Advancements in Common Shock modeling

May 15th, 2016

Paul Ferrara, Ming Li

Tokio Marine Technologies LLC

Acknowledgments:

This work was sponsored by:

- *Tokio Marine Technologies LLC*
- *Tokio Millennium Re*

Co-Researchers:

- Phui Jing Lee (PJ) *Senior Consultant* Deloitte Bermuda
- Jimmy Su *PhD & SVP*, Tokio Marine Technologies

Special Thanks to:

- Steven B. White, *FCAS & Chief Actuary*, Guy Carpenter LLC.
- Benedict M. Escoto, *PhD FCAS*, Aon Benfield
- Stu Sadwin FCAS, *SVP & Chief Corporate Actuary*, Philadelphia Insurance Co.

Background:

Relevant literature:

- *Advanced Correlations* (2012 MetaRisk® Conference), *Steve White*
- *The Common Shock Model* (Variance Vol. 1/Issue 1 1997) *Glenn Meyers*
- The Calculation of Aggregate Loss Distribution from Claim Severity and Claim Count distributions (PCAS, LXX, 1983), *Philip Heckman*, *Glenn Meyers*

Common Shock modeling *(a.k.a. Contagion modeling)*

Main purpose: to account for the *systematic* uncertainty within Insurance data.

- Can be applied within both Frequency and Severity modeling:
  - **Claim Counts (Frequency) distributions:**
    - Exposure, changes over-time.
    - IBNR claims must be estimated.
    - External drivers can cause change in claim frequencies:
      - Severe recession → increase fire claims
  - **Claim Size (Severity) distributions:**
    - External drivers of severities:
      - Inflation
      - Underwriting cycle
      - Macroeconomic factors

Common Shock Model – drivers

- Industry
  - Generally similar risk characteristics (e.g. long vs short tail), market cycles
- Claims Inflation
  - Losses tend to increase over time for all or most lines of business due to economic, legal, judicial and social inflation
- Underwriting Cycle
  - Inadequate pricing and reserve weakening tend to occur in more than one line of business in soft market
- Model or Pricing Bias
  - Inadequate pricing and reserve deficiency tend to occur in more than one line of business due to same or similar models, assumptions, processes and approaches used

Common Shock modeling

Frequency

Let N be the claim count RV", for a *single* line of business: (Ex. Poisson dist)

$$f(N) = \frac{\lambda^N e^{-\lambda}}{N}$$

Introduce another *RV*: *C* …. *'apply'* C to the parameters of the distribution of N

$$f\big|_C(N) = \frac{(C\lambda)^N e^{-(C\lambda)}}{N} \qquad \text{where:} \quad C \sim Dist(\,mean = 1, var = c\,)$$

….. $c = Var(C)$ is a *scalar valued* parameter, the "*Frequency Contagion parameter*".

Severity

Let: X be the loss size *R.V.*, for a *single* line of business

Introduce another *RV*: β …. multiply each realization of X, by the *same* random draw of β

$$\beta \cdot X_k \qquad \text{where:} \quad \beta \sim Dist(mean = 1, var = b\ )$$

…. $b = Var(\beta)$ is called *Severity Contagion parameter.*

The effect on Frequency

Now, denote by N^* the claim count, under common shock, i.e. $f(N^*) = f|_C(N)$.

- Common shock ***preserves** the **mean***:

$$E(N^*) = E_C[E_N(N \mid C)] = \lambda = E(N)$$

- However, the ***variance*** will be increased: (depends on distribution used)

$$Var(N^*) = Var_C\big[E[N|C]\big] + E_C\big[Var[N|C]\big] = \lambda(1 + c \cdot \lambda) \qquad (N \sim \boldsymbol{Poisson}(\lambda))$$

$$Var(N^*) = \lambda\big(1 + \lambda(c + c\gamma + \gamma)\big) \qquad (N \sim \boldsymbol{NegBin}(\lambda_i, \gamma_i))$$

$$Var(N^*) = \frac{\hat{n}\hat{p}\,(1-\hat{p}) + c\hat{p}^*\left[\hat{n}\hat{p}\left(1-\frac{\hat{p}_i}{\hat{p}^*}\right)+(\hat{n}\hat{p})^2\left(\frac{1}{\hat{p}^*}-1\right)\right]}{1+c\hat{p}^*} \qquad (\hat{N} \sim \boldsymbol{Binomial}(\hat{n}, \hat{p}))$$

- The "frequency contagion RV", C, can follow *<u>any</u>* distributional form.

  The only restrictions are:

  1. The distribution must have ***positive support***
  2. The mean must be 1: $E[C] = 1$

The effect on Severity

As was the case for the frequency contagion RV, the *Severity Contagion RV* $\boldsymbol{\beta}$ can follow <u>*any*</u> distributional form. The only restrictions are:

1. The distribution of $\boldsymbol{\beta}$ must have ***positive support***
2. The *mean* of $\boldsymbol{\beta}$ must be 1: $E[\beta] = 1$

To denote the (arbitrary) distribution used to model severity, we write:

$$X_i \sim RanDist\left(E[X_i] = \mu_{x_i}, Var[X_i] = \sigma_{x_i}^2 \right)$$

Severity Common Shock has the following impact:

I. The *mean* Severity remains <u>unchanged</u>:

$$E[\beta X_i] = E[\beta]E[X_i] = 1 \cdot \mu_{x_i} = \mu_{x_i} = E(X_i)$$

II. However, the *dispersion* of the distribution of severities is <u>increased</u>:

$$Var[\beta X_i] = \sigma_{x_i}^2 + b\left(\mu_{x_i}{}^2 + \sigma_{x_i}^2\right)$$

The effect on the Collective Risk (Aggregate Loss)

Let S^* denote the Aggregate loss, under both Frequency and Severity common shock:

$$S^* = \sum_{i=1}^{N^*} \beta X_i$$

Then, it can be shown that:

$$\boxed{Var[S^*] = \underbrace{Var[X_i]E(\,N^*) + (E[X_i])^2 Var(\,N^*)}_{\text{Traditional Collective Risk model (slightly inflated due to increased } Var[\,N^*] \text{ from frequency contagion)}} + \underbrace{b \cdot \{Var[X_i]E(\,N^*) + (E[X_i])^2 E(N^{*2})\}}_{\text{Additional dispersion due to severity contagion } b \text{ and frequency contagion } E[N^{*2}]}}$$

Note: This equation incorporates both frequency and severity contagion, and *does* <u>*not*</u> depend on the distributional form of the claim count *RV* (Poisson, Negative Binomial, or Binomial).

In the particular case that N follows a Poisson distribution, this becomes:

$$Var[S^*] = \lambda(1+b)(\mu_x^2 + \sigma_x^2) + \lambda^2 \mu_x^2 [b + c + bc]$$

Since for a Poisson distribution: $E(\,N^*) = \lambda$ and $Var(\,N^*) = \lambda(1 + c \cdot \lambda)$

Implementation – Across LOBs

Assume that:

- There are K lines-of-business, and
- The claim count for line i is modeled with a *Poisson distribution* with mean λ_i , $i = 1, 2 \cdots, K$
- The values of the Frequency distribution parameters λ_i , $i = 1, 2 \cdots, K$, and
- The best-fitting Severity distribution, for each line, and values of the parameters (μ_i , σ_i) $i = 1, 2 \cdots, K$ have been determined, in some manner.

Set values c and b for the Frequency & Severity common shock.

On each iteration:

Step 1: Randomly draw a values of the common shock *RVs*: Ex: $\boldsymbol{C'} = 1.05$ and $\boldsymbol{\beta'} = 0.99$

$$C \sim Dist(\ E[C] = 1, Var[C] = c\ )$$
$$\beta \sim Dist(\ E[\beta] = 1, Var[\beta] = b\ )$$

Step 2: Generate the number of claims for each LOB: (Note: same value of $\boldsymbol{C'}$ used)

$$\boldsymbol{N'_1} \sim f(N_1) = (\boldsymbol{C'}\lambda_1)^{N_1} e^{-(\boldsymbol{C'}\lambda_1)} / N_1 \quad \rightarrow \quad \boldsymbol{N'_1 = 3}$$
$$\boldsymbol{N'_2} \sim f(N_2) = (\boldsymbol{C'}\lambda_2)^{N_2} e^{-(\boldsymbol{C'}\lambda_2)} / N_2 \quad \rightarrow \quad \boldsymbol{N'_2 = 7}$$
$$\vdots$$
$$\boldsymbol{N'_k} \sim f(N_k) = (\boldsymbol{C'}\lambda_k)^{N_k} e^{-(\boldsymbol{C'}\lambda_k)} / N_k \quad \rightarrow \quad \boldsymbol{N'_k = 4}$$

Implementation – Across LOBs

Step 3: Randomly draw N_i' values from the Severity distribution for line i, for $i = 1, 2 \cdots, K$.

$$N_1' = 3 \quad \longrightarrow \quad X_{1_1}, X_{1_2}, X_{1_3} \sim Dist_1\left(\mu_1, \sigma_{x_1}^2\right)$$
$$N_2' = 7 \quad \longrightarrow \quad X_{2_1}, X_{2_2}, \cdots\cdots, X_{2_7} \sim Dist_2\left(\mu_2, \sigma_{x_2}^2\right)$$
$$\vdots$$
$$N_k' = 4 \quad \longrightarrow \quad X_{k_1}, X_{k_2}, X_{k_3}, X_{k_4} \sim Dist_k\left(\mu_k, \sigma_{x_k}^2\right)$$

Step 4: Multiply all severity values, drawn from each line, by $\boldsymbol{\beta}' = 0.99$

Line 1: $\beta' X_{1_1},\ \beta' X_{1_2},\ \beta' X_{1_3}$

Line2: $\beta' X_{2_1}, \cdots\cdots, \beta' X_{2_7}$

$\vdots$

Line k: $\beta' X_{k_1},\ \beta' X_{k_2},\ \beta' X_{k_3},\ \beta' X_{k_4}$

Note: Since the same value of β' is used for each line-of-business, *correlation* is induced.

Repeat steps 1-4.

Correlation between Claim counts RVs

- The induced correlations will <u>only</u> depend on: (Regardless of the frequency distribution used)
  1. *Parameters* of the Frequency distribution
  2. The "*contagion parameter*" value, c.

Ex: Poisson: $\rho_{N_i,N_j} = \sqrt{\frac{c\lambda_i}{1+c\lambda_i}}\sqrt{\frac{c\lambda_j}{1+c\lambda_j}}$

- Even though the same contagion parameter (c) is used, *across all lines-of-business*:
  - ρ_{N_i,N_j} will <u>NOT</u> equal ρ_{N_m,N_n} , **unless** $(\lambda_i, \lambda_j) = (\lambda_m, \lambda_n)$
    - Hence, ***only one parameter***, c, will induce a whole, non-constant, correlation matrix.
    - The induced correlation matrix will be *"automatically" determined,* by:
      - The distribution, of each line.
      - The value of the *contagion parameter*, c.

Correlation between Aggregate Losses

Let S_k^* denote the aggregate losses, for the k^{th} LOB, under both Frequency and Severity common shock:

$$S_k^* = \sum_{i=1}^{N^*} \beta Z_i$$

Then, the correlation between the Aggregate losses, from lines k and j, under both frequency and severity common shock is:

$$\rho_{S_k^*, S_j^*} = \frac{Cov\left(S_k^*, S_j^*\right)}{\sqrt{Var\left[S_k^*\right] Var\left[S_j^*\right]}} = \frac{\lambda_k \mu_k \lambda_j \mu_j \cdot (cb + b + c)}{\sqrt{(\Sigma_k + b \cdot [\Sigma_k + (\mu_k \lambda_k)^2])\left(\Sigma_j + b \cdot \left[\Sigma_j + \left(\mu_j \lambda_j\right)^2\right]\right)}}$$

where:

- $\lambda_{\mathbf{k}} = E(N)$
- $\mu_{\mathbf{k}} = E[Z_i]$
- $\Sigma_{\mathbf{k}} = Var[Z_i]E(N^*) + (E[Z_i])^2 Var(N^*)$
- $c =$ frequency contagion parameter
- $b =$ severity contagion parameter

Proposed **version of** (severity) **Common-Shock**

Tokio Marine Technologies LLC

Motivation for Proposed method

Recall: If one fits X_k to the empirical data, and then applies common shock, then:

$$Var[\boldsymbol{\beta} X_k] = \sigma_{x_k}^2 + b\left(\mu_k^2 + \sigma_{x_k}^2\right) > \sigma_{x_k}^2 = Var[data]$$

***i.e.*:** Introducing common shock to a claim size RV that has been, strictly, calibrated to the observed data, results in a set of simulated losses that have a larger variance than that of the observed data.*

- $Dist_{X_k}(\mu, \sigma_{X_k}^2)$ is the best-fitting Severity distribution to the observed (sample) data, from the k^{th} line-of-business.

Proposed method:

1. Provides a methodology that can be used to induce correlation across a set of claim size RVs, *without* inflating the variance.

2. For practical purposes, boils-down to a *calibration* problem.
   - Calibration schemes, which are consistent with the view that the observed loss severities, X_k, arise from a loss process with a common shock structure, are presented.

* This is can be viewed as a desireable charactistic of common shock.

Proposed version severity common shock.

The proposed approach to severity common shock is a *refinement* of the approach in the literature, namely: $\boldsymbol{\beta} X_k$.

View the observed loss severities X_k as having been produced by the product of two RVs: $\boldsymbol{\beta} Z_k$, and calibrate such that the overall resultant variance of the product approximates the observed variance:

$$Var(X_k) \approx Var[\beta Z_k]$$

Where:

- X_k is the observed, empirical, claim severity data.
- β is the *severity contagion* RV, s.t. $\beta \sim Dist(E(\beta) = 1, Var(\beta) = b)$

  β represents the ***systematic*** component of the losses process.
- Z_k the *unobserved*, true underlying, loss process.

  Z_k represents the ***idiosyncratic*** component of the losses process .

We investigate one, straight-forward, calibration approach in the following.

Case Studies

Case Study #1: Property Natural Peril Severity Data

We first investigate the assumptions for a *single line-of-business*:

Data: 670 claims between years 2003 and 2012

- Property Natural Peril (severe convective storm)
- For a single company.
- Occurrence basis.
- Losses over 10-years: 2003 – 2012
- Loss sizes are in units of $1,000.

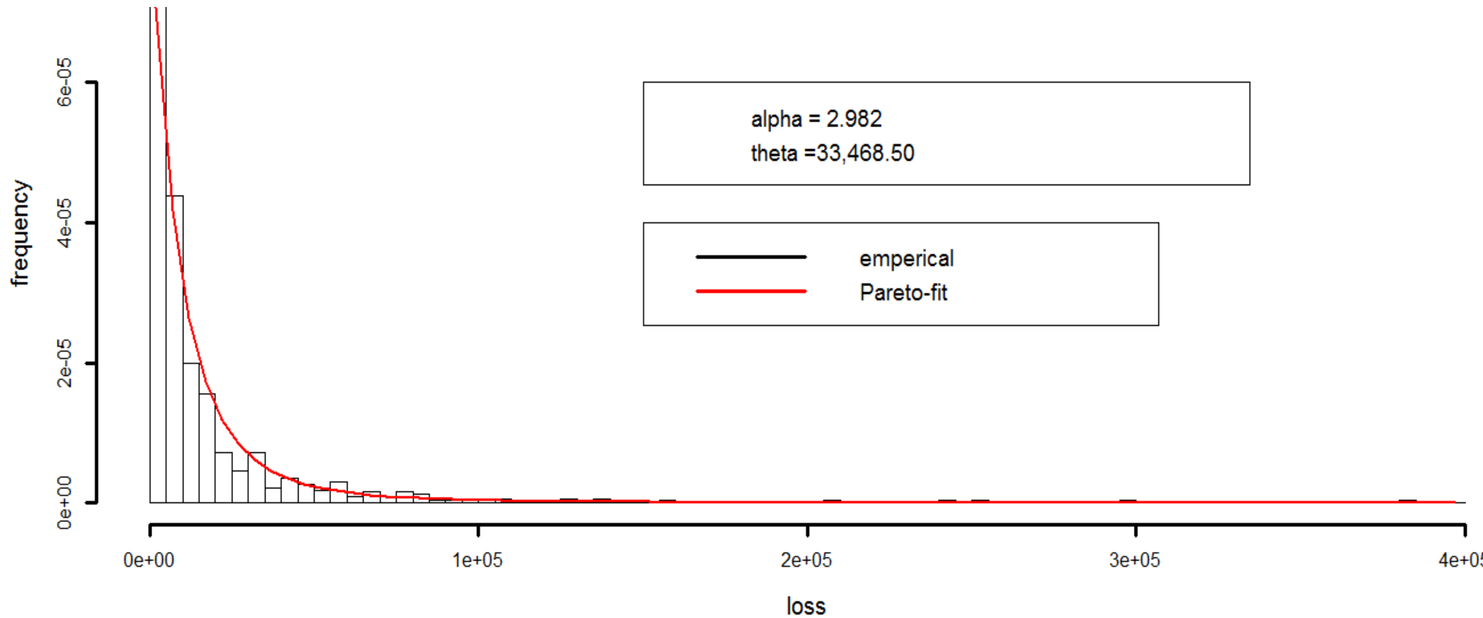
Frequency calibration:

- We use a Poisson distribution for the claim counts.
- The parameter value λ is set equal to the empirical average annual claim counts:
  - $\lambda = \bar{X} = 67.$
  - Solve: $Var(N) = \lambda(1 + c \cdot \lambda)$ for c:
    - $c = \frac{\widehat{Var}(N)}{\lambda^2} - \frac{1}{\lambda} = \approx 0.115$
      - $\widehat{Var}(N) = sample\ variance$ and $\lambda = \bar{X} = 67$

Case Study #1: Property Natural Peril Severity Data

Severity calibration:

- Using the per-claim severity data, over the full 10-years:
  - The best-fitting distribution to the per-claim severity, $\boldsymbol{X}$, is:
    - ***Pareto*** *distribution,* with
      - $\alpha_{MLE} = \textit{shape parameter} = 2.982$, and
      - $\theta_{MLE} = \textit{scale parameter} = 33{,}468$



Case Study #1: Property Natural Peril Severity Data

But ….. we need the distribution of the *pure*, *underlying*, loss process; $\boldsymbol{Z}$

Let:

$$S^* = \sum_{i=1}^{N^*} \beta Z_i$$

$$\Longrightarrow \; Var[S^*] = Var[Z_i]E(\,N^*) + (E[Z_i])^2\; Var(\,N^*) + b \cdot \{Var[Z_i]E(\,N^*) + (E[Z_i])^2 E(N^{*2})\}$$

Note: This equation incorporates both frequency and severity contagion, and does not depend on the distributional form of the claim count RV (Poisson, Negative Binomial, or Binomial).

And:

- $E(\,N^*) = \lambda$
- $Var(\,N^*) = \lambda(1 + c \cdot \lambda)$

Hence:

$$Var[S^*] = \sigma_z^2\lambda + \mu_z^2\lambda(1 + c\lambda) + b \cdot \{\sigma_z^2\lambda + \mu_z^2[\lambda(1 + c\lambda) + \lambda^2)]\} = \lambda(1 + b)(\mu_z^2 + \sigma_z^2) + \lambda^2\mu_z^2[b + c + bc]$$

$$\Longrightarrow \; b = \frac{Var[S^*] - \lambda\sigma_z^2 - \lambda b(\mu_z^2 + \sigma_z^2) - \lambda\mu_z^2 - \lambda^2\mu_z^2 c}{\lambda^2\mu_z^2(1 + c)}$$

Case Study #1: Property Natural Peril Severity Data

Calibration of the *pure*, *underlying*, loss process; $\boldsymbol{Z}$

Set: $Var[S^*] = \widehat{Var}[S^*] =$ *empirical* variance of the ***annual aggregate losses***:

$$\Longrightarrow \quad b = \frac{\widehat{Var}[S^*] - \lambda\sigma_z^2 - \lambda b(\mu_z^2 + \sigma_z^2) - \lambda\mu_z^2 - \lambda^2\mu_z^2 c}{\lambda^2\mu_z^2(1+c)}$$

And use the following values (all based on the sample):

- The mean of the Poisson frequency distribution ($\lambda = 67$)
- The modeled mean of the per-claim severity distribution ($\mu_z = 17{,}842$)
- The modeled variance of the per-claim severity ($Var[\beta Z] = \sigma_x^2 = 32{,}329^2$)
- The *empirical* variance of the annual aggregate losses ($Var[S^*] = 697{,}245^2$)
- The frequency contagion parameter ($c = 0.115$)

$$\Longrightarrow \qquad b = \frac{Var[S^*] - \lambda \cdot Var[\beta Z] - \lambda\mu_z^2 - \lambda^2\mu_z^2 c}{\lambda^2\mu_z^2(1+c)} \approx \mathbf{0.13}$$

And:

$$\Longrightarrow \qquad \sigma_{\boldsymbol{z}}^2 = \frac{Var[\beta Z] - b\mu_z^2}{1+b} = 29{,}634^2$$

Case Study #1: Property Natural Peril Severity Data

Finally, using:

- $\mu_z = 17{,}842$
- $\sigma_z^2 = 29{,}634^2$

The distribution of the *pure*, *underlying*, loss process; $\boldsymbol{Z} \sim Pareto(\alpha_z, \theta_z)$ are determined to be:

- $\alpha_z = 3.137$, and
- $\theta_z = 38{,}133$

Now, we perform a simulation study of the Aggregate Annual *Layered* Losses

$$S^* = \sum_{i=1}^{N^*} \beta Z_i$$

Using the parameter values calibrated from the actual, empirical, data:

- $\boldsymbol{N}|\boldsymbol{C} \sim Poisson(\boldsymbol{C} \cdot \mathbf{67})$ and $\boldsymbol{C} \sim Gamma(\, E[C] = 1, Var[C] = \mathbf{0.115}\,)$
- $Z_i \sim Pareto(\mathbf{3.137},\ \mathbf{38{,}133}\,)$ and $\beta \sim Gamma(E(\beta) = 1,\ \ Var(\beta) = \mathbf{0.13})$

Case Study #1: Property Natural Peril Severity Data

We use the following Layers, of the *AAL*:

$$
\begin{cases}
Layer1: \mathbf{0 - 7.5M} \\
Layer2: \boldsymbol{7.5M - 20M} \\
Layer3: \boldsymbol{20M - 45M} \\
Layer4: \boldsymbol{45M - 70M} \\
Layer5: \boldsymbol{70M - 100M} \\
Layer6: \boldsymbol{100M - 200M}
\end{cases}
$$

Simulation procedure:

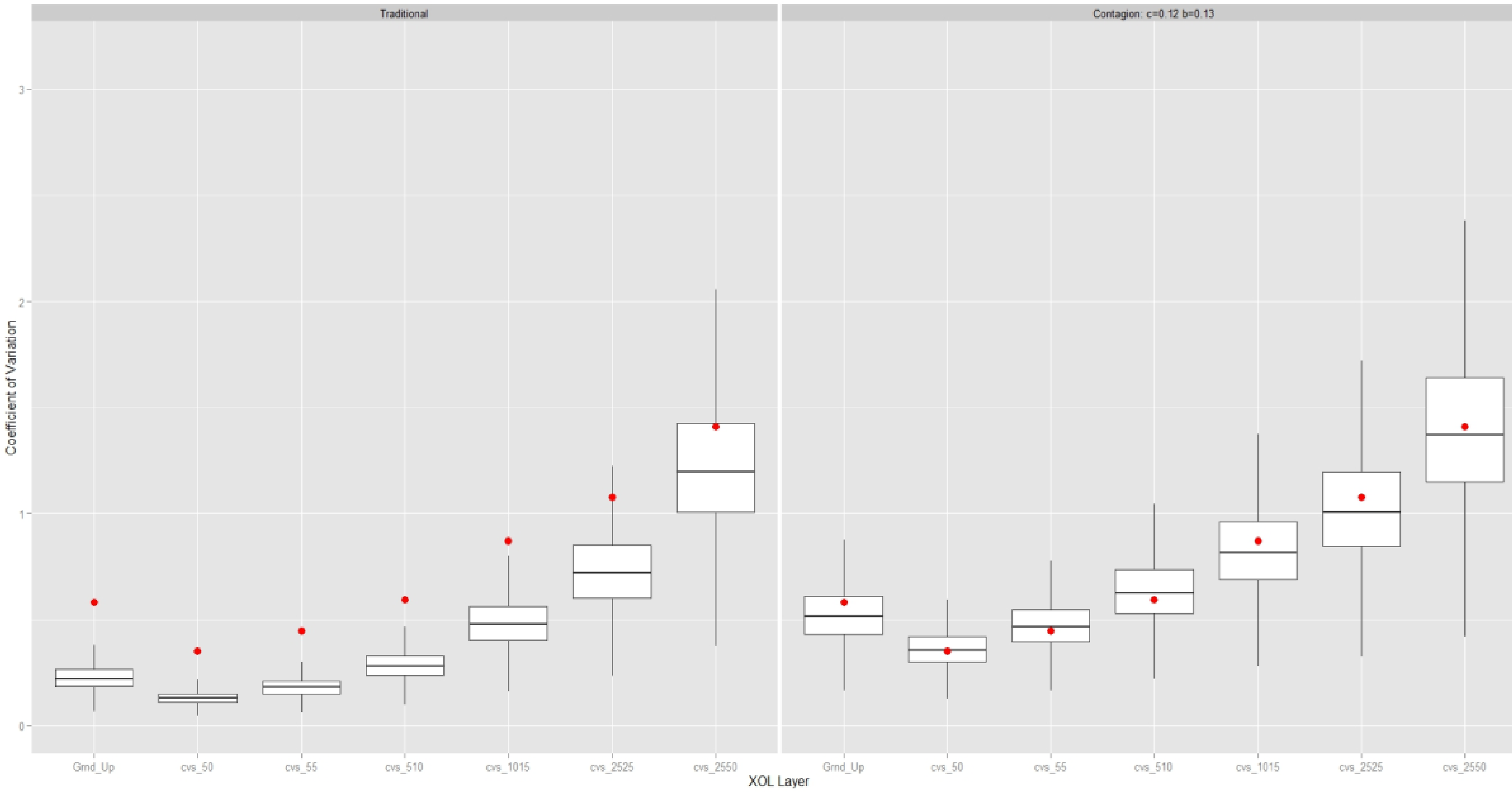
1. For each iteration of the simulation, generate 10 years of Aggregate Annual Losses under both:
   - The Traditional method
   - The Contagion method.
2. For each of the pre-defined Loss Layers (above) calculate the ***Annual Aggregate losses*** within each layer.
3. For each layer, calculate the CV of the Aggregate Annual losses, over the 10-years.

Repeat 100,000 times.

At this point, we have $100{,}000$ $10\text{-year}\ CV$ estimates, ***for each layer***

Case Study #1: Results

Traditional Collective Risk model | ***Proposed* Aggregate Contagion method**

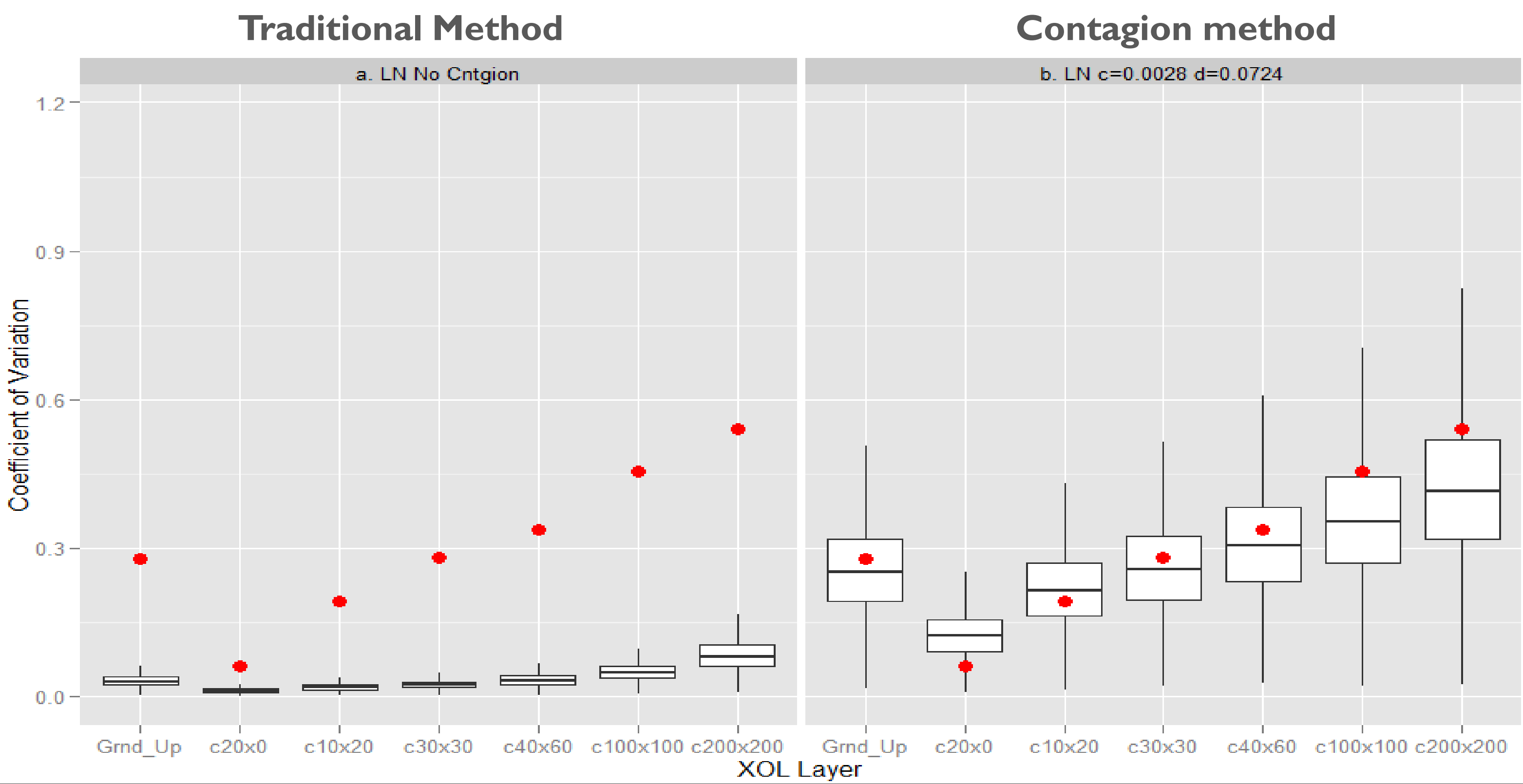


TOKIO MARINE TECHNOLOGIES

Case Study #2: XYZ Insurance *GL* Claims Data

Provided loss data on:

- LOB: GL claims on transaction-level
- Losses over 5-years: 2009 – 2013
- Occurrence basis, Losses recorded after policy-limits and deductibles



Case Study: Summary

- **Traditional Collective Risk modeling**
  - No Frequency Correlation available
  - No Severity Correlation available within the LOBs
  - Produces *flawed* estimates of the Variation of Aggregate Annual claims.
    - Underestimates the Variation.

- **Traditional Collective Risk *underestimates*:**
  - Volatility of aggregate losses.
  - Volatility of aggregate losses within XOL layers.
  - Risk measure in terms of Spectral risk, TVaR or VaR.

- **Effect of Traditional Collective Risk Model:**
  - May *underestimate* Capital Requirements.
  - May *underprice* reinsurance contracts.

Part 4 : Reinsurance Pricing Application

Tokio Marine Technologies LLC

Background

- **Some reinsurance programs may consist of multiple sections and/or two or more layers covering the same or similar underlying exposure across portfolios.**
  - These layers are **closely related** to one another
  - Sections across treaties may be related or correlated.
- **However, pricing actuaries price those layers treaty by treaty independently.**
- **In order to price multi-section and/or multi-layered loss properly, one has to model the inter-layer and inter sectional relationship.**

Input

❑ Treaty Information

- Required field: Number of trials

Treaty Information					
Office		Cedent		Start Date	
Treaty Type	XL	Broker		End Date	
Simulation Date		Number of Trials	10,000	Total Premium	
Description					

❑ Contract Detailed Information

XL Info									
Layer	Attachment	Limit	Premium	TMR %	Reinstatements				
					Num of Rates	Rate (%)	Rate (%)	Rate (%)	
1									
2									
3									
4									
5									
6									
7									
8									
9									
10									

Input--Continued

- **Contagion Environment**
  - World wide Contagion environments covering
    - Industry
    - Lines of Business
    - Region
    - Sub-region
    - Pricing
    - Etc.

Contagion Environment		
Environment	Frequency, c	Severity, b
Industry Env 0	0.01	0.01
Industry Env 1	0.01	0.01
Geo Env 1	0.01	0.01
Pricing Env 1	0.05	0.05
Pricing Env 2	0.05	0.05

Input--Continued

- **Suggested Contagion Environments**
  - Frequency Environment and/or Severity Environment
  - Overall Contagion Environment Level
    - Low [0, 0.05]
    - Medium [0.05, 0.1]
    - High [0.1, 0.5]
    - Extreme [0.5, 1.0]
  - 5 Major Groups of Underlying Contagion Environments
    - Independent
    - Multiplicative

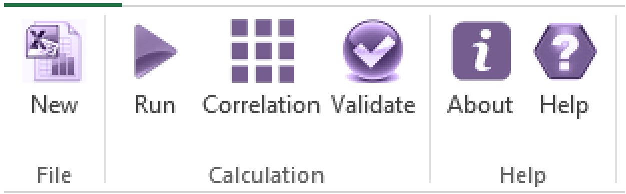
Input--Continued

- **Section Profile (Part 1)**
  - Risk Name
  - Each Risk Contagion Environment
    - One contagion environment should not appear more than once in one risk due to the independent assumption.
    - Blank is allowed indicating the risk is NOT exposed to contagion environment

Name	Contagion Environment		
	Industry	Geo/Legal	Pricing
Attritional Loss	Industry Env 0	Geo Env 1	Pricing Env 2
Large Loss		Geo Env 1	
Other Loss	Industry Env 1		Pricing Env 1
Motor loss	Industry Env 0	Geo Env 1	Pricing Env 2

Execution Button

- ***Correlation* button to generate the implied loss size correlation and analytical mean/STDEV.**
  - Correlation is NOT rank correlation but linear correlation of loss size
  - Correlation depends on the environment and the loss profile (frequency and severity distribution) settings
  - Complete frequency and severity distribution is required
- ***Run* button to generate simulation results.**
- ***Validate* button to validate all input cells**
- ***Help* button to bring up the users manual.**



TOKIO MARINE TECHNOLOGIES

Output

Implied Theoretic Correlation Matrix

Implied Correlation							
Section	Attritional Loss	Large Loss	Other Loss	Motor loss			
Attritional Loss	1.00	0.0469	0.00	0.2138			
Large Loss	0.0469	1.00	0.00	0.0828			
Other Loss	0.00	0.00	1.00	0.00			
Motor loss	0.2138	0.0828	0.00	1.00			

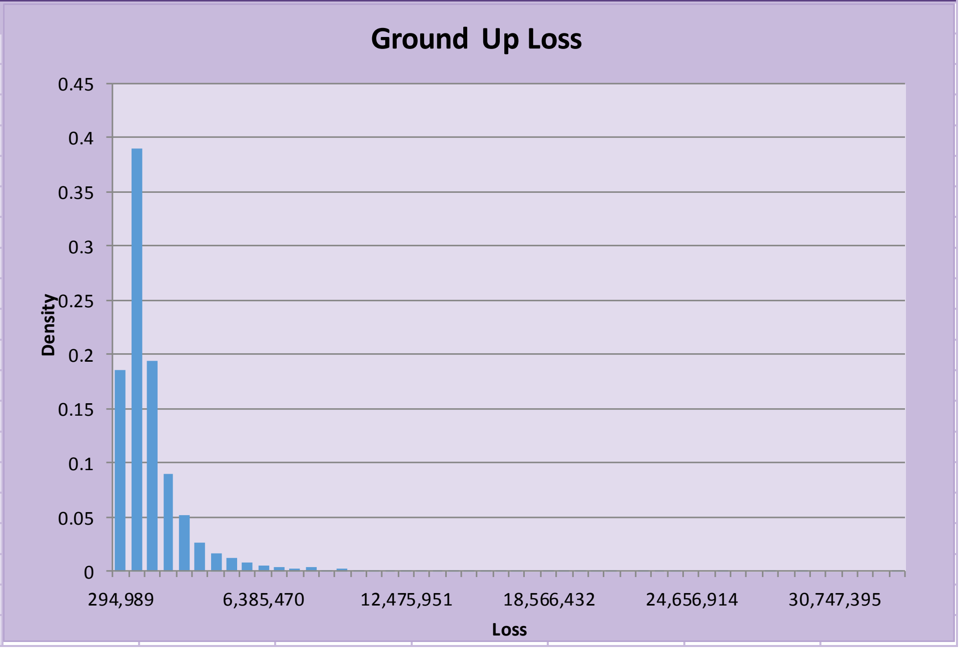
Analytical Mean and STDEV

Contagion Effect		
Agg Mean	Agg STDV wo	Agg STDV w
1,989,874	1,707,743	1,925,323
2,904	1,048	1,131
702	212	330
1,333	632	831

Output--Continued

❑ Achieved aggregated and each risk loss distribution

Ground Up Loss	
Statistic	Value
Mean	1,990,797.71
STDV	1,856,712.42
Minimum	294,988.91
1st Percentile	546,902.40
5th Percentile	712,221.02
95th Percentile	4,739,750.89
99th Percentile	9,306,905.82
Maximum	34,130,995.34
99% Shortfall	14,677,773.32



Part 5 : Concluding Remarks

Contagion Model: Summary

- **Contagion Model**
  - Induce frequency correlation using frequency contagion
  - Induce severity correlation through severity contagion
- **Easy to understand through implied correlation and volatility**
- **Easy to implement within high performance simulation**
- **Represent the state-of-the-art in correlation treatment (*Meta Risk, ReMetrica*)**
- **Have demonstrated that contagion exists using real life data.**
- **Have showed that the contagion can better estimate:**
  - Volatility of aggregate losses
  - Risk measure in terms of Spectral risk, TVaR or VaR
  - Expected loss of XOL layers

Common Shock Model Overview

❑ Common Shock model

- Is an approach to model the correlation of losses within a line of business or between lines of business
- Assumes that different losses are linked by a common shock (or variation) in the parameters of each line's loss model
- Models the correlation on loss sizes and/or frequency counts
- Is free of loss distribution
- Achieved correlation is dependent on underlying loss volatility and Contagion parameters
- Strong (100%) correlation may not be achievable if underlying losses have huge volatilities
- This limitation is true due to the linear correlation measure
- Achieved correlation is known prior simulation

Appendix – Details Traditional method

Appendix – Correlation frequency (Poissons)

Since the same *Frequency Contagion RV*, C, is used within each N_i:

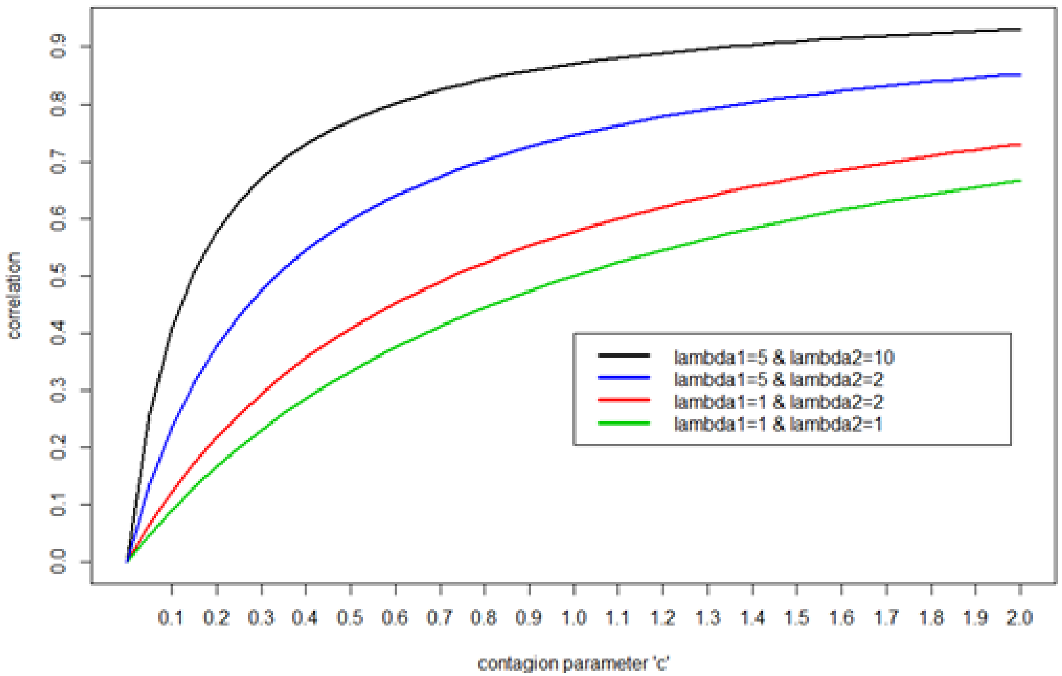
- N_i, for $i = 1, 2 \cdots, K$ are *correlated*:

If $N_{\boldsymbol{i}} \sim \boldsymbol{Poisson}(\lambda_{\boldsymbol{i}})$, then $define$: $N_{\boldsymbol{i}} | \boldsymbol{C} \sim Poisson(\boldsymbol{C}\lambda_{\boldsymbol{i}})$ where $\boldsymbol{C} \sim Dist(\, E[C] = 1, Var[C] = c\,)$

Then, for $1 \leq i, j \leq K$, and $i \neq j$, the correlation between N_i, N_j is:

$$\rho_{N_i,N_j} = \sqrt{\frac{c\lambda_i}{1 + c\lambda_i}}\sqrt{\frac{c\lambda_j}{1 + c\lambda_j}}$$

- As $c \to 0 \implies \rho_{N_i,N_j} \to 0$
  - weak, or absent, contagious environment
- As $c \to \infty \implies \rho_{N_i,N_j} \to 1$
  - a strong contagious environment



TOKIO MARINE TECHNOLOGIES

Appendix – Correlation frequency (Neg Bin's)

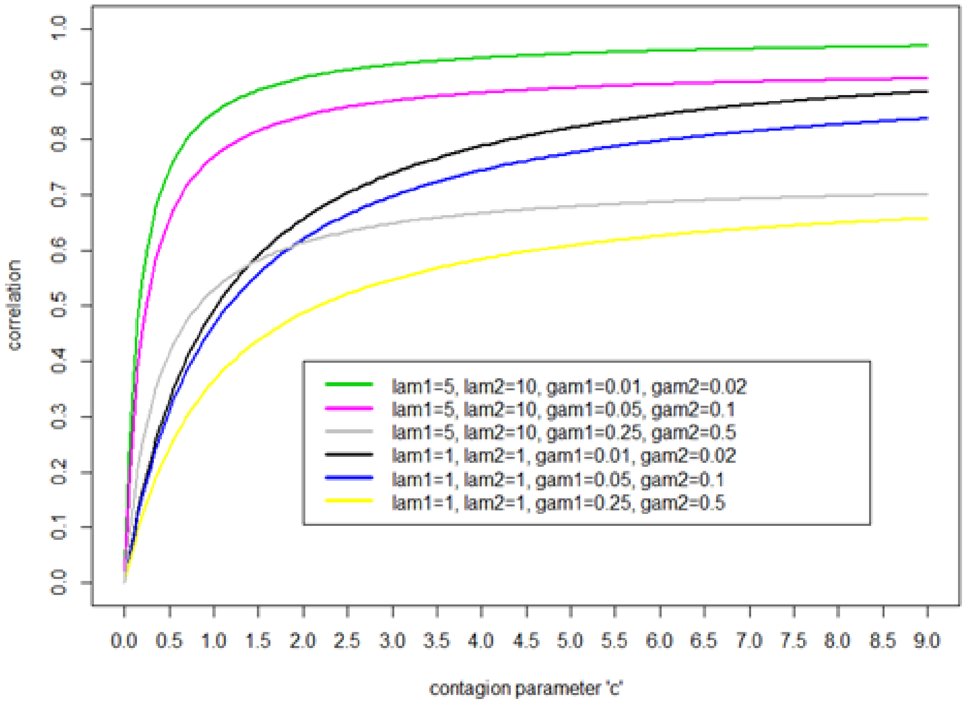
If $N_i \sim \boldsymbol{NegBinomial}(\lambda_i, \ \gamma_i)$ then define $N_i | \boldsymbol{C} \sim NegBin(\boldsymbol{C}\lambda_i, \ \gamma_i)$

where: $C \sim Dist(\ E[C] = 1, Var[C] = c\ )$ and $(\lambda_i = \text{mean} \quad \gamma_i = \text{dispersion parmeter})$

For $1 \leq i, j \leq K$, and $i \neq j$, the correlation between N_i, N_j is:

$$\rho_{N_i,N_j} = \sqrt{\frac{c\lambda_i}{1 + \lambda_i(c + c\gamma_i + \gamma_i)}} \sqrt{\frac{c\lambda_j}{1 + \lambda_j(c + c\gamma_j + \gamma_j)}}$$

- As $c \to 0 \implies \rho_{N_i,N_j} \to 0$
  - weak, or absent, contagious environment
- As $c \to \infty \implies \rho_{N_i,N_j} \to \sqrt{\frac{1}{1+\gamma_1}} \sqrt{\frac{1}{1+\gamma_2}}$
  - *Strong* contagious environment



Appendix – application to Binomials

- Denote the best-fitting Binomial distributions to each of the K lines, by:
  - $\widehat{N}_i \sim \boldsymbol{Bin}(\hat{n}_i, \hat{p}_i)$ for $1 \leq i \leq K$

- Let: $\hat{p}^* = \max(\hat{p}_1, \hat{p}_2, \dots, \hat{p}_n)$

To simulate Claim counts from the line:

- Adjust the parameter, p, of each distribution, by the constant ratio: $\hat{p}_i/\hat{p}^*$

$$N_i|p \sim Bin\left(\hat{n}_i, \left(\frac{\hat{p}_i}{\hat{p}^*}\right)p\right) \quad \text{where} \quad p \sim Beta\left(\alpha = \frac{1}{c},\ \beta = \frac{1}{c}\left(\frac{1-\hat{p}^*}{\hat{p}^*}\right)\right)$$

Hence:

- $E(N_i) = E_p\left[E_{N_i}(N_i \mid p)\right] = \hat{n}_i \cdot \hat{p}_i$ (the mean of the best-fitting Binomial distribution, to that line)

$$Var(N) = \frac{\hat{n}_i\hat{p}_i(1-\hat{p}_i) + c\hat{p}^*\left[\hat{n}_i\hat{p}_i\left(1-\frac{\hat{p}_i}{\hat{p}^*}\right) + (\hat{n}_i)^2(\hat{p}_i)^2\left(\frac{1}{\hat{p}^*}-1\right)\right]}{1+c\hat{p}^*}$$

Appendix - Variance Binomials under common shock

$$Var(N) = \frac{\hat{n}_i\hat{p}_i(1-\hat{p}_i) + c\hat{p}^*\left[\hat{n}_i\hat{p}_i\left(1-\frac{\hat{p}_i}{\hat{p}^*}\right) + (\hat{n}_i)^2(\hat{p}_i)^2\left(\frac{1}{\hat{p}^*}-1\right)\right]}{1+c\hat{p}^*}$$

Several observations can be made from this expression:

- $c \to 0 \implies Var(N_i) \to \hat{n}_i\hat{p}_i(1-\hat{p}_i)$ (variance of *best-fitting* Binomial, to business line i)
- $Var(N_i)$ is an *increasing* function of c.
- $c \to \infty \implies Var(N_i) \to \hat{n}_i\hat{p}_i\left(1-\frac{\hat{p}_i}{\hat{p}^*}\right) + (\hat{n}_i)^2(\hat{p}_i)^2\left(\frac{1}{\hat{p}^*}-1\right)$

Note:

- If $\hat{p}_i = \hat{p}^*$, for line i:
- $c \to \infty \implies Var(N_i) \to \hat{n}_i\hat{p}_i(0) + (\hat{n}_i)^2(\hat{p}_i)^2\left(\frac{1}{\hat{p}_i}-1\right) = (\hat{n}_i)^2\hat{p}_i(1-\hat{p}_i) > \hat{n}_i\hat{p}_i(1-\hat{p}_i)$

Appendix – Correlation frequency (Binomials)

For $1 \leq i, j \leq K$, and $i \neq j$:

$$\rho_{N_1,N_2} = \frac{\hat{n}_i \hat{p}_i \hat{n}_j \hat{p}_j \left(\frac{c(1-\hat{p}^*)}{1+c\hat{p}^*}\right)}{\sigma_i \sigma_j}$$

Where:

$$\sigma_i = \sqrt{\frac{\hat{n}_i \hat{p}_i (1-\hat{p}_i) + c\hat{p}^* \left[\hat{n}_i \hat{p}_i \left(1 - \frac{\hat{p}_i}{\hat{p}^*}\right) + (\hat{n}_i)^2 (\hat{p}_i)^2 \left(\frac{1}{\hat{p}^*} - 1\right)\right]}{1 + c\hat{p}^*}}$$

Observations:

- $c \to 0 \implies \rho_{N_1,N_2} \to 0.$
- ρ_{N_1,N_2} in an *increasing* function of c.
- $c \to \infty \implies \rho_{N_i,N_j} \to \frac{1}{\sqrt{\left(1+\frac{(\hat{p}^* - \hat{p}_i)}{\hat{n}_i \hat{p}_i (1-\hat{p}^*)}\right)\left(1+\frac{(\hat{p}^* - \hat{p}_j)}{\hat{n}_j \hat{p}_j (1-\hat{p}^*)}\right)}}$

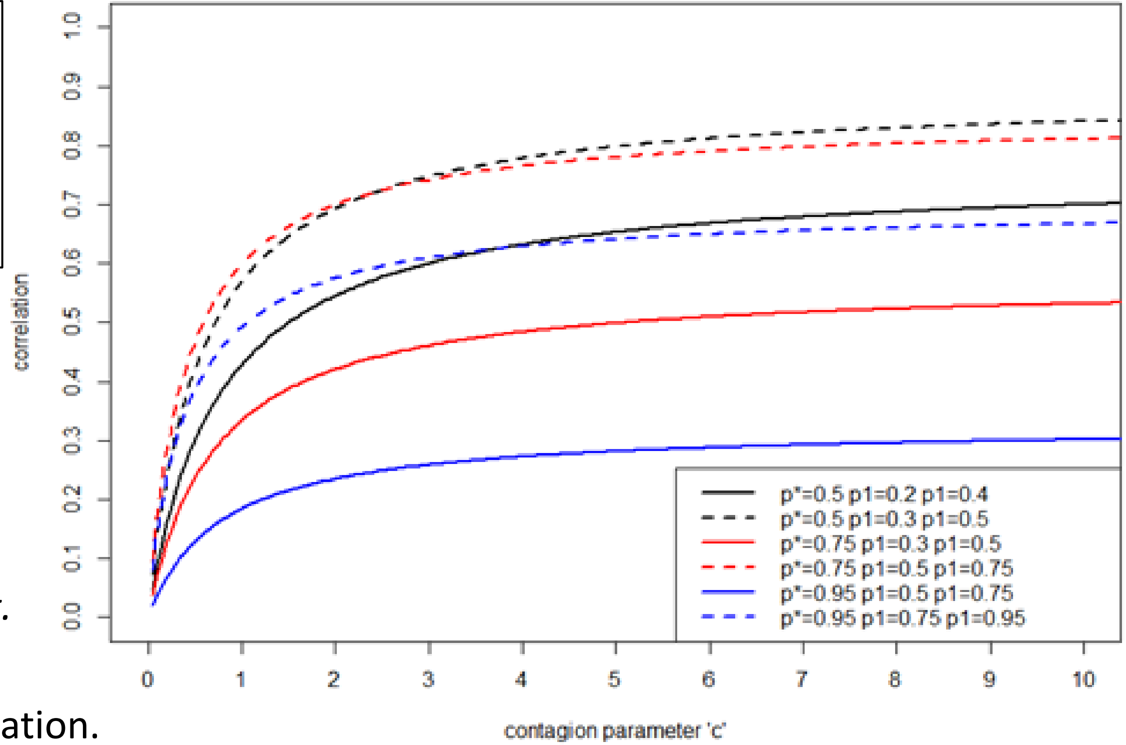
Appendix – Binomial: correlation *vs. c*

(per level of Binomial parameters, and *p**)

Recall:

As: $c \to \infty \implies$

$$\rho_{N_i,N_j} \to \frac{1}{\sqrt{\left(1+\frac{(\hat{p}^*-\hat{p}_i)}{\hat{n}_i\hat{p}_i(1-\hat{p}^*)}\right)\left(1+\frac{(\hat{p}^*-\hat{p}_j)}{\hat{n}_j\hat{p}_j(1-\hat{p}^*)}\right)}}$$



Observations:

- For a given value of $\hat{p}^* < 1$:
  $\hat{p}_i$ and $\hat{p}_j$ larger $\implies$
  $\implies$ ***higher*** correlation.
- For fixed values of $\hat{p}_i$ and $\hat{p}_j$, *s.t.* $\min\{\hat{p}_i, \hat{p}_j\} < \hat{p}^*$:
  - higher $\hat{p}^* \implies$ ***lower*** correlation.
- As $c \to \infty$:
  - If $\hat{p}_i = \hat{p}_j = \hat{p}^* < 1 \implies \rho_{N_i,N_j} \to 1$
  - If $\min\{\hat{p}_i, \hat{p}_j\} < \hat{p}^*$, then $\hat{p}^* \to 1 \implies \rho_{N_i,N_j} \to 0$

Appendix – Proposed severity contagion

Proposed method

Proposed severity Contagion - requirements

Requirements for the proposed Aggregate contagion model:

1. β be *independent* of Z_k, and
2. $E(Z_k) = \mu_{Z_k} = \mu_k = E(X_k)$
3. The distribution of β must have ***positive support***
4. The mean must be 1: $E[\beta] = 1$

These conditions ensure that:

- $E[\beta Z_k] = E[\beta]E[Z_k] = 1 \cdot \mu_{Z_k} = \mu_k = E(Z_k)$
- $Var[\beta Z_k] = \sigma_{Z_k}^2 + b\left(\mu_k^2 + \sigma_{Z_k}^2\right)$

In summary, the *Severity component* of the proposed Aggregate Contagion model is s.t.:

$\boldsymbol{\beta} Z_k$ where: $\begin{cases} Z_k \sim RanDist_k\left(E[Z_k] = \mu_k,\ Var[Z_k] = \sigma_{Z_k}^2\right) \\ \beta \sim RanDist(E(\beta) = 1,\ Var(\beta) = b),\ \text{with } b \geq 0 \end{cases}$

- such that: $E(Z_k) = \mu_{Z_k} = \mu_k = E(X_k)$ and $Var(Z_k) = \sigma_{Z_k}^2 \leq \sigma_{X_k}^2 = Var(X_k)$

Proposed severity Contagion - consistency

The proposed severity contagion model is consistent with the common shock/ contagion modeling paradigm, since:

- In the absence of a contagious environment, it should be inferred that; $\sigma_{Z_k}^2 \approx \sigma_{X_k}^2$, and hence:
  - $\sigma_{X_k}^2 = Var(X_k) \approx Var[\beta Z_k] = \sigma_{Z_k}^2 + b(\mu_k^2 + \sigma_{Z_k}^2)$, which implies that; $b \approx 0$.
- Conversely, in the presence of a strong contagious environment, it should be inferred that:
  - $\sigma_{Z_k}^2 \ll \sigma_{X_k}^2$, which, by the same argument, implies that $Var[\beta] \gg 0$, or $b \gg 0$.

Conversely, under the same assumption that: $Var(X_k) \approx Var[\beta Z_k]$, we have that:

- $b \approx 0$ implies that $\sigma_{Z_k}^2 = Var[\beta Z_k] \approx Var(X_k) = \sigma_{X_k}^2$, which implies a weak contagious environment, and:
- $b \gg 0 \implies \sigma_{X_k}^2 = Var(X_k) \approx Var[\beta Z_k] = \sigma_{Z_k}^2 + b(\mu_k^2 + \sigma_{Z_k}^2) \gg \sigma_{Z_k}^2 \implies$ strong contagious environment.