Analysis of bivariate excess losses

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Outline of the Presentation

Univariate excess losses

Let X be a random loss variable taking nonnegative values and having cumulative distribution function F and survival function S. Then the limited loss up to a retention level d is defined by

$$X_0^d = \begin{cases} X & \text{if } X \le d \\ d & \text{if } X > d \end{cases} .$$

Univariate excess losses

The loss in the layer (d, l) is defined by

$$X_d^l = X_0^l - X_0^d = \begin{cases} 0 & \text{if } X \leq d \\ X - d & \text{if } d < X \leq l \\ l - d & \text{if } X > l \end{cases} .$$

Univariate excess losses

The excess loss over a limit d is defined by

$$X_d^\infty = (X-d)_+ = X - X_0^d = \begin{cases} 0 & \text{if } X \le d \\ X-d & \text{if } X > d \end{cases}.$$

Table M charge

$$\phi(d) = \frac{\mathbb{E}[X_d^\infty]}{\mathbb{E}[X]}$$

Introduction

Excess of Loss

- The mathematics of excess losses has been studied extensively in the property and casualty insurance literature.
- See for example, Lee (1988) and Halliwell (2012), Bahnemann (2017).

Univariate excess losses

- The first moment of the excess loss has been tabulated into the Table of Insurance Charges (Table M) for use in NCCI retrospective rating plan.
- Higher moments of excess losses can be used to measure the volatility of excess losses
- They seem to be much harder to obtain and formulas for them are not readily available in the property casualty actuarial literature. One could refer to Section 2 of Miccolis (1977), Bahnemann (2017).

Purpose of the talk

- Introduce a simple formula for calculating the higher moments of the excess losses
- Show that the higher moments can be obtained directly from the Table of Insurance Charges (Table M).
- Generalize the concept of excess losses to a bivariate scenario.
- Show some applications of the theory of bivariate excess losses.

Univariate excess losses

It is well known that (see for example, Lee 1985)

$$\mathbb{E}(X_0^d) = \int_0^d S(u)du.$$

Because $X_d^l = X_0^l - X_0^d$, we

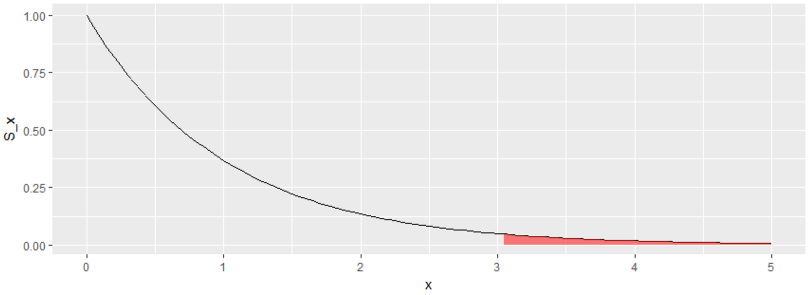
$$\mathbb{E}[X_d^l] = \int_d^l S(u)du.$$

and

$$\mathbb{E}[X_l^\infty] = \int_l^\infty S(u)du. \tag{1}$$

Univariate excess losses

Figure 1: Expected excess losses.



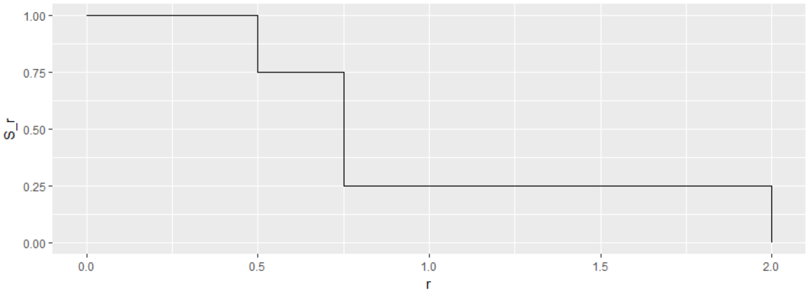
Univariate excess losses

- Consider problem 4 of Brosius (2002). Let X represent the loss ratio for a homogeneous group of insureds and was observed to have values 30%, 45%, 45% and 120% respectively.
- Let $Y = X/\mathbb{E}(X)$ be the corresponding entry ratios and thus take values 0.5, 0.75, 0.75, 2.
- Then the insurance charge at entry ratio r is just the mean excess loss function of Y evaluated at r , i.e.,

$$\phi(d) = \frac{\mathbb{E}[X_d^\infty]}{\mathbb{E}[X]} = \mathbb{E}[Y_r^\infty] = \mathbb{E}[(Y-r)_+] = R_1(r) = \int_r^\infty S_Y(y)dy.$$

Univariate excess losses

Figure 2: THe survival function of Y.



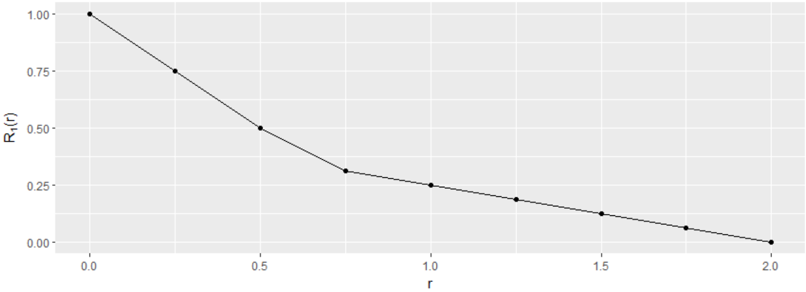
Univariate excess losses

Table 1: Calculating Table M Charge

Entry ratio (r)	# of risks	$S_Y(r)$	$R_1(r)$
0	0	1	1
0.25	0	1	0.75
0.5	1	0.75	0.5
0.75	2	0.25	0.3125
1	0	0.25	0.25
1.25	0	0.25	0.1875
1.5	0	0.25	0.125
1.75	0	0.25	0.0625
2	1	0	0

Univariate excess losses

Figure 3: Table M charge: $R_1(r)$.



Univariate excess losses

It is known in the literature that the higher moments of the excess loss X_1^∞ can be calculated iteratively from $R_1(l)$ as follows:

Proposition 2.1

Let

$$R_1(l) = \mathbb{E}[X_l^\infty],$$

and for $i \geq 1$, let

$$R_{i+1}(l) = \int_l^\infty R_i(u)du.$$

Then

$$\mathbb{E}[(X_l^\infty)^i] = (i!)R_i(l), \quad \text{for } i \geq 1. \tag{2}$$

Univariate excess losses

Proof: We use mathematical induction. For $i = 1$, equation (2) is true by definition. Assume that it is true for i, then

$$\begin{aligned} R_{i+1}(l) &= \int_l^\infty R_i(u)du \\ &= \int_l^\infty \frac{1}{i!}\mathbb{E}[(X-u)_+^i]du \\ &= \frac{1}{i!}\mathbb{E}\left[\int_l^\infty (X-u)_+^i du\right] \\ &= \frac{1}{(i+1)!}\mathbb{E}\left[(X-l)_+^{i+1}\right] \\ &= \frac{1}{(i+1)!}\mathbb{E}\left[(X_l^\infty)^{i+1}\right]. \quad \blacksquare \end{aligned}$$

Univariate excess losses

- The second moment of the excess losses $\mathbb{E}\left[(Y_r^\infty)^2\right]$ may simply be obtained by numerically integrating $R_1(r)$ and then multiplying the result by $2!$.
- Realizing that $R_1(r)$ is piecewise linear between entry ratio values, the numerical integration is implemented by

$$R_2(r) = \sum_{k\geq 0} \frac{R_1(r+k\Delta) + R_1(r+(k+1)\Delta)}{2}\Delta,$$

  where Δ is the interval between entry ratio values.

Univariate excess losses

Table 2: Calculating higher moments of excess losses using Table M

(r)	# of risks	$R_1(r)$	R_2 in layer	$R_2(r)$	$E[(Y_r^\infty)^2]$
0	0	1	0.21875	0.671875	1.34375
0.25	0	0.75	0.15625	0.453125	0.90625
0.5	1	0.5	0.1015625	0.296875	0.59375
0.75	2	0.3125	0.0703125	0.1953125	0.390625
1	0	0.25	0.0546875	0.125	0.25
1.25	0	0.1875	0.0390625	0.0703125	0.140625
1.5	0	0.125	0.0234375	0.03125	0.0625
1.75	0	0.0625	0.0078125	0.0078125	0.015625
2	1	0	0	0	0

Univariate excess losses

The second moment of the layered losses $\mathbb{E}\left[(X_d^l)^2\right]$ is also of interest. We have

$$\begin{aligned}
\mathbb{E}\left[(X_d^l)^2\right] &= \mathbb{E}\left[(X_d^\infty - X_l^\infty)^2\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] + \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^\infty)(X_l^\infty)\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] + \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^l + X_l^\infty)(X_l^\infty)\right] \\
&= \mathbb{E}\left[(X_d^\infty)^2\right] - \mathbb{E}\left[(X_l^\infty)^2\right] - 2\mathbb{E}\left[(X_d^l)(X_l^\infty)\right]. \qquad (3)
\end{aligned}$$

Bivariate excess losses

- Let (X, Y) be a pair of random loss random variables with joint distribution function $F(x, y) = \mathbb{P}(X \leq x, Y \leq y)$ and joint survival function $S(x, y) = \mathbb{P}(X > x, Y > y)$.
- We intend to calculate

$$\mathbb{E}[X_{d_x}^{l_x} Y_{d_y}^{l_y}]$$

Bivariate excess losses

Proposition 3.1

$$\mathbb{E}[X_{d_x}^{l_x} Y_{d_y}^{l_y}] = \int_{d_x}^{l_x} \int_{d_y}^{l_y} S(u,v)\,dv\,du. \tag{4}$$

Bivariate excess losses

Proof:

- first notice that

$$X_{d_x}^{l_x} Y_{d_y}^{l_y} = \int_{d_x}^{l_x} I(X > u)du \int_{d_y}^{l_y} I(Y > v)dv = \int_{d_x}^{l_x} \int_{d_y}^{l_y} I(X > u)I(Y > v)dvdu.$$

- Then we have

$$\mathbb{E}\left[X_{d_x}^{l_x} Y_{d_y}^{l_y}\right] = \int_{d_x}^{l_x} \int_{d_y}^{l_y} \mathbb{E}\left[I(x > u)I(y > v)\right] dvdu = \int_{d_x}^{l_x} \int_{d_y}^{l_y} S(u, v)dvdu. \quad \blacksquare$$

Bivariate excess losses

- With this, the covariance between $X_{d_x}^{l_x}$ and $Y_{d_y}^{l_y}$ is given by

$$Cov(X_{d_x}^{l_x}, Y_{d_y}^{l_y}) = \int_{d_x}^{l_x} \int_{d_y}^{l_y} S(u,v)dvdu - \int_{d_x}^{l_x} S_x(u)du \int_{d_y}^{l_y} S_y(v)dv,$$

- A somewhat similar formula can be found in Dhaene et. al. (1996).

Bivariate excess losses

As shown in Denuit et. al (1999), higher joint moments of the bivariate excess losses can be computed using the following result.

Proposition 3.2

Let

$$R_{11}(l_x, l_y) = \int_{l_x}^{\infty} \int_{l_y}^{\infty} S(u, v) dv du \tag{5}$$

and for $(i, j) > (1, 1)$, *let*

$$R_{ij}(l_x, l_y) = \int_{l_x}^{\infty} R_{i-1,j}(u, l_y) du = \int_{l_y}^{\infty} R_{i,j-1}(l_x, v) dv.$$

Then,

$$R_{ij}(l_x, l_y) = \frac{1}{i!j!} \mathbb{E}[(X_{l_x}^{\infty})^i (Y_{l_y}^{\infty})^j]. \tag{6}$$

Application 1: Correlation among layers

- Setting $X = Y$, we have
  $S(u,v) = P[X > u, Y > v] = P[X > \max(u,v)] = S_x(\max(u,v))$.
- Then for two non-overlapping layers (d_1, l_1) and (d_2, l_2) of X with $d_2 \geq l_1$, we have
-

$$\begin{aligned}
\mathbb{E}[X_{d_1}^{l_1} X_{d_2}^{l_2}] &= \int_{d_1}^{l_1} \int_{d_2}^{l_2} S(u,v)dvdu \\
&= \int_{d_1}^{l_1} \int_{d_2}^{l_2} S_x(v)dvdu \\
&= (l_1 - d_1)\mathbb{E}[X_{d_2}^{l_2}]. \qquad (7)
\end{aligned}$$

Application 1: Correlation among layers

Consequently

- $$Cov[X_{d_1}^{l_1} X_{d_2}^{l_2}] = \left(l_1 - d_1 - \mathbb{E}[X_{d_1}^{l_1}]\right) \mathbb{E}[X_{d_2}^{l_2}], \qquad (8)$$

  which is Equation (39) of Miccolis (1977).

- $$\mathbb{E}[X_d^l X_l^\infty] = (l - d)\mathbb{E}[X_l^\infty]. \qquad (9)$$

Application 1: Correlation among layers

- applying it to (3) yields

$$\mathbb{E}\left[(X_d^l)^2\right] = \mathbb{E}\left[(X_d^\infty)^2\right] - \mathbb{E}\left[(X_l^\infty)^2\right] - 2(l-d)E[X_l^\infty]. \quad (10)$$

- Notice that all three terms on the right hand side of (10) can be obtained from Table M.

Application 2: Bivariate Table M

Assume that one observes a sample of a pair of bivariate loss ratio random variables (X, Y) as shown in the Table 3.

X	0.6	0.8	1.2	1.4
Y	0.4	0.6	1.4	1.6

Table 3: Sample of Bivariate Loss Ratios

Spread Sheet

Application 3: Bivariate Pareto Distribution

- Following Wang (1998), assume that there exists a random parameter Λ such that for $i = 1, 2$, $X_i | \Lambda = \lambda$ are independent and exponentially distributed with rate parameter λ / θ_i.
- Assume that Λ follows a Gamma $(\alpha, 1)$ distribution with moment generating function $M_\Lambda(t) = (1 - t)^{-\alpha}$. Then the unconditional distribution of (X_1, X_2) is a bivariate Pareto with the joint survival function

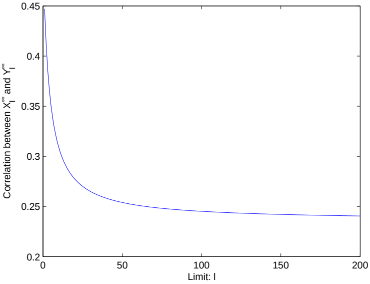
$$S(x, y) = \left(1 + \frac{x}{\theta_1} + \frac{y}{\theta_2}\right)^{-\alpha}. \tag{11}$$

Application 3: Bivariate Pareto Distribution

- $$\mathbb{E}(X_l^\infty) = \frac{\theta_1}{(\alpha-1)}\left(1+\frac{l}{\theta_1}\right)^{-\alpha+1},$$
- $$\mathbb{E}(X_{l_x}^\infty Y_{l_y}^\infty) = \frac{\theta_1\theta_2}{(\alpha-1)(\alpha-2)}\left(1+\frac{l_x}{\theta_1}+\frac{l_y}{\theta_2}\right)^{-\alpha+2},$$
- $$\begin{aligned}\mathbb{E}[(X_l^\infty)^2] &= 2\int_l^\infty (x-l)\left(1+\frac{x}{\theta_1}\right)^{-\alpha} dx \\ &= \frac{2\theta_1^2}{(\alpha-1)(\alpha-2)}\left(\frac{\theta_1+l}{\theta_1}\right)^{-\alpha+2}.\end{aligned}$$

Application 3: Bivariate Pareto Distribution

Figure 4: The correlation between X_l^∞ and Y_l^∞ as a function of l.



Application 4: Per-occurrence and stop–loss coverage

- This example follows the one in Section 2 of Homer and Clark (2002) with some modifications. Assume that the size of Workers Compensation losses from a fictional large insured ABC, denoted by Z, follow a Pareto distribution with the survival function
$$S(x) = \left(1 + \frac{x}{\theta}\right)^{-\alpha},$$
where $\alpha = 3$ and $\theta = \$100,000$.
- Assume that the number of losses N follows a negative binomial distribution with parameters where $\beta = 0.2$ and $r = 25$.

Application 4: Per-occurrence and stop–loss coverage

- As an actuary of XYZ, you are asked to consider a per–occurrence coverage of $\$50,000$ excess of d_0 and then a stop–loss coverage on an aggregate basis of $\$500,000$ excess of d_1 .
- You are trying to determine an optimal combination of d_0 and d_1, so that your objective function– the ratio between the expected payments and the standard deviation of the payments, – is maximized.

Application 4: Per-occurrence and stop–loss coverage

- The amount that ABC has to pay per occurrence:

$$Z_A = Z_0^{d_0} + Z_{d_0+50}^{\infty}.$$

- The amount that XYZ has to pay per occurrence:

$$Z_X = Z_{d_0}^{d_0+50}.$$

Application 4: Per-occurrence and stop–loss coverage

- The aggregate amount that XYZ pays for the per–occurrence coverage

$$V = \sum_{i=1}^{N} Z_{X,i}$$

- The aggregate amount ABC pays after the per–occurrence coverage but before the stop–loss coverage:

$$U = \sum_{i=1}^{N} Z_{A,i}$$

- Then the total amount XYZ has to pay under the insurance treaty is given by

$$W = V + U_{d_1}^{d_1+500}.$$

Application 4: Per-occurrence and stop–loss coverage

Our goal is to select values of d_0 and d_1 so that the objective function $\mathbb{E}[W]/\sigma_W$, where σ_W stands for the standard deviation of W, is maximized.

Application 4: Per-occurrence and stop–loss coverage

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	60.5477	51.4147	50.1536	49.8230	49.7006
100	36.9328	25.5104	24.0836	23.7256	23.5960
150	28.3191	15.3471	13.8257	13.4540	13.3209
200	24.6302	10.5683	8.9845	8.6046	8.4696
250	22.8897	8.0352	6.4053	6.0201	5.8840
300	22.0208	6.5733	4.9065	4.5175	4.3806

Table 4: The expected value of W (in thousands).

Application 4: Per-occurrence and stop–loss coverage

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	86.2705	56.5318	50.3110	48.4753	47.7640
100	83.5429	45.7247	36.9180	34.2140	33.1540
150	82.5044	40.6877	29.9994	26.5188	25.1145
200	81.8811	38.0508	26.0496	21.9057	20.1714
250	81.4313	36.5486	23.6461	18.9579	16.9179
300	81.0813	35.6331	22.1120	16.9900	14.6755

Table 5: The standard deviation of W (in thousands).

Application 4: Per-occurrence and stop–loss coverage

$d_0 \backslash d_1$	500	1000	1500	2000	2500
50	0.7018	0.9095	0.9969	1.0278	1.0405
100	0.4421	0.5579	0.6524	0.6934	0.7117
150	0.3432	0.3772	0.4609	0.5073	0.5304
200	0.3008	0.2777	0.3449	0.3928	0.4199
250	0.2811	0.2198	0.2709	0.3175	0.3478
300	0.2716	0.1845	0.2219	0.2659	0.2985

Table 6: The ratio of the mean and the standard deviation of W.

Conclusions

- Higher moments of excess losses may be obtained from Table M.
- The concept of bivariate excess losses may be useful.

Conclusions

Thank you!

References:

J. Brosius. Table M construction. *CAS Exam 8 Study Notes*, 2002.

M. Denuit, C. Lefevre, M. Shaked, et al. The s-convex orders among real random variables, with applications. *Mathematical Inequalities and Their Applications*, 1(4):585–613, 1998.

M. Denuit, C. Lefevre, and M. Mesfioui. A class of bivariate stochastic orderings, with applications in actuarial sciences. *Insurance: Mathematics and Economics*, 24(1):31–50, 1999.

J. Dhaene and M. J. Goovaerts. Dependency of risks and stop-loss order. *ASTIN BULLETIN*, 26(2):201–212, 1996.

R. Gupta and P. Sankaran. Bivariate equilibrium distribution and its applications to reliability. *Communications in Statistics-Theory and Methods*, 27(2):385–394, 1998.

L. J. Halliwell. The mathematics of excess losses. *Variance*, 6: 32–47, 2012.

W. Hürlimann. Higher degree stop-loss transforms and stochastic orders(i) theory. *Blätter der DGVFM*, 24(3):449–463, 2000.

Y.-S. Lee. The mathematics of excess of loss coverages and retrospective rating-a graphical approach. *PCAS LXXV*, page 49, 1988.

R. S. Miccolis. On the theory of increased limits and excess of loss pricing. *PCAS LXIV*, page 27, 1977.

S. Wang. Aggregation of correlated risk portfolios: models and algorithms. In *Proceedings of the Casualty Actuarial society*, volume 85, pages 848–939, 1998.

H. Zahedi. Some new classes of multivariate survival distribution functions. *Journal of Statistical Planning and Inference*, 11(2): 171–188, 1985.