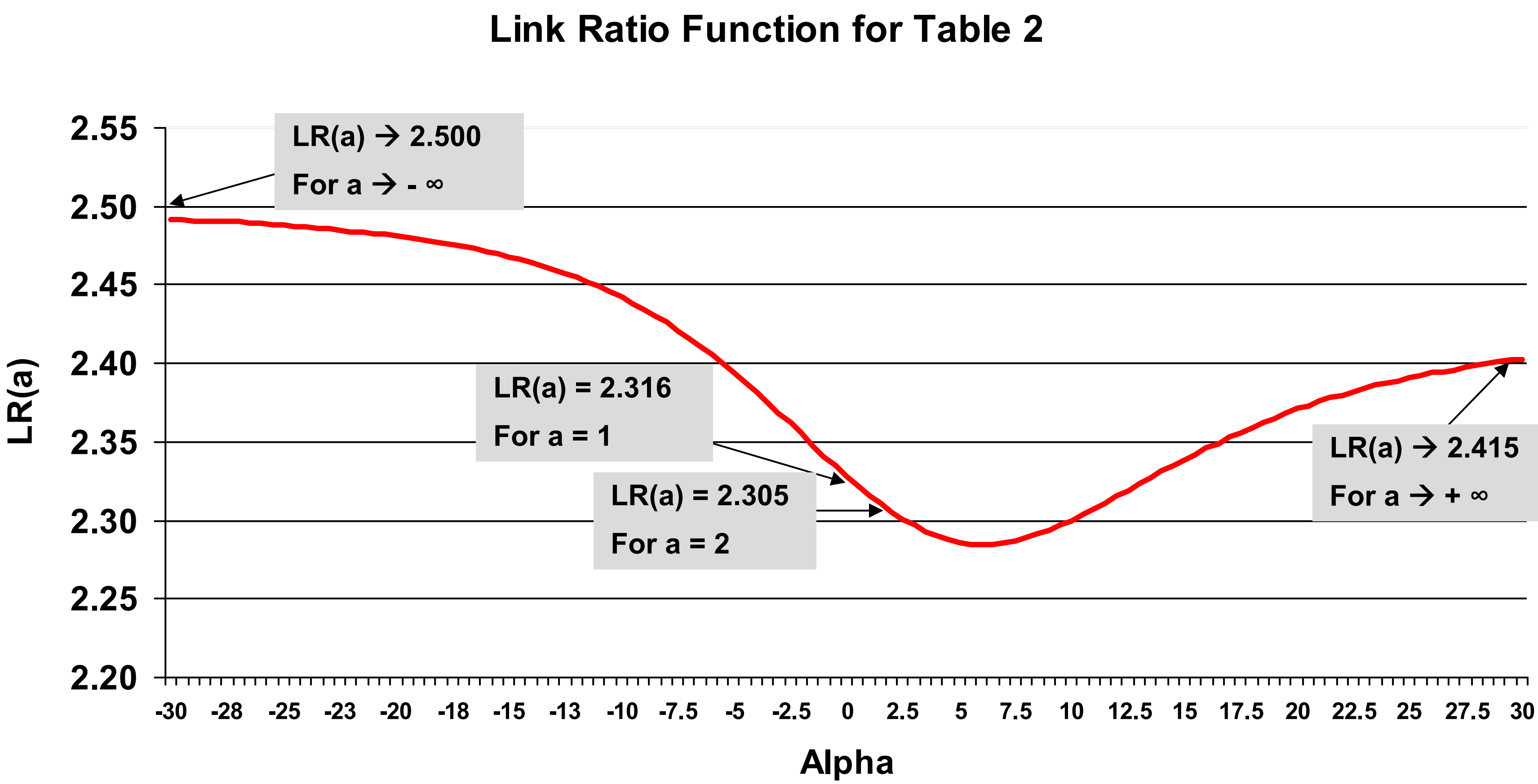
Illustrative Example #2

Table 2: Development Period Losses

$C_{i,j}$	j=1	j=2	$F_{i,1}$
j=1	280	680	2.429
j=2	250	550	2.200
j=3	300	750	2.500
j=4	235	466	1.983
j=5	207	500	2.415
	Volume Weighted Avg.		2.316
	Simple Avg.		2.305

- $LR_k(1.000) = 2.316$
- $LR_k(2.000) = 2.305$
- The Link Ratio Function is not necessarily a monotonic function
- The image of the link ratio function is not the entire real line, in fact the min link ratio is “off-the-chart” in this example

Illustrative Example #2 continued



- The Link Ratio Function is asymptotic to the y = 2.500 and y = 2.415 lines

Asymptotic properties of the Link Ratio Function

- When $a \to +\infty$ the BLUE of a link ratio approaches the link ratio experienced by the accident year with the **smallest** value of loss at the beginning of the development period

- When $a \to +\infty$ the BLUE of a link ratio approaches the link ratio experienced by the accident year with the **largest** value of loss at the beginning of the development period

- The alpha function is defined in the whole real line, but not all possible link ratios correspond to an alpha value

Bridge to the Variance concept

- Variance estimation between the Mack/Murphy and CLFM follows similar logic
- One exception relates to the estimation of the process risk
- The process risk $\Gamma(C)^2$ is a function of the E{Var($C_{i,k+1}$)} or E($\sigma_k^2 C_{i,k}^a$)
  - Calculations are difficult for any number other than 0 and 1 since the expectation of

    E(C^a) is different from E(C)^a
- Our paper suggests a “helper” function Ψ such as:
  - E(C^α) = $\Psi(\alpha,\kappa)$ x E(C)$^\alpha$; where k is the coefficient of variation of C
- For small integer values of alpha Ψ can be easily calculated:
  - $\Psi(0,\kappa) = \Psi(1,\kappa) = 1$
  - $\Psi(2,\kappa) = 1 + \kappa^2$, since E(C^2) = E^2(C)+Var(C), or E(C^2)/E^2(C) = 1+ CV^2(C)

Helper Ψ function for integer values

- For larger integer values we have:

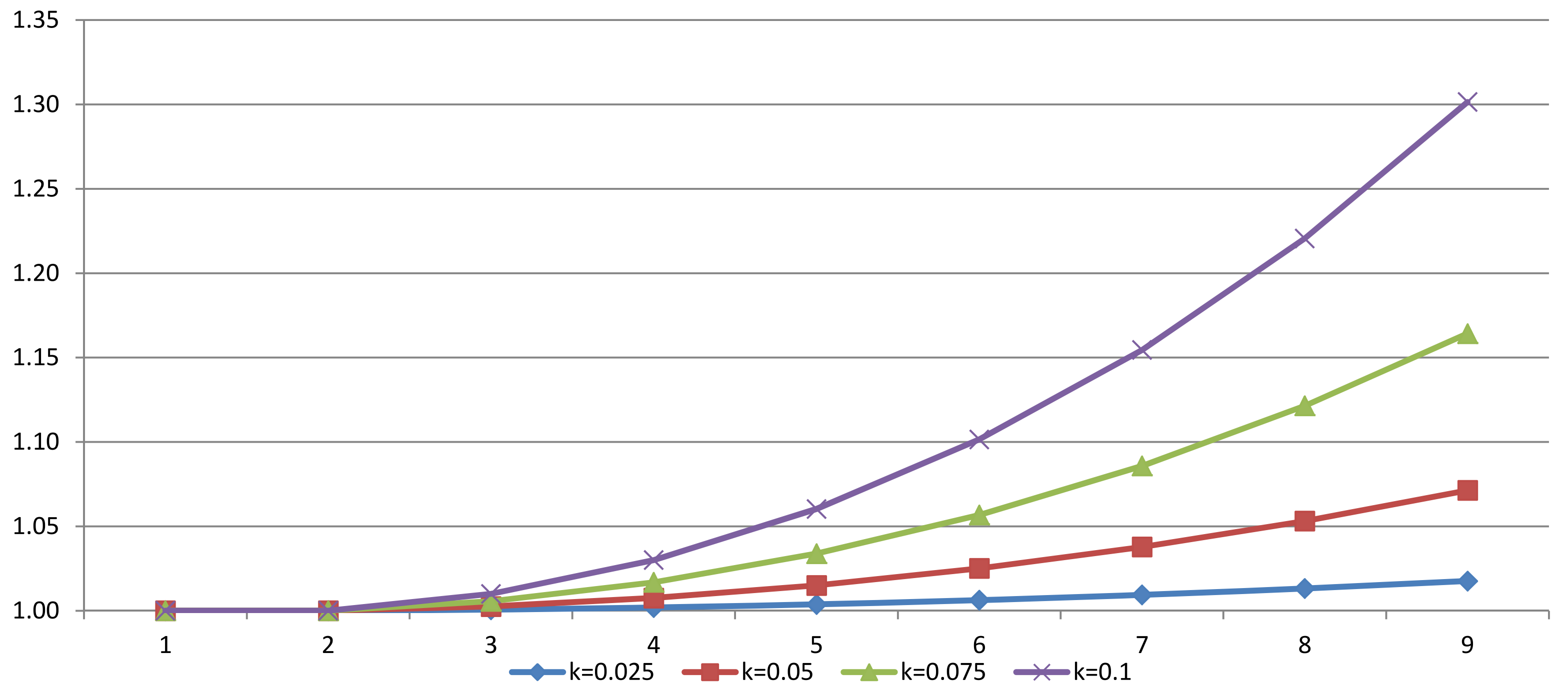
α	Raw Moment	In terms of coefficient of variation κ
0	1	1
1	μ	μ
2	$\mu^2 + \sigma^2$	$\mu^2(1 + \kappa^2)$
3	$\mu^3 + 3\mu\sigma^2$	$\mu^3(1 + 3\kappa^2)$
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$\mu^4(1 + 6\kappa^2 + 3\kappa^4)$
5	$\mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	$\mu^5(1 + 10\kappa^2 + 15\kappa^4)$
6	$\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$	$\mu^6(1 + 15\kappa^2 + 45\kappa^4 + 15\kappa^6)$
7	$\mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$	$\mu^7(1 + 21\kappa^2 + 105\kappa^4 + 105\kappa^6)$
8	$\mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\sigma^8$	$\mu^8(1 + 28\kappa^2 + 210\kappa^4 + 420\kappa^6 + 105\kappa^8)$

- And generally:

$$\Psi(n,\kappa) = \sum_{\substack{j=0 \\ j\,even}}^{n} \frac{1*n*(n-1)*\ldots*(n-(j-1))}{2^{j/2}(\frac{j}{2})!} \cdot k^{j}$$

Helper Ψ function for non-integer values

- For positive values we suggest interpolation between integer values
- For negative values we suggest approximating Y by employing simulations



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Questions and discussion